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ROOT CUBE MEAN CORDIAL LABELING OF $C_n \vee C_m$, FOR $n, m \in \mathbb{N}$

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Abstract. All the graphs considered in this article are simple and undirected. Let G = (V(G), E(G)) be a simple undirected Graph. A function $f : V(G) \to \{0,1,2\}$ is called root cube mean cordial labeling if the induced function $f^* : E(G) \to \{0,1,2\}$ defined by $f^*(uv) = \lfloor \sqrt{\frac{((f(u))^3+(f(v))^3}{2}} \rfloor$ satisfies the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x \rfloor$ denotes the greatest integer less than or equals to x. A Graph G is called root cube mean cordial if it admits root cube mean cordial labeling. In this article we have shown that the join of two cycles $C_n \vee C_m$ is not a root cube mean cordial and also we have provided graph which is root cube mean cordial.

Keywords: Cycle, root cube mean cordial labeling, Join of two graphs $G \vee H$, labeling, corona of graphs.

AMS Subject Classification: 05C78.

1. Introduction

All the graphs considered in this article are simple, undirected and finite. Recall from [1] that for two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the union of G_1 and G_2 is denoted by $G_1 \cup G_2$ is a graph whose vertex set is $V_1 \cup V_2$ and edge set is $E_1 \cup E_2$ and if G_1 and G_2 are vertex disjoint, then $G_1 \cup G_2$ is called sum of G_1 and G_2 and it is denoted by $G_1 + G_2$. Recall from [1], Def. 1.8.3 that the *join of two graphs* G and H denoted as $G \vee H$ is a supergraph of $G + H$ in which every vertex of G is adjacent to each vertex of H. Note that $|V(G \vee H)| = |V(G)| + |V(H)|$ and $|E(G \vee H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$. Let G = (V(G), E(G)) be a simple undirected Graph. Recall from [4] that a function $f : V(G) \to \{0,1,2\}$ is called *root cube mean cordial labeling* if the induced function

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$f^* : E(G) \to \{0,1,2\}$ defined by $f^*(uv) = \lfloor\sqrt{\frac{((f(u))^3+(f(v))^3}{2}}\rfloor$ satisfied the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and number of edges with label x respectively and $\lfloor x\rfloor$ denotes the greatest integer less than or equals to x. A Graph G is called *root cube mean cordial* if it admits root cube mean cordial labeling. In [4], the authors defined root cube mean cordial labeling and they have proved some interesting results. Motivated by the results proved in [4], in this article we have proved that the join of two cycles is not root cube mean cordial. Let G be a graph and $\{v_1, v_2, ..., v_n\} \subseteq V(G)$. We called $v_1, v_2, ..., v_n$ are in *sequence with respect to label x* if $v_1, v_2, ..., v_n$ forms a path. For the sake of convenience of the reader, we use abbreviation RCMC for root cube meal cordial labeling.

2. Main Results

Remark 2.1. *If all the vertices with labels 1 and 2 are in sequence in cycle C_n, then it is clear that all the vertices with labels 0 are in sequence in cycle C_n. So, to prove all the vertices are in sequence in cycle C_n, it is enough to prove that all the vertices with labels 1 and 2 are in sequence in cycle C_n. Now, it is clear that all the vertices with label 2 are in sequence in cycle C_n, then it produces a minimum number of edges with label 2 in cycle C_n and when all the vertices with label 1 are in sequence in cycle C_n, then it produces a maximum number of edges with label 1 in cycle C_n. So, this is the best possible situation in which $|e_f(2) - e_f(1)|$ is minimum in cycle C_n. So, now onwards, we have considered all the vertices with labels 1 and 2 are in sequence in cycle C_n. Hence, all the vertices with labels 0, 1 and 2 are in sequence in cycle C_n.*

Remark 2.2. *Let $p, q \equiv 0 \pmod 3$. If all the vertices in C_p or C_q have the same labels x; for some $x \in \{0,1,2\}$, then $C_p \vee C_q$ is not RCMC.*

Proof. Let $p = 3n$ and $q = 3m$ for some $m, n \in \mathbb{N}$. Without loss of generality, we may assume that $n < m$. Note that $|V(C_p \vee C_q)| = 3m + 3n$. Suppose that $C_p \vee C_q$ is RCMC. Then we have $v_f(0) = v_f(1) = v_f(2) = n + m$.

Case (I) All the vertices in C_p have the label 0

Then in C_q, we have $m - 2n$ number of vertices with label 0, $m + n$ number of vertices with label 1 and $m + n$ number of vertices with label 2. Note that

$e_f(1) = m + n - 1$ and

$e_f(2) = m + n + 1 + 3n(m + n) = m + n + 1 + 3mn + 3n^2$.

So, $e_f(2) - e_f(1) = m + n + 1 + 3mn + 3n^2 - m - n + 1 = 3mn + 3n^2 + 2 > 1$.

Case (II) All the vertices in C_p have the label 1

Then in C_q, we have $m + n$ number of vertices with label 0, $m - 2n$ number of vertices with label 1 and $m + n$ number of vertices with label 2. Note that

$e_f(1) = 3n + m - 2n - 1 + 3n(m - 2n) = m + n - 1 + 3mn - 6n^2$ and

$e_f(2) = m + n + 1 + 3n(m + n) = m + n + 3mn + 3n^2 + 1$.

So, $e_f(2) - e_f(1) = m + n + 1 + 3mn + 3n^2 - m - n + 1 - 3mn + 6n^2 = 9n^2 + 2 > 1$.

Case (III) All the vertices in C_p have the label 2

Then in C_q, we have $m+n$ number of vertices with label 0, $m+n$ number of vertices with label 1 and $m - 2n$ number of vertices with label 2. Note that

$e_f(1) = m + n - 1$ and

$e_f(2) = 3n + m - 2n + 1 + 3n(3m) = m + n + 9mn + 1$.

So, $e_f(2) - e_f(1) = m + n + 9mn + 1 - m - n + 1 = 9mn + 2 > 1$.

Thus, in all the Cases, we have $e_f(2) - e_f(1) > 1$. Hence, $C_p \vee C_q$ is not RCMC. □

Remark 2.3. *Let $p \equiv 0 \ (mod\ 3)$ and $q \equiv 1 \ (mod\ 3)$. If all the vertices in C_p or C_q have the same labels x; for some $x \in \{0,1,2\}$, then $C_p \vee C_q$ is not RCMC.*

Proof. Let $p = 3n$ and $q = 3m+1$ for some $m, n \in \mathbb{N}$. Suppose that $C_p \vee C_q$ is RCMC. Without loss of generality, we may assume that $n < m$. Note that $|V(C_p \vee C_q)| = 3m + 3n + 1$.

Case (I) All the vertices in C_p have the label 0
As $C_p \vee C_q$ is RCMC, we have the following three possibilities :
(i) $v_f(0) = n+m+1, v_f(1) = v_f(2) = n+m$
(ii) $v_f(0) = v_f(2) = n+m, v_f(1) = n+m+1$
(iii) $v_f(0) = v_f(1) = n+m, v_f(2) = n+m+1$.

Subcase (i) $v_f(0) = n+m+1, v_f(1) = v_f(2) = n+m$
Note that in C_q, we have $m-2n+1$ number of vertices with label 0, $m+n$ number of vertices with label 1 and $m+n$ number of vertices with label 2. Note that
$e_f(1) = m+n-1$ and
$e_f(2) = m+n+1+3n(m+n) = m+n+1+3mn+3n^2$.
So, by Case (I) of Remark 2.2, we have $e_f(2) - e_f(1) > 1$.

Subcase (ii) $v_f(0) = v_f(2) = n+m, v_f(1) = n+m+1$
Note that in C_q, we have $m-2n$ number of vertices with label 0, $m+n+1$ number of vertices with label 1 and $m+n$ number of vertices with label 2. Note that
$e_f(1) = m+n$ and
$e_f(2) = m+n+1+3n(m+n) = m+n+1+3mn+3n^2$.
So, $e_f(2) - e_f(1) = m+n+1+3mn+3n^2-m-n = 3mn+3n^2+1 > 1$.

Subcase (iii) $v_f(0) = v_f(1) = n+m, v_f(2) = n+m+1$
Note that in C_q, we have $m-2n$ number of vertices with label 0, $n+m$ number of vertices with label 1 and $m+n+1$ number of vertices with label 2. Note that
$e_f(1) = m+n-1$ and
$e_f(2) = m+n+2+3n(m+n+1) = m+4n+2+3mn+3n^2$.
So, $e_f(2) - e_f(1) = m+4n+2+3mn+3n^2-m-n+1 = 3mn+3n^2+3n+3 > 1$.

Case (II) All the vertices in C_p have the label 1
In this Case, we have the following three subcases :

Subcase (i) $v_f(0) = n+m+1, v_f(1) = v_f(2) = n+m$
Note that in C_q, we have $m+n+1$ number of vertices with label 0, $m-2n$ number of vertices with label 1 and $m+n$ number of vertices with label 2. Note that
$e_f(1) = 3n+m-2n-1+3n(m-2n) = m+n-1+3mn-6n^2$ and
$e_f(2) = m+n+1+3n(m+n) = m+n+1+3mn+3n^2$.
So, $e_f(2) - e_f(1) = m+n+1+3mn+3n^2-m-n+1-3mn+6n^2 = 9n^2+2 > 1$.

Subcase (ii) $v_f(0) = v_f(2) = n+m, v_f(1) = n+m+1$
Note that in C_q, we have $m+n$ number of vertices with label 0, $m-2n+1$ number of vertices with label 1 and $m+n$ number of vertices with label 2. Note that
$e_f(1) = 3n+m-2n+3n(m-2n+1) = m+4n+3mn-6n^2$ and
$e_f(2) = m+n+1+3n(m+n) = m+n+1+3mn+3n^2$.
So, $e_f(2) - e_f(1) = m+n+1+3mn+3n^2-m-4n-3mn+6n^2 = 9n^2-3n+1 > 1$.

Subcase (iii) $v_f(0) = v_f(1) = n+m, v_f(2) = n+m+1$
Note that in C_q, we have $m+n$ number of vertices with label 0, $m-2n$ number of vertices with label 1 and $m+n+1$ number of vertices with label 2. Note that
$e_f(1) = 3n+m-2n-1+3n(m-2n) = m+n-1+3mn-6n^2$ and
$e_f(2) = m+n+2+3n(m+n+1) = m+4n+2+3mn+3n^2$.
So, $e_f(2) - e_f(1) = m+4n+2+3mn+3n^2-m-n+1-3mn+6n^2 = 9n^2+3n+3 > 1$.

Case (III) All the vertices in C_p have the label 2
In this Case, we have the following three subcases :
Subcase (i) $v_f(0) = n + m + 1, v_f(1) = v_f(2) = n + m$
Note that in C_q, we have $m + n + 1$ number of vertices with label 0, $m + n$ number of vertices with label 1 and $m - 2n$ number of vertices with label 2. Note that
$e_f(1) = m + n - 1$ and
$e_f(2) = 3n + m - 2n + 1 + 3n(3m + 1) = m + 4n + 1 + 9mn.$
So, $e_f(2) - e_f(1) = m + 4n + 1 + 9mn - m - n + 1 = 9mn + 3n + 2 > 1.$
Subcase (ii) $v_f(0) = v_f(2) = n + m, v_f(1) = n + m + 1$
Note that in C_q, we have $m + n$ number of vertices with label 0, $m + n + 1$ number of vertices with label 1 and $m - 2n$ number of vertices with label 2. Note that
$e_f(1) = m + n$ and
$e_f(2) = 3n + m - 2n + 1 + 3n(3m + 1) = m + 4n + 1 + 9mn.$
So, $e_f(2) - e_f(1) = m + 4n + 1 + 9mn - m - n = 9mn + 3n + 1 > 1.$
Subcase (iii) $v_f(0) = v_f(1) = n + m, v_f(2) = n + m + 1$
Note that in C_q, we have $m + n$ number of vertices with label 0, $m + n$ number of vertices with label 1 and $m - 2n + 1$ number of vertices with label 2. Note that
$e_f(1) = m + n - 1$ and
$e_f(2) = 3n + m - 2n + 2 + 3n(3m + 1) = m + 4n + 2 + 9mn.$
So, $e_f(2) - e_f(1) = m + 4n + 1 + 9mn - m - n + 1 = 9mn + 3n + 2 > 1.$
Thus, in all the Cases, we get $e_f(2) - e_f(1) > 1$. Hence, $C_p \vee C_q$ is not RCMC. □

Remark 2.4. *Let $p \equiv 0 \ (mod\ 3)$ and $q \equiv 2 \ (mod\ 3)$. If all the vertices in C_p or C_q have the same labels x; for some $x \in \{0, 1, 2\}$, then $C_p \vee C_q$ is not RCMC.*

Proof. Let $p = 3n$ and $q = 3m + 2$ for some $m, n \in \mathbb{N}$. Without loss of generality, we may assume that $n < m$. Note that $|V(C_p \vee C_q)| = 3m + 3n + 2$. Suppose that $C_p \vee C_q$ is RCMC.
Case (I) All the vertices in C_p have the label 0
As $C_p \vee C_q$ is RCMC, we have the following three possibilities :
(i) $v_f(0) = v_f(1) = n + m + 1, v_f(2) = n + m$
(ii) $v_f(0) = n + m, v_f(1) = v_f(2) = n + m + 1$
(iii) $v_f(0) = v_f(2) = n + m + 1, v_f(1) = n + m.$
Subcase (i) $v_f(0) = v_f(1) = n + m + 1, v_f(2) = n + m$
In C_q, we have $m - 2n + 1$ number of vertices with label 0, $m + n + 1$ number of vertices with label 1 and $m + n$ number of vertices with label 2. Note that
$e_f(1) = m + n$ and
$e_f(2) = m + n + 1 + 3n(m + n) = m + n + 1 + 3mn + 3n^2.$
So, $e_f(2) - e_f(1) = m + n + 1 + 3mn + 3n^2 - m - n = 3mn + 3n^2 + 1 > 1.$
Subcase (ii) $v_f(0) = n + m, v_f(1) = v_f(2) = n + m + 1$
In C_q, we have $m - 2n$ number of vertices with label 0, $n + m + 1$ number of vertices with label 1 and $m + n + 1$ number of vertices with label 2. Note that
$e_f(1) = m + n$ and
$e_f(2) = m + n + 2 + 3n(m + n + 1) = m + 4n + 2 + 3mn + 3n^2.$
So, $e_f(2) - e_f(1) = m + 4n + 2 + 3mn + 3n^2 - m - n = 3mn + 3n^2 + 3n + 2 > 1.$
Subcase (iii) $v_f(0) = v_f(2) = n + m + 1, v_f(1) = n + m$
In C_q, we have $m - 2n + 1$ number of vertices with label 0, $n + m$ number of vertices with label 1 and $m + n + 1$ number of vertices with label 2. Note that
$e_f(1) = m + n - 1$ and
$e_f(2) = m + n + 2 + 3n(m + n + 1) = m + 4n + 2 + 3mn + 3n^2.$

So, $e_f(2) - e_f(1) = m + 4n + 2 + 3mn + 3n^2 - m - n + 1 = 3mn + 3n^2 + 3n + 3 > 1$.

Case (II) All the vertices in C_p have the label 1

In this Case, we have the following three subcases :

Subcase (i) $v_f(0) = v_f(1) = n + m + 1, v_f(2) = n + m$

In C_q, we have $m + n + 1$ number of vertices with label 0, $m - 2n + 1$ number of vertices with label 1 and $m + n$ number of vertices with label 2. Note that

$e_f(1) = 3n + m - 2n + 3n(m - 2n + 1) = m + 4n + 3mn - 6n^2$ and

$e_f(2) = m + n + 1 + 3n(m + n) = m + n + 1 + 3mn + 3n^2$.

So, $e_f(2) - e_f(1) = m + n + 1 + 3mn + 3n^2 - m - 4n - 3mn + 6n^2 = 9n^2 - 3n + 1 > 1$.

Subcase (ii) $v_f(0) = n + m, v_f(1) = v_f(2) = n + m + 1$

In C_q, we have $m + n$ number of vertices with label 0, $m - 2n + 1$ number of vertices with label 1 and $m + n + 1$ number of vertices with label 2. Note that

$e_f(1) = 3n + m - 2n + 3n(m - 2n + 1) = m + 4n + 3mn - 6n^2$ and

$e_f(2) = m + n + 2 + 3n(m + n + 1) = m + 4n + 2 + 3mn + 3n^2$.

So, $e_f(2) - e_f(1) = m + 4n + 2 + 3mn + 3n^2 - m - 4n - 3mn + 6n^2 = 9n^2 + 2 > 1$.

Subcase (iii) $v_f(0) = v_f(2) = n + m + 1, v_f(1) = n + m$

In C_q, we have $m + n + 1$ number of vertices with label 0, $m - 2n$ number of vertices with label 1 and $n + m + 1$ number of vertices with label 2. Note that

$e_f(1) = 3n + m - 2n - 1 + 3n(m - 2n) = m + n - 1 + 3mn - 6n^2$ and

$e_f(2) = m + n + 2 + 3n(m + n + 1) = m + 4n + 2 + 3mn + 3n^2$.

So, $e_f(2) - e_f(1) = m + 4n + 2 + 3mn + 3n^2 - m - n + 1 - 3mn + 6n^2 = 9n^2 + 3n + 3 > 1$.

Case (III) All the vertices in C_p have the label 2

In this Case, we have the following three subcases :

Subcase (i) $v_f(0) = v_f(1) = n + m + 1, v_f(2) = n + m$

In C_q, we have $m + n + 1$ number of vertices with label 0, $m + n + 1$ number of vertices with label 1 and $m - 2n$ number of vertices with label 2. Note that

$e_f(1) = m + n$ and

$e_f(2) = 3n + m - 2n + 1 + 3n(3m + 2) = m + 7n + 1 + 9mn$.

So, $e_f(2) - e_f(1) = m + 7n + 1 + 9mn - m - n = 9mn + 6n + 2 > 1$.

Subcase (ii) $v_f(0) = n + m, v_f(1) = v_f(2) = n + m + 1$

In C_q, we have $m + n$ number of vertices with label 0, $m + n + 1$ number of vertices with label 1 and $m - 2n + 1$ number of vertices with label 2. Note that

$e_f(1) = m + n$ and

$e_f(2) = 3n + m - 2n + 2 + 3n(3m + 2) = m + 7n + 2 + 9mn$.

So, $e_f(2) - e_f(1) = m + 7n + 2 + 9mn - m - n = 9mn + 6n + 2 > 1$.

Subcase (iii) $v_f(0) = v_f(2) = n + m + 1, v_f(1) = n + m$

In C_q, we have $m + n + 1$ number of vertices with label 0, $m + n$ number of vertices with label 1 and $m - 2n + 1$ number of vertices with label 2. Note that

$e_f(1) = m + n - 1$ and

$e_f(2) = 3n + m - 2n + 2 + 3n(3m + 2) = m + 7n + 2 + 9mn$.

So, $e_f(2) - e_f(1) = m + 7n + 1 + 9mn - m - n + 1 = 9mn + 6n + 2 > 1$.

Thus, in all the Cases, we get $e_f(2) - e_f(1) > 1$. Hence, $C_p \vee C_q$ is not RCMC.

□

Remark 2.5. *Let $p, q \equiv 1 \ (mod\ 3)$. If all the vertices in C_p or C_q have the same labels x; for some $x \in \{0, 1, 2\}$, then $C_p \vee C_q$ is not RCMC.*

Proof. Let $p = 3n + 1$ and $q = 3m + 1$ for some $m, n \in \mathbb{N}$. Without loss of generality, we may assume that $n < m$. Note that $|V(C_p \vee C_q)| = 3m + 3n + 2$. Suppose that $C_p \vee C_q$ is RCMC.

Case (I) All the vertices in C_p have the label 0
As $C_p \vee C_q$ is RCMC, we have the following two subcases :
(i) $v_f(0) = v_f(1) = n + m + 1, v_f(2) = n + m$
(ii) $v_f(0) = v_f(2) = n + m + 1, v_f(1) = n + m$.
Subcase (i) $v_f(0) = v_f(1) = n + m + 1, v_f(2) = n + m$
Note that in C_q, we have $m - 2n$ number of vertices with label 0, $m + n + 1$ number of vertices with label 1 and $m + n$ number of vertices with label 2. Note that
$e_f(1) = m + n$ and
$e_f(2) = m + n + 1 + (3n + 1)(m + n) = 2m + 2n + 1 + 3mn + 3n^2$.
So, $e_f(2) - e_f(1) = 2m + 2n + 1 + 3mn + 3n^2 - m - n = 3mn + 3n^2 + m + n + 1 > 1$.
Subcase (ii) $v_f(0) = v_f(2) = n + m + 1, v_f(1) = n + m$
Note that in C_q, we have $m-2n$ number of vertices with label 0, $m+n$ number of vertices with label 1 and $m + n + 1$ number of vertices with label 2. Note that
$e_f(1) = m + n - 1$ and
$e_f(2) = m + n + 2 + (3n + 1)(m + n + 1) = 2m + 5n + 3 + 3mn + 3n^2$.
So, $e_f(2) - e_f(1) = 2m + 5n + 3 + 3mn + 3n^2 - m - n + 1 = 3mn + 3n^2 + m + 4n + 4 > 1$.
Case (II) All the vertices in C_p have the label 1
As $C_p \vee C_q$ is RCMC, we have the following two subcases :
(i) $v_f(1) = v_f(0) = n + m + 1, v_f(2) = n + m$
(ii) $v_f(1) = v_f(2) = n + m + 1, v_f(0) = n + m$.
Subcase (i) $v_f(1) = v_f(0) = n + m + 1, v_f(2) = n + m$
Note that in C_q, we have $m + n + 1$ number of vertices with label 0, $m - 2n$ number of vertices with label 1 and $m + n$ number of vertices with label 2. Note that
$e_f(1) = 3n + 1 + m - 2n - 1 + (3n + 1)(m - 2n) = 2m - n + 3mn - 6n^2$ and
$e_f(2) = m + n + 1 + (3n + 1)(m + n) = 2m + 2n + 1 + 3mn + 3n^2$.
So, $e_f(2) - e_f(1) = 2m + 2n + 1 + 3mn + 3n^2 - 2m + n - 3mn + 6n^2 = 9n^2 + 3n + 1 > 1$.
Subcase (ii) $v_f(1) = v_f(2) = n + m + 1, v_f(0) = n + m$
Note that in C_q, we have $m+n$ number of vertices with label 0, $m-2n$ number of vertices with label 1 and $m + n + 1$ number of vertices with label 2. Note that
$e_f(1) = 3n + 1 + m - 2n - 1 + (3n + 1)(m - 2n) = 2m - n + 3mn - 6n^2$ and
$e_f(2) = m + n + 2 + (3n + 1)(m + n + 1) = 2m + 5n + 3 + 3mn + 3n^2$.
So, $e_f(2) - e_f(1) = 2m + 5n + 3 + 3mn + 3n^2 - 2m + n - 3mn + 6n^2 = 9n^2 + 6n + 3 > 1$.
Case (III) All the vertices in C_p have the label 2
As $C_p \vee C_q$ is RCMC, we have the folllowing two subcases :
(i) $v_f(2) = v_f(0) = n + m + 1, v_f(1) = n + m$
(ii) $v_f(2) = v_f(1) = n + m + 1, v_f(0) = n + m$.
Subcase (i) $v_f(2) = v_f(0) = n + m + 1, v_f(1) = n + m$
Note that in C_q, we have $m + n + 1$ number of vertices with label 0, $m + n$ number of vertices with label 1 and $m - 2n$ number of vertices with label 2. Note that
$e_f(1) = m + n - 1$ and
$e_f(2) = 3n + 1 + m - 2n + 1 + (3n + 1)(3m + 1) = 4m + 4n + 3 + 9mn$.
So, $e_f(2) - e_f(1) = 4m + 4n + 3 + 9mn - m - n + 1 = 9mn + 3m + 3n + 4 > 1$.
Subcase (ii) $v_f(2) = v_f(1) = n + m + 1, v_f(0) = n + m$
Note that in C_q, we have $m + n$ number of vertices with label 0, $m + n + 1$ number of vertices with label 1 and $m - 2n$ number of vertices with label 2. Note that
$e_f(1) = m + n$ and
$e_f(2) = 3n + 1 + m - 2n + 1 + (3n + 1)(3m + 1) = 4m + 4n + 3 + 9mn$.
So, $e_f(2) - e_f(1) = 4m + 4n + 3 + 9mn - m - n = 9mn + 3m + 3n + 3 > 1$.
Thus, in all the Cases, we have $e_f(2) - e_f(1) > 1$. Hence, $C_p \vee C_q$ is not RCMC. □

Remark 2.6. *Let $p \equiv 1 \ (mod\ 3)$ and $q \equiv 2 \ (mod\ 3)$. If all the vertices in C_p or C_q have the same labels x; for some $x \in \{0,1,2\}$, then $C_p \vee C_q$ is not RCMC .*

Proof. Let $p = 3n+1$ and $q = 3m+2$ for some $m, n \in \mathbb{N}$. Without loss of generality, we may assume that $n < m$. Note that $|V(C_p \vee C_q)| = 3m+3n+3$. Suppose that $C_p \vee C_q$ is RCMC. Then we have $v_f(0) = v_f(1) = v_f(2) = n+m+1$

Case (I) All the vertices in C_p have the label 0

Then in C_q, we have $m-2n$ number of vertices with label 0, $m+n+1$ number of vertices with label 1 and $m+n+1$ number of vertices with label 2. Note that

$e_f(1) = m+n$ and

$e_f(2) = m+n+2+(3n+1)(m+n+1) = 2m+5n+3+3mn+3n^2$.

So, $e_f(2) - e_f(1) = 2m+5n+3+3mn+3n^2 - m - n = 3mn+3n^2+m+4n+3 > 1$.

Case (II) All the vertices in C_p have the label 1

Then in C_q, we have $m+n+1$ number of vertices with label 0, $m-2n$ number of vertices with label 1 and $m+n+1$ number of vertices with label 2. Note that

$e_f(1) = 3n+1+m-2n-1+(3n+1)(m-2n) = 2m-n+3mn-6n^2$ and

$e_f(2) = m+n+2+(3n+1)(m+n+1) = 2m+5n+3+3mn+3n^2$.

So, $e_f(2)-e_f(1) = 2m+5n+3+3mn+3n^2-2m+n-3mn+6n^2 = 9n^2+6n+3 > 1$.

Case (III) All the vertices in C_p have the label 2

Then in C_q, we have $m+n+1$ number of vertices with label 0, $m+n+1$ number of vertices with label 1 and $m-2n$ number of vertices with label 2. Note that

$e_f(1) = m+n$ and

$e_f(2) = 3n+1+m-2n+1+(3n+1)(3m+2) = 4m+7n+4+9mn$.

So, $e_f(2)-e_f(1) = 4m+7n+4+9mn-m-n = 9mn+3m+6n+4 > 1$.

Thus, in all the Cases, we have $e_f(2)-e_f(1) > 1$. Hence, $C_p \vee C_q$ is not RCMC.

□

Remark 2.7. *Let $p, q \equiv 2 \ (mod\ 3)$. If all the vertices in C_p or C_q have the label x; for some $x \in \{0,1,2\}$, then $C_p \vee C_q$ is not RCMC.*

Proof. Let $p = 3n+2$ and $q = 3m+2$ for some $m, n \in \mathbb{N}$. Without loss of generality, we may assume that $n < m$. Note that $|V(C_p \vee C_q)| = 3m+3n+2$. Suppose that $C_p \vee C_q$ is RCMC.

Case (I) All the vertices in C_p have the label 0

As $C_p \vee C_q$ is RCMC, we have $v_f(0) = n+m+2, v_f(1) = v_f(2) = n+m+1$

Note that in C_q, we have $m-2n$ number of vertices with label 0, $m+n+1$ number of vertices with label 1 and $m+n+1$ number of vertices with label 2. Note that

$e_f(1) = m+n$ and

$e_f(2) = m+n+2+(3n+2)(m+n+1) = 3m+6n+4+3mn+3n^2$.

So, $e_f(2)-e_f(1) = 3m+6n+4+3mn+3n^2-m-n = 3mn+3n^2+2m+5n+4 > 1$.

Case (II) All the vertices in C_p have the label 1

As $C_p \vee C_q$ is RCMC, we have $v_f(1) = n+m+2, v_f(0) = v_f(2) = n+m+1$

Note that in C_q, we have $m+n+1$ number of vertices with label 0, $m-2n$ number of vertices with label 1 and $m+n+1$ number of vertices with label 2. Note that

$e_f(1) = 3n+2+m-2n-1+(3n+2)(m-2n) = 3m-3n+3mn-6n^2+1$ and

$e_f(2) = m+n+2+(3n+2)(m+n+1) = 3m+6n+4+3mn+3n^2$.

So, $e_f(2)-e_f(1) = 3m+6n+4+3mn+3n^2-3m+3n-3mn+6n^2-1 = 9n^2+9n+5n+3 > 1$.

Case (III) All the vertices in C_p have the label 2

As $C_p \vee C_q$ is RCMC, we have $v_f(2) = n+m+2, v_f(0) = v_f(1) = n+m+1$

Note that in C_q, we have $m+n+1$ number of vertices with label 0, $m+n+1$ number of

vertices with label 1 and $m-2n$ number of vertices with label 2. Note that
$e_f(1) = m+n$ and
$e_f(2) = 3n+2+m-2n+1+(3n+2)(3m+2) = 7m+7n+7+9mn$.
So, $e_f(2) - e_f(1) = 7m+7n+7+9mn-m-n = 9mn+6m+6n+7 > 1$.
Thus, in all the Cases, we have $e_f(2) - e_f(1) > 1$. Hence, $C_p \vee C_q$ is not RCMC. □

Theorem 2.1. *$C_p \vee C_q$ is not RCMC, for any $p, q \in \mathbb{N}$.*

Proof. Suppose that $C_p \vee C_q$ is RCMC. By Remark 2.1 it is now clear that all the vertices in C_p and C_q must be in sequence with respect to each label x, for $x \in \{0,1,2\}$. Hence, throughout the proof we consider that all the vertices with respect to each label x, for $x \in \{0,1,2\}$ are in sequence.

Case (I) $p \equiv 0 \ (mod\ 3)$, $q \equiv 0 \ (mod\ 3)$

Let $p = 3n$ and $q = 3m$ for some $n, m \in \mathbb{N}$. Note that $|V(C_p \vee C_q)| = 3(n+m)$. As $C_p \vee C_q$ is RCMC, we have $v_f(0) = v_f(1) = v_f(2) = n+m$. Suppose that in C_{3n} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m} we have, $v_f(0) = m+n-t$, $v_f(1) = m+n-s$, $v_f(2) = m+n-r$. Also, we have, $t+s+r = 3n$. Now, in C_{3n} we have, $e_f(0) = t$, $e_f(1) = s-1$ and in C_{3m} we have $e_f(0) = m+n-t$, $e_f(1) = m+n-s-1$. Therefore, in $C_{3n} \vee C_{3m}$ we have,

$$\begin{aligned} e_f(0) &= t+m+n-t+t(m+n-t)+t(m+n-s)+s(m+n-t) \\ &= m+n+2tm+2tn-t^2-2ts+sm+sn \text{ and} \end{aligned}$$

$e_f(1) = s-1+m+n-s-1+s(m+n-s) = m+n-2+sm+sn-s^2$. Now,

$$\begin{aligned} |e_f(0)-e_f(1)| &= |m+n+2tm+2tn-t^2-2ts+sm+sn-m-n+2-sm-sn+s^2| \\ &= |2tm+2tn+s^2-2st-t^2+2| \\ &= |2tm+2tn+s^2-2st+t^2-2t^2+2| \\ &= |2tm+2tn+(s-t)^2-2t^2+2| \\ &> |2tm+2tn+(s-t)^2-2tn+2| \qquad (t<n \Rightarrow -2tn < -2tt) \\ &= |2tm+(s-t)^2+2| \\ &> 2. \end{aligned}$$

Case (II) $p \equiv 0 \ (mod\ 3)$, $q \equiv 1 \ (mod\ 3)$

Let $p = 3n$ and $q = 3m+1$ for some $n, m \in \mathbb{N}$. Note that $|V(C_p \vee C_q)| = 3(n+m)+1$. So, we have the following three subcases in this Case :
(1) $v_f(0) = m+n+1$, $v_f(1) = m+n$ and $v_f(2) = m+n$
(2) $v_f(0) = m+n$, $v_f(1) = m+n+1$ and $v_f(2) = m+n$
(3) $v_f(0) = m+n$, $v_f(1) = m+n$ and $v_f(2) = m+n+1$

Subcase (i) $v_f(0) = m+n+1$, $v_f(1) = m+n$ and $v_f(2) = m+n$

Suppose that in C_{3n} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+1} we have, $v_f(0) = m+n-t+1$, $v_f(1) = m+n-s$, $v_f(2) = m+n-r$. Also, we have $t+s+r = 3n$. Now in C_{3n}, we have, $e_f(0) = t$, $e_f(1) = s-1$ and in C_{3m+1} we have $e_f(0) = m+n-t+1$, $e_f(1) = m+n-s-1$. Hence, in $C_{3n} \vee C_{3m+1}$ we have,

$$\begin{aligned} e_f(0) &= t+m+n-t+1+t(m+n-t+1)+t(m+n-s)+s(m+n-t+1) \\ &= m+n+2tm+2tn-t^2+t+s-2ts+sm+sn+1 \text{ and} \end{aligned}$$

$e_f(1) = s-1+m+n-s-1+s(m+n-s) = m+n-2+sm+sn-s^2$. Now

$$\begin{aligned} |e_f(0)-e_f(1)| &= |m+n+2tm+2tn-t^2+t+s-2ts+sm+sn+1-m-n+2-sm-sn+s^2| \\ &= |2tm+2tn+s^2-2st-t^2+t+s+3| \\ &= |2tm+2tn+s^2-2st+t^2-2t^2+t+s+3| \\ &= |2tm+2tn+(s-t)^2-2t^2+t+s+3| \\ &> |2tm+2tn+(s-t)^2-2tn+t+s+3| \qquad (t<n \Rightarrow -2tn < -2tt) \\ &= |2tm+(s-t)^2+t+s+3| \\ &> 3. \end{aligned}$$

Subcase (ii) $v_f(0) = m+n$, $v_f(1) = m+n+1$ and $v_f(2) = m+n$
Suppose that in C_{3n} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+1} we have, $v_f(0) = m+n-t$, $v_f(1) = m+n-s+1$, $v_f(2) = m+n-r$. Also, we have $t+s+r = 3n$. Note that in C_{3n} we have $e_f(0) = t$, $e_f(1) = s-1$ and in C_{3m+1} we have, $e_f(0) = m+n-t$, $e_f(1) = m+n-s$. Hence, in $C_{3n} \vee C_{3m}$ we have,
$e_f(0) = t+m+n-t+t(m+n-t)+t(m+n-s+1)+s(m+n-t)$
$\quad = m+n+2tm+2tn-t^2+t-2ts+sm+sn$ and
$e_f(1) = s-1+m+n-s+s(m+n-s+1) = m+n+sm+sn+s-s^2-1$. Now,
$|e_f(0)-e_f(1)| = |m+n+2tm+2tn-t^2+t-2ts+sm+sn-m-n-sm-sn-s+s^2+1|$
$= |2tm+2tn+s^2-2ts-t^2+t-s+1|$ ....(1)
$= |2tm+2tn+(s-t)^2-2t^2+t-s+1|$
$> |2tm+2tn+(s-t)^2-2tn+t-s+1|$
$= |2tm+(s-t)^2+t-s+1|$
If $s \le t$,then $t-s \ge 0$. So, $|e_f(0)-e_f(1)| > 2$.
If $s > t$, then from equation (1),
$|e_f(0)-e_f(1)| = |2tm+2tn-2ts+s^2-t^2+t-s+1|$
$= |2tm+2tn-2ts+(s-t)(s+t)-(s-t)+1|$
$= |2tm+2tn-2ts+(s-t)(s+t-1)+1|$
$> |2tm+2tn-2tn+(s-t)(s+t-1)+1|$ $(s < n \Rightarrow -2tn < -2ts)$
$= |2tm+(s-t)(s+t-1)+1|$
> 1.

Subcase (iii) $v_f(0) = m+n$, $v_f(1) = m+n$ and $v_f(2) = m+n+1$
Suppose that in C_{3n}, we have $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+1}, we have $v_f(0) = m+n-t$, $v_f(1) = m+n-s$, $v_f(2) = m+n-r+1$. Note that the numbers of vertices with labels 0 and with labels 1 in this Subcase are the same as those in the Case (I). So, in this Case, we have, $|e_f(0)-e_f(1)| > 1$.

Case (III) $p \equiv 0 \ (mod\ 3)$**,** $q \equiv 2 \ (mod\ 3)$
Let $p = 3n$ and $q = 3m+2$ for some $n, m \in \mathbb{N}$. Note that $|V(C_p \vee C_q)| = 3(n+m)+2$. As So, we have the following three subcases in this Case :

Subcase (i) $v_f(0) = m+n+1$, $v_f(1) = m+n+1$, $v_f(2) = m+n$
Suppose that in C_{3n} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2}, we have $v_f(0) = m+n-t+1$, $v_f(1) = m+n-s+1$, $v_f(2) = m+n-r$. Also, we have $t+s+r = 3n$. Note that in C_{3n} we have, $e_f(0) = t$, $e_f(1) = s-1$ and in C_{3m+2}, we have $e_f(0) = m+n-t+1$, $e_f(1) = m+n-s$. Note that $C_{3n} \vee C_{3m+2}$, we have
$e_f(0) = t+m+n-t+1+t(m+n-t+1)+t(m+n-s+1)+s(m+n-t+1)$
$\quad = m+n+2tm+2tn-t^2+2t+s-2ts+sm+sn+1$ and
$e_f(1) = s-1+m+n-s+s(m+n-s+1) = m+n+sm+sn-s^2+s-1$. Now,
$|e_f(0)-e_f(1)| = |m+n+2tm+2tn-t^2+2t+s-2ts+sm+sn+1-m-n-sm-sn+s^2-s+1|$
$= |2tm+2tn+s^2-2st-t^2+2t+2|$
$= |2tm+2tn+s^2-2st+t^2-2t^2+2t+2|$
$= |2tm+2tn+(s-t)^2-2t^2+2t+2|$
$> |2tm+2tn+(s-t)^2-2tn+2t+2|$ $(t < n \Rightarrow -2tn < -2tt)$
$= |2tm+(s-t)^2+2t+2|$
> 2.

Subcase (ii) $v_f(0) = m+n$, $v_f(1) = m+n+1$ and $v_f(2) = m+n+1$
Suppose that in C_{3n} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2} we have, $v_f(0) = m+n-t$, $v_f(1) = m+n-s+1$, $v_f(2) = m+n-r+1$. Note that the numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Subcase (ii) of Case (II). So, in this Case, we have $|e_f(0)-e_f(1)| > 1$.

Subcase (iii) $v_f(0) = m + n + 1$, $v_f(1) = m + n$, $v_f(2) = m + n + 1$
Suppose that in C_{3n} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2} we have, $v_f(0) = m + n - t + 1$, $v_f(1) = m + n - s$, $v_f(2) = m + n - r + 1$. Note that the numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Subcase (i) of Case (II). So, in this Case, we have $|e_f(0) - e_f(1)| > 1$.

Case (IV) : $p \equiv 1\ (mod\ 3)$**,** $q \equiv 1\ (mod\ 3)$
Let $p = 3n+1$ and $q = 3m+1$ for some $n, m \in \mathbb{N}$. Note that $|V(C_p \vee C_q)| = 3(n+m)+2$. So, we have the following three subcases in this Case :

Subcase (i) $v_f(0) = m + n + 1$, $v_f(1) = m + n + 1$, $v_f(2) = m + n$
Suppose that in C_{3n+1} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+1} we have, $v_f(0) = m + n - t + 1$, $v_f(1) = m + n - s + 1$, $v_f(2) = m + n - r$. Also, we have, $t + s + r = 3n + 1$. Note that the numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Subcase (i) of Case (III). So, in this Case, we have $|e_f(0) - e_f(1)| > 1$.

Subcase (ii) $v_f(0) = m + n$, $v_f(1) = m + n + 1$, $v_f(2) = m + n + 1$
Suppose that in C_{3n+1} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+1} we have, $v_f(0) = m + n - t$, $v_f(1) = m + n - s + 1$, $v_f(2) = m + n - r + 1$. Also, we have, $t + s + r = 3n + 1$. Note that the numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Subcase (ii) of Case (III). So, in this Case, we have $|e_f(0) - e_f(1)| > 1$.

Subcase (iii) $v_f(0) = m + n + 1$, $v_f(1) = m + n$, $v_f(2) = m + n + 1$
Suppose that in C_{3n+1} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+1} we have, $v_f(0) = m + n - t + 1$, $v_f(1) = m + n - s$, $v_f(2) = m + n - r + 1$. Also, we have, $t + s + r = 3n + 1$. Note that the numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Subcase (iii) of Case (III). So, in this Case, we have $|e_f(0) - e_f(1)| > 1$.

Case (V) $p \equiv 1\ (mod\ 3)$**,** $q \equiv 2\ (mod\ 3)$
Let $p = 3n+1$ and $q = 3m+2$ for some $n, m \in \mathbb{N}$. Note that $|V(C_p \vee C_q)| = 3(n+m)+3$. So, we have $v_f(0) = v_f(1) = v_f(2) = n+m+1$. Suppose that in C_{3n+1} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2} we have, $v_f(0) = m+n-t+1$, $v_f(1) = m+n-s+1$, $v_f(2) = m+n-r+1$. Note that the numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Subcase (i) of Case (IV). So, in this Case, we have $|e_f(0) - e_f(1)| > 1$.

Case (VI) : $p \equiv 2\ (mod\ 3)$**,** $q \equiv 2\ (mod\ 3)$
Let $p = 3n+2$ and $q = 3m+2$ for some $n, m \in \mathbb{N}$. Note that $|V(C_p \vee C_q)| = 3(n+m)+4$. So, we have the following three subcases in this Case :

Subcase (i) $v_f(0) = m + n + 2$, $v_f(1) = m + n + 1$, $v_f(2) = m + n + 1$
Suppose that in C_{3n+2} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2} we have, $v_f(0) = m + n - t + 2$, $v_f(1) = m + n - s + 1$, $v_f(2) = m + n - r + 1$. Also, we have $t + s + r = 3n + 2$. Note that in C_{3n+2} we have $e_f(0) = t$, $e_f(1) = s - 1$ and in C_{3m+2}, we have $e_f(0) = m + n - t + 2$, $e_f(1) = m + n - s$. So, in $C_{3n+2} \vee C_{3m+2}$ we have,

$$\begin{aligned} e_f(0) &= t + m + n - t + 2 + t(m + n - t + 2) + t(m + n - s + 1) + s(m + n - t + 2) \\ &= m + n + 2tm + 2tn - t^2 - 2ts + sm + sn + 3t + 2s + 2 \text{ and} \end{aligned}$$

$e_f(1) = s - 1 + m + n - s + s(m + n - s + 1) = m + n + sm + sn - s^2 + s - 1$. Now

$$\begin{aligned} |e_f(0)-e_f(1)| &= |m+n+2tm+2tn-t^2-2ts+sm+sn+3t+2s+2-m-n-sm-sn+s^2-s+1| \\ &= |2tm + 2tn + s^2 - 2st - t^2 + 3t + s + 3| \\ &= |2tm + 2tn + s^2 - 2st + t^2 - t^2 + 3t + s + 3| \\ &= |2tm + 2tn + (s - t)^2 - 2t^2 + 3t + s + 3| \\ &> |2tm + 2tn + (s - t)^2 - 2tn + 3t + s + 3| \qquad (t < n \Rightarrow -2tn < -2tt) \end{aligned}$$

$= |2tm + (s-t)^2 + 3t + s + 3|$
$> 3.$

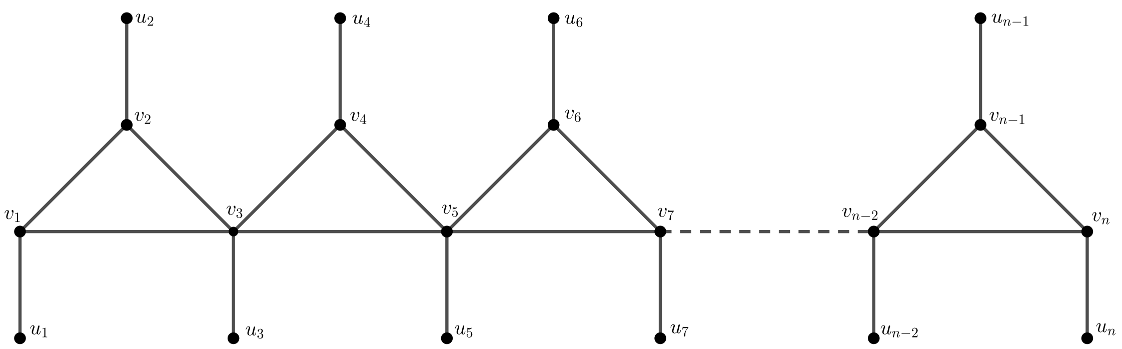
Subcase (ii) $v_f(0) = m+n+1$, $v_f(1) = m+n+2$, $v_f(2) = m+n+1$
Suppose that in C_{3n+2} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2} we have, $v_f(0) = m+n-t+1$, $v_f(1) = m+n-s+2$, $v_f(2) = m+n-r+1$. Also, we have $t+s+r = 3n+2$. Note that in C_{3n+2} we have, $e_f(0) = t$, $e_f(1) = s-1$ and in C_{3m+2} we have, $e_f(0) = m+n-t+1$, $e_f(1) = m+n-s+1$. Hence, in $C_{3n+2} \vee C_{3m+2}$ we have,
$e_f(0) = t+m+n-t+1+t(m+n-t+1)+t(m+n-s+2)+s(m+n-t+1)$
$= m+n+2tm+2tn-t^2-2ts+sm+sn+3t+s+1$ and
$e_f(1) = s-1+m+n-s+1+s(m+n-s+2) = m+n+sm+sn-s^2+2s$. Now,
$|e_f(0)-e_f(1)| = |m+n+2tm+2tn-t^2-2ts+sm+sn+3t+s+1-m-n-sm-sn+s^2-2s|$
$= |2tm+2tn+s^2-2ts-t^2+3t-s+1|$
$= |2tm+2tn+(s-t)^2-2t^2+3t-s+1|$
$> |2tm+2tn+(s-t)^2-2tn+3t-s+1| \qquad (s<n, t<n \Rightarrow -2tn < -2t^2)$
$= |2tm+(s-t)^2+3t-s| \qquad ....(1)$
If $s \le t$, then $|e_f(0) - e_f(1)| > 1$.
If $t < s$, then from equation (1),
$|e_f(0) - e_f(1)| > |2tm + (s-t)^2 + 3t - s|$
$= |2tm + 2t + (s-t)^2 - (s-t)|$
$= |2tm + 2t + (s-t)(s-t-1)|$
$> 1.$

Subcase (iii) $v_f(0) = m+n+1$, $v_f(1) = m+n+1$, $v_f(2) = m+n+2$
Suppose that in C_{3n+2} we have, $v_f(0) = t$, $v_f(1) = s$, $v_f(2) = r$. Then in C_{3m+2} we have, $v_f(0) = m+n-t+1$, $v_f(1) = m+n-s+1$, $v_f(2) = m+n-r+2$. Note that numbers of vertices with labels 0 and labels 1 in this Subcase are the same as those in the Case (V). So, in this Case we have $|e_f(0) - e_f(1)| > 1$.
Therefore, $C_p \vee C_q$ is not RCMC.

□

Theorem 2.2. *$TS_n \odot K_1$ is RCMC for $n = 3k+1$, $n \in \mathbb{N}$ and $k \equiv 0 \ (mod\ 2)$.*

Proof. Note that $|V(TS_n \odot K_1)| = 2n$ and $|E(TS_n \odot K_1)| = \frac{3n-3}{2} + n = \frac{5n-3}{2}$. Let $V(TS_n) = \{v_1, v_2, ..., v_n\}$ be the vertex set of TS_n and u_i be the pendant vertices of $TS_n \odot K_1$ adjacent to v_i for $1 \le i \le n$ as shown in the following figure.

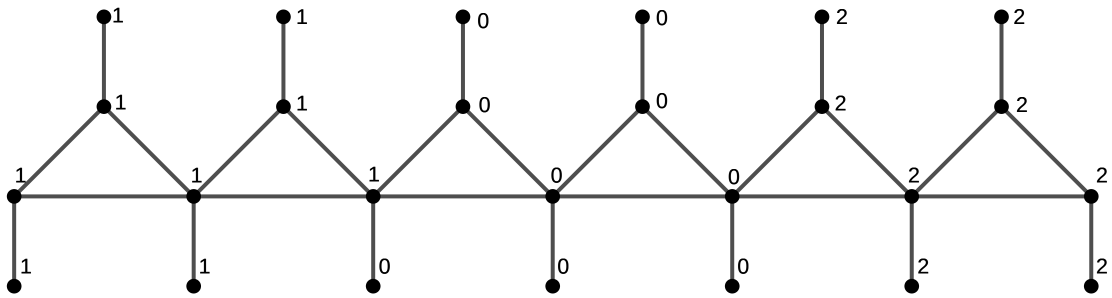


$TS_n \odot K_1$

Define a labeling function $f : V(TS_n) \to \{0,1,2\}$ as follows :
$f(v_i) = 1$ if $1 \le i \le k+1$
$= 0$ if $k+1 < i \le 2k+1$

$= 2$ if $2k+1 < i \le 3k+1$
$f(u_i) = 1$ if $1 \le i \le k$
$= 0$ if $k < i \le 2k+1$
$= 2$ if $2k+1 < i \le 3k+1$
Note that $v_f(0) = 2k+1$, $v_f(1) = 2k+1$, $v_f(2) = 2k$, $e_f(0) = \frac{3k}{2}+k+1$, $e_f(1) = \frac{3k}{2}+k$ and $e_f(2) = \frac{3k}{2}+k$. Thus, $|v_f(i) - v_f(j)| \le 1$ and $|e_f(i) - e_f(j)| \le 1$. Hence, $TS_n \odot K_1$ is RCMC for $n = 3k+1$, $n \in \mathbb{N}$ and $k \equiv 0 \ (mod\ 2)$. □

Example 2.1. *RCMC labeling of $TS_{13} \odot K_1$ is shown in the following figure.*



$TS_{13} \odot K_1$

Remark 2.8. *Here, we are mentioning some of the families of graphs that can be studied by interested researchers, as an open problem .*
(1) Join of graphs
(2) Product of graphs
(3) Family of cycle related graphs like Wheel graph, Helm graph, Closed Helm Graph.

3. Conclusion

In this article, we have proved that the Join of two cycles $C_n \vee C_m$ is not a Root Cube Mean Cordial labeling. Also, we have provided a graph which is RCMC.

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