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A NOVEL STOCHASTIC APPROACH TO BUFFER STOCK PROBLEM

Z. HANALIOGLU[1], A. POLADOVA[2], B. GEVER[3], T. KHANIYEV[2,4*], §

ABSTRACT. In this paper, the stochastic fluctuation of buffer stock level at time t is investigated. Therefore, random walk processes $X(t)$ and $Y(t)$ with two specific barriers have been defined to describe the stochastic fluctuation of the product level. Here $X(t) \equiv Y(t) - a$ and the parameter a specifies half capacity of the buffer stock warehouse. Next, the one-dimensional distribution of the process $X(t)$ has calculated. Moreover, the ergodicity of the process $X(t)$ has been proven and the exact formula for the characteristic function has been found. Then, the weak convergence theorem has been proven for the standardized process $W(t) \equiv X(t)/a$, as $a \to \infty$. Additionally, exact and asymptotic expressions for the ergodic moments of the processes $X(t)$ and $Y(t)$ are obtained.

Keywords: Random walk with two barriers, buffer stock problem, stationary distribution, weak convergence, asymptotic expansion.

AMS Subject Classification: 60G50, 60K15.

1. Introduction

This article discusses an innovative approach to the buffer stock problem. A buffer stock is necessary to avoid delays on the production line and reduce economic losses. By adding a buffer between two machines, it is possible to optimize the production line and reduce system downtime (Lv et al. [17]). Numerous eminent scientists have studied the buffer stock dilemma in their research (see, Cochran et al. [4]; He et al. [10]; Kokangul et al. [15]; Lv et al. [17]; Smith and Cruz [20]; and etc.). In this article, it is aimed to create a

[1] Karabuk University, Department of Actuarial Sciences, Demir Celik Campus, Klavuzlar Koyu, 78050, Karabuk, Turkey.
e-mail: zulfiyyamammadova@karabuk.edu.tr; ORCID: http://orcid.org/0000-0003-1197-9421.
[2] TOBB University of Economics and Technology, Department of Industrial Engineering, Sogutozu Cad. 43, 06560, Ankara, Turkey.
e-mail: apoladova@etu.edu.tr; ORCID: https://orcid.org/0000-0003-1366-8619.
e-mail: tahirkhaniyev@etu.edu.tr; ORCID: http://orcid.org/0000-0003-1974-0140.
[*] Corresponding author.
[3] University of Turkish Aeronautical Association, Department of Industrial Engineering, Bahcekapi Mah., Okul Sk., No.11, 06790, Etimesgut Ankara, Turkey.
e-mail: bgever@thk.edu.tr; ORCID: https://orcid.org/0000-0001-6414-508X.
[4] Department of Digital Technology and Applied Informatics, Azerbaijan State University of Economics, Istiglaliyyet Str. 6, Baku, Azerbaijan.
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mathematical substructure related to the buffer stock problem in a production system. It is assumed that all the machines in this system produce at the same speed and the downtimes are random. In addition, the production line is equipped with a limited capacity buffer stock. Also, the level of stock in the buffer warehouse fluctuates stochastically in the interval $[0, 2a]$. Here, $2a$ denotes the maximum capacity of the buffer stock warehouse. The raw materials are first included in the production line and processed on machine 1, then stored in the buffer stock warehouse. The product is then removed from the buffer stock warehouse and transferred to machine 2. After the product is processed in machine 2, it is placed in the finished product warehouse (see, Fig.1). Random breakdowns occur in both machines and these malfunctions are repaired at random times. When machine 1 breaks down and the product runs out in the buffer stock warehouse, a "starvation" begins for machine 2. Similarly, when machine 2 breaks down and the buffer stock depot is full, "saturation" begins for machine 1. In the event of starvation, the production process is halted and restarted at the beginning level a, which is the buffer stock's midpoint. Therefore, the most undesirable situation for the production system operating with these rules is that the buffer stock level reaches the maximum level $(2a)$ or the lower (0) level. Because when the stock level reaches $2a$, the production goes into the "saturation" state, similarly, when the buffer stock level drops to 0, the production goes into the "starvation" state and therefore the production process stops in both cases. As a result, the production process is restarted from the midpoint (a) of the buffer stock to reduce the probability of undesirable events.

For the above reasons, in this study, the production process is started with buffer stock level (a) at the start of each cycle. The fluctuations in the buffer stock level are mathematically expressed by a random walk process with two specific barriers. Figure 1 shows a graphical representation of an investigated buffer stock problem.

This study provides a mathematical basis for estimating the optimal capacity of the buffer stock under the above-mentioned assumptions. Thus, in this study, the fluctuation in buffer stock level is represented by a random walk process $(Y(t))$ with two specific barriers. There are several number of significant studies on random walk, renewal and renewal-reward processes in the literature (see, Aliyev and Khaniyev [1]; Borovkov [2]; Chang and Peres [3]; Feller [5]; Gihman and Skorohod [6]; Gokpinar et al. [7]; Hanalioglu et al. [8]; Janssen and Leeuwarden [11]; Khaniyev et al. [14]; Kokangul et al. [15]; Lotov [16]; Poladova et al. [19], Kamislik et al. [12]; Hanalioglu et al. [9]; Marandi et al. [18]; Khaniyev and Aksop [13], etc.). However, examining $Y(t)$ is mathematically challenging. First, a stochastic process defined as $X(t) \equiv Y(t) - a$ is investigated.

The remainder of the article is organized as follows. In Section 2, the stochastic processes $X(t)$ and $Y(t)$ is mathematically defined. In Section 3, the one-dimensional distribution of $X(t)$ is discussed. In Section 4 the ergodicity of $X(t)$ and the characteristic function of its stationary distribution is examined. Moreover, in Section 5 the weak convergence theorem for stationary distribution of the "standardized" process $W(t) \equiv X(t)/a$ is proved. Finally, in Section 6 and Section 7 the exact and asymptotic results are obtained for the moments of the stationary distributions of $X(t)$ and $Y(t)$. Conclusion is given in Section 8.

Now, let us define $X(t)$ and $Y(t)$ mathematically.

2. Mathematical Construction of Processes $X(t)$ and $Y(t)$

Let $\{\xi_n\}$ and $\{\eta_n\}$, $n \geq 1$, be sequences of independently distributed random variables with identical distributions specified on probability space $\{\Omega, F, P\}$. Moreover, let the random variable ξ_n take only positive values, but the random variable η_n take both positive and negative values. Suppose that the random variables ξ_1 and η_1 are independent from

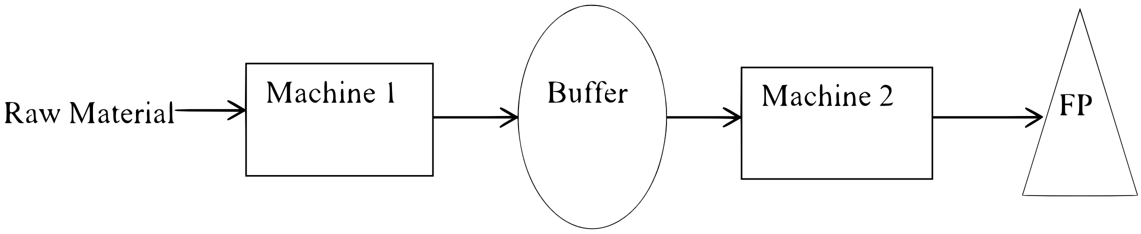

Figure 1. Two machine serial production system

each other. In addition, ξ_1 and η_1 are the random variables with distribution functions $\Phi \equiv P\{\xi_1 \leq t\}$, $t \geq 0$, and $F(x) \equiv P\{\eta_1 \leq x\}$, $x \in R$, respectively.

Let the renewal sequence $\{T_n\}$ and random walk $\{S_n\}$ be defined as follows:

$$T_0 = S_0 = 0;\ T_n \equiv \sum_{i=1}^{n} \xi_i,\ S_n \equiv \sum_{i=1}^{n} \eta_i,\ n = 1, 2, ...$$

Moreover, define a sequence $\{N_n\}$, $n \geq 0$ as:

$$\begin{aligned} N_0 &= 0;\ N_1 \equiv N = \inf\{k \geq 1 : S_k \notin [-a, a]\},\ a > 0, \\ N_{n+1} &= \inf\{k \geq N_n + 1 : S_k - S_{N_n} \notin [-a, a]\},\ n = 1, 2, ... \end{aligned}$$

Let $\tau_0 = 0$, $\tau_1 = T_N = \sum_{i=1}^{N} \xi_i$, $\tau_n = \sum_{i=1}^{N_n} \xi_i$ and define $\nu(t)$ as follows:

$$\nu(t) = \max\{n \geq 0 : T_n \leq t\},\ t > 0.$$

Using the above notations, the desired stochastic process $X(t)$ can be expressed mathematically as follows:

$$X(t) = S_{\nu(t)} - S_{N_n},\ \ \tau_n \leq t < \tau_{n+1},\ \ n = 0, 1, 2, ... \tag{1}$$

The process $X(t)$ is interpreted as a semi-Markovian random walk process with two special barriers. A representation of $X(t)$ is given in Figure 2.

In this study, it is assumed that the random variable η_1 has a bilateral exponential distribution with parameter $\lambda = 1$. In other words, density function of η_1 is $f_\eta(x) = \frac{1}{2}e^{-|x|}$, $x \in R$ and characteristic function of η_1 has the following form:

$$\varphi_\eta(\theta) \equiv E(e^{i\theta\eta_1}) = \frac{1}{1+\theta^2},\ \theta \in R. \tag{2}$$

Thus, the buffer stock level $Y(t)$ can be represented as:

$$Y(t) \equiv X(t) + a \tag{3}$$

The main goal of this research is to analyze the probability and numerical characteristics of stochastic process $Y(t)$. For this purpose, firstly, the one-dimensional distribution of $X(t)$ is examined in the following section.

3. One-Dimensional Distribution of $X(t)$

As is well known, the most general probability characteristic of a stochastic process is its n-dimensional distributions. Accordingly, this section will focus on calculating the one-dimensional distribution of $X(t)$.

The one-dimensional distribution of $X(t)$ can be defined as:

$$Q(t; x) \equiv P\{X(t) \leq x\};\ x \in [-a, a],\ t > 0.$$

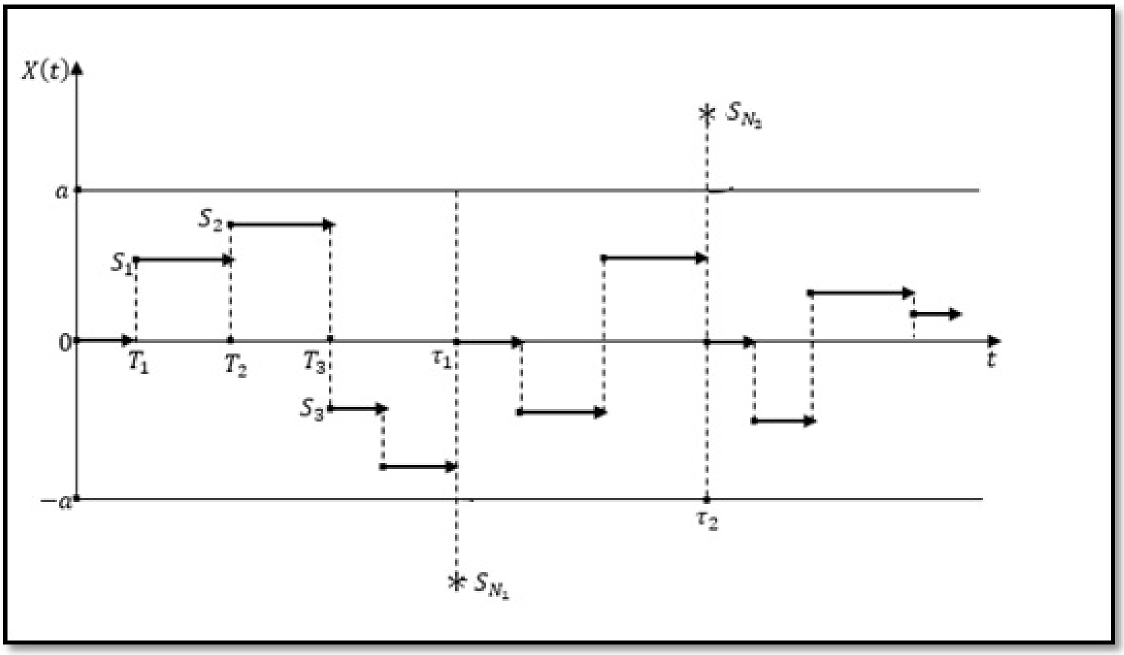

FIGURE 2. A sample graph of the process $X(t)$

Theorem 3.1. *The exact expression of the one-dimensional distribution of* $X(t)$ *can be written as follows:*

$$Q(t;x) = G(t,x) + G(t,x) * \sum_{n=1}^{\infty} R^{*(n)}(t);$$

where $G(t,x) \equiv \sum_{n=0}^{\infty} a_n(x)\Delta\Phi_n(t);\quad R(t) \equiv \sum_{n=1}^{\infty} b_n\Phi_n(t);$

$$\begin{aligned}
a_n(x) &\equiv P\{S_i \in [-a,a];\ i=\overline{1,n};\ S_n \leq x\},\ x \in [-a,a];\\
b_n(x) &\equiv P\{S_i \in [-a,a];\ i=\overline{1,n-1};\ S_n \notin [-a,a]\};\\
\Phi_n(t) &\equiv P\{T_n \leq t\},\ n=1,2,...;\quad \Phi_0(t)=1,\ \ if\ \ t \geq 0\ \ and\ \ \Phi_0(t)=0,\ \ if\ \ t<0;\\
\Delta\Phi_n(t) &\equiv \Phi_n(t)-\Phi_{n+1}(t),\ n=0,1,2,...;\ \ M_1(t)*M_2(t) = \int_0^t M_1(t-s)dM_2(s).
\end{aligned}$$

Proof. The following equation can be used to calculate the one-dimensional distribution of $X(t)$ based on the total probability formula:

$$Q(t,x) \equiv P\{X(t) \leq x\} = \sum_{n=1}^{\infty} P\left\{\tau_{n-1} \leq t \leq \tau_n; X(t) \leq x\right\} \tag{4}$$

Let us define the following notation in order to calculate the first term in the Eq.(4):

$$G(t,x) \equiv G_1(t;x) \equiv P\{\tau_1 > t,\ X(t) \leq x\}.$$

Using the total probability formula, we can calculate $G(t;x)$ as:

$$\begin{aligned} G(t,x) &\equiv G_1(t;x) = \sum_{n=0}^{\infty} P\{\nu(t)=n;\ \tau_1 > t,\ X(t)\le x\}. \\ &= \sum_{n=0}^{\infty} P\left\{T_n \le t < T_{n+1}\right\} P\left\{N_1 > n; S_n \le x\right\} \\ &= \sum_{n=0}^{\infty} \left(\Phi_n(t) - \Phi_{n+1}(t)\right) P\left\{S_i \in [-a;a], i=\overline{1,n};\ S_n \le x\right\} = \sum_{n=0}^{\infty} a_n(x)\Delta\Phi_n(t), \end{aligned}$$

where $N_1 \equiv \min\{n \ge 1 : S_n \notin [-a;a]\}$ and $S_n = \sum_{i=1}^{n} \eta_i$.

Now, we can calculate the second term in the Eq.(4) as follows:

$$\begin{aligned} G_2(t;x) &\equiv P\left\{\tau_1 \le t < \tau_2; X(t) \le x\right\} = \int_{s=0}^{t} P\left\{\tau_1 \in ds; s \le t < \tau_2,\ X(t) \le x\right\} \\ &= \int_{s=0}^{t} G(t-s;x)R(ds) \equiv G(t;x) * R(t), \end{aligned}$$

where $R(ds) \equiv P\{\tau_1 \in ds\};\ R(t) \equiv \sum_{n=1}^{\infty} b_n \Phi_n(t)$.

In a similar manner, we can derive the following result:

$$G_n(t;x) \equiv P\left\{\tau_{n-1} \le t < \tau_n;\ X(t) \le x\right\} = G(t;x) * \left(R(t)\right)^{*(n-1)},\ n = 2,3,...$$

Using the above results, we obtain the exact expression for the one-dimensional distribution of $X(t)$ as:

$$\begin{aligned} Q(t;x) &= G(t,x) + G(t,x) * R(t) + \ldots + G(t,x) * R^{*(n)}(t) + \ldots \\ &= G(t,x) + G(t,x) * \sum_{n=1}^{\infty} R^{*(n)}(t). \end{aligned}$$

Thus the proof of Theorem 3.1 is proved. □

Remark 3.1. *As can be seen from Theorem 3.1, calculating the finite-dimensional distribution of $X(t)$ is extremely complicated due to infinite convolution multiplications. To overcome this difficulty, it is useful to examine the stationary characteristics of $X(t)$. For this purpose, in the following section, the ergodicity of $X(t)$ and the characteristic function of its ergodic distribution will be examined.*

4. Characteristic Function of Ergodic Distribution of $X(t)$

For shortness, represent the ergodic distribution ($Q_X(x)$) and characteristic function ($\varphi_X(\theta)$) of ergodic distribution of the process $X(t)$ as follows:

$$Q_X(x) \equiv \lim_{t\to\infty} Q(t;x) \equiv \lim_{t\to\infty} P\{X(t) \le x\},\ \varphi_X(\theta) \equiv \lim_{t\to\infty} E(\exp(i\theta X(t))).$$

Now, give the proposition about the ergodicity of $X(t)$.

Proposition 4.1. *Suppose the following supplementary conditions are met for sequence of random pairs $\{(\xi_n;\eta_n),\ n \ge 1\}$:*

(i) $0 < E(\xi_1) < \infty$;

(ii) η_1 *has a bilateral exponential distribution with parameter* $\lambda = 1$.

Then, the process $X(t)$ is ergodic.

Proof. According to Smith's "key renewal theorem", $X(t)$ is ergodic if the following two assumptions are met (Gihman and Skorohod [6], p.243).

Assumption 1. According to Assumption 1, such a random time sequence should be chosen that the values of the process $X(t)$ at these times form an embedded ergodic Markov chain. For this purpose, the random time sequence $\{\tau_n,\ n = 1, 2, ...\}$, defined in Section 2, can be selected. At these moments of time, the values of the process $X(t)$ are equal to $X(\tau_0) = X(\tau_1) = ... = X(\tau_n)$, $n = 1, 2, ...$ and these values form an embedded ergodic Markov chain. This embedded Markov chain $\{X(\tau_n),\ n = 1, 2, ...\}$ is ergodic and a stationary distribution is as $\pi(z) = 0$, if $z < 0$, $\pi(z) = 1$, if $z \geq 0$. Thus, the first assumption of the general ergodic theorem is satisfied (see, Gihman and Skorohod [6], p.243).

Assumption 2. According to Assumption 2, it is necessary to have a finite expected values for the time intervals between successive intervention moments $\{\tau_n,\ n = 1, 2, ...\}$. In other words, $E(\tau_n - \tau_{n-1})$must be finite, for $n = 1, 2, \ldots$.

According to the conditions of Proposition 4.1, $E(\xi_1) < \infty$. On the other hand, according to the Wald identity (see, Feller [5]) the following equality can be written:

$$E(\tau_1) = E\left(\sum_{i=1}^{N} \xi_i\right) = E(\xi_1)E(N).$$

and, $E(N) = \frac{a^2}{2} + a + 1$ (see, Feller [5], p.601). Therefore, $E(\tau_1) < \infty$.

Similarly, it can be shown that $E(\tau_n - \tau_{n-1}) < \infty$ for each $n = 2, 3, ...$ Then, Assumption 2 also holds. Therefore, the process $X(t)$ is ergodic. Hence, Proposition 4.1 has been proven. □

Theorem 4.1. *Suppose that the assumptions of Proposition 4.1 are correct. Then, the exact formula for the characteristic function ($\varphi_X(\theta)$) of the stationary distribution of $X(t)$ is as follows:*

$$\varphi_X(\theta) \equiv \lim_{t\to\infty} E\left(e^{i\theta X(t)}\right) = \frac{1}{E(N)}\left\{1 + \frac{\sin(\theta a)}{\theta} + \frac{1 - \cos(\theta a)}{\theta^2}\right\},\ \theta \in R. \tag{5}$$

Proof. In study Aliyev and Khaniyev [1], the general expression of the characteristic function for the stationary distribution of $X(t)$ is written as:

$$\varphi_X(\theta) = \frac{1}{E(N)} \int_{-\infty}^{+\infty} e^{i\theta z} \frac{\varphi_{S_{N(z)}}(-\theta) - 1}{\varphi_\eta(-\theta) - 1} d\pi(z),\ \ \theta \neq 0. \tag{6}$$

Here, $\varphi_\eta(-\theta) = E\left(\exp\left(-i\theta\eta_1\right)\right)$; $\varphi_{S_{N(z)}}(-\theta) = E\left(\exp\left(-i\theta S_{N(z)}\right)\right)$, $\theta \in R$.

In the considered case, the distribution $\pi(z)$, which expresses the discrete interference of chance, is as follows: $\pi(z) = 1$, if $z \geq 0$ and $\pi(z) = 0$, if $z < 0$.

Substituting the expression of $\pi(z)$ into Eq.(6), the characteristic function of the stationary distribution of $X(t)$ is found as:

$$\varphi_X(\theta) = \frac{1}{E(N)} \frac{\varphi_{S_N}(-\theta) - 1}{\varphi_\eta(-\theta) - 1},\ \theta \neq 0. \tag{7}$$

Here, $N = \inf\{k \geq 1 : S_k \notin [-a; a]\}$; $S_N = \sum_{i=1}^{N} \eta_i$.

$E(N)$, is an expected value of N. The exact expression for the expected value of N is equal to $E(N) = \frac{a^2}{2} + a + 1$ (see, Feller [5], p. 601). Additionally, the exact formula for the characteristic function ($\varphi_\eta(\theta)$) of the random variable η_1 is given in Eq.(2) as:

$$\varphi_\eta(\theta) \equiv E\left(\exp\left(i\theta\eta_1\right)\right) = \frac{1}{1 + \theta^2},\ \theta \in R. \tag{8}$$

Moreover, the exact expression for the characteristic function ($\varphi_{S_N}(\theta)$) of S_N is as follows (see, Feller [5], p. 600)):

$$\varphi_{S_N}(\theta) = E\left(e^{i\theta S_N}\right) = \frac{1}{2}\left[\frac{e^{-i\theta a}}{1+i\theta} + \frac{e^{i\theta a}}{1-i\theta}\right] \tag{9}$$

Using Euler identity, we can rewrite Eq.(9) as follows:

$$\varphi_{S_N}(\theta) = \frac{\cos(\theta a) - \theta\sin(\theta a)}{1+\theta^2} \tag{10}$$

The exact formula of the characteristic function of the stationary distribution of $X(t)$ is derived by considering Eq.(8) and Eq.(10) in Eq.(7). Hence, the proof of Theorem 4.1 is finalized. □

The weak convergence theorem for the stationary distribution of standardized process $W(t) \equiv X(t)/a$, as $a \to \infty$, is discussed in the following section.

5. Weak Convergence Theorem for Stationary Distribution of $W(t)$

One of the main purposes of this article is to find the limit form of stationary distributions of $X(t)$ and $Y(t)$. To achieve this, we must first examine the weak convergence of the stationary distribution of the "standardized" process $W(t) \equiv X(t)/a$, as $a \to \infty$.

For brevity, let us represent the stationary distribution and characteristic function of $W(t)$ as follows: $Q_W(x) \equiv \lim_{t\to\infty} P\{W(t) \le x\}$; $\varphi_W(\theta) \equiv \lim_{t\to\infty} E(\exp\{i\theta W(t)\})$.

Theorem 5.1. *Under the conditions of Proposition 4.1, the characteristic function ($\varphi_W(\theta)$) of the stationary distribution of $W(t)$ converges to the following limit, when $a \to \infty$:*

$$\lim_{a\to\infty} \varphi_W(\theta) = \frac{2(1-\cos(\theta))}{\theta^2},\ \theta \ne 0.$$

Proof. By definition, we can represent the characteristic function ($\varphi_W(\theta)$) of the stationary distribution of $W(t)$ as follows:

$$\varphi_W(\theta) \equiv \lim_{t\to\infty} E(\exp\{i\theta W(t)\}) = \lim_{t\to\infty} E(\exp\{(i\theta/a)X(t)\}) = \varphi_X\left(\frac{\theta}{a}\right) \tag{11}$$

Using Eq.(5), we can rewrite Eq.(11) as follows:

$$\varphi_W(\theta) = \varphi_X\left(\frac{\theta}{a}\right) = \frac{1}{E(N)}\left\{1 + a\frac{\sin(\theta)}{\theta} + a^2\frac{1-\cos(\theta)}{\theta^2}\right\} \tag{12}$$

Substituting the exact expression of $E(N)$ into Eq.(12), we get the following result for $\varphi_W(\theta)$:

$$\varphi_W(\theta) = \frac{1}{\frac{a^2}{2} + a + 1}\left\{1 + a\frac{\sin(\theta)}{\theta} + a^2\frac{1-\cos(\theta)}{\theta^2}\right\} \tag{13}$$

Note that, $\left|\frac{\sin(\theta)}{\theta}\right| \le 1$, for all $\theta \in R/\{0\}$.

Therefore, the following limit form for the characteristic function ($\varphi_W(\theta)$) of the stationary distribution of $W(t)$ is immediately obtained from the Eq.(13), when $a \to \infty$:

$$\lim_{a\to\infty} \varphi_W(\theta) = \frac{2(1-\cos(\theta))}{\theta^2},\ \theta \ne 0. \tag{14}$$

Thus, the proof of Theorem 5.1 is completed. □

The below weak convergence theorem is stated based on Theorem 5.1 and Continuity Theorem (see, Feller [5]).

Theorem 5.2. *Let the conditions of the Proposition 4.1 be satisfied. In this case, the stationary distribution* $(Q_W(x))$ *of the process* $W(t)$ *weakly converges to the symmetrical triangular distribution in the interval* $[-1, 1]$*, when* $a \to \infty$*. In other words, the following asymptotic relation can be written:*

$$\lim_{a\to\infty} Q_W(x) \equiv Q_T(x) \equiv P\{T \leq x\} = \begin{cases} \frac{(1+x)^2}{2}; & x \in [-1, 0) \\ 1 - \frac{(1-x)^2}{2}; & x \in [0, 1] \end{cases}$$

Here, the random variable T *has symmetrical triangular distribution in the interval* $[-1, 1]$.

Proof. Note that, the characteristic function of symmetrical triangular distribution in the interval $[-1; 1]$ can be calculated as:

$$\begin{aligned} \varphi_T(\theta) &\equiv E\left(e^{i\theta T}\right) = \int_{-1}^{1} e^{i\theta x} f_T(x)dx = \int_{-1}^{1} \cos(\theta x) f_T(x)dx \\ &= 2\int_0^1 \cos(\theta x)(1-x)dx = \frac{2(1-\cos(\theta))}{\theta^2} \end{aligned} \tag{15}$$

Here $f_T(x)$ is a probability density function of the random variable T, i.e., $f_T(x) = 1 - |x|$, $x \in [-1, 1]$.

Comparing the result of Eq.(15) and Theorem 5.1, the following result is obtained:

$$\lim_{a\to\infty} \varphi_W(\theta) = \frac{2(1-\cos(\theta))}{\theta^2} = \varphi_T(\theta) \tag{16}$$

Thus, according to the continuity theorem, the stationary distribution $(Q_W(x))$ of $W(t)$ weakly converges to the distribution $(Q_T(x))$ of the random variable T, when $a \to \infty$ (see, Feller [5]). Thus, the proof of Theorem 5.2 is completed. □

Remark 5.1. *Using the definition of* $W(t)$ *and Theorem 5.2, it can be shown that the ergodic distributions of* $X(t)$ *and* $Y(t)$ *close to the symmetrical triangular distributions in the intervals* $[-a, a]$ *and* $[0, 2a]$*, respectively, when capacity of the buffer stock warehouse is sufficiently large.*

In Section 6, the moments of the stationary distribution of $X(t)$ are discussed in detail.

6. Exact and Asymptotic Results for Moments of Stationary Distribution of $X(t)$

The primary goal of this section is to examine the moments of the stationary distribution of $X(t)$, using Theorem 4.1.

Put $E(X^n) \equiv \lim_{t\to\infty} E(X^n(t))$; $n = 1, 2, ...$ and the following theorem can be stated about the exact expressions for the moments of the stationary distribution of $X(t)$.

Theorem 6.1. *Let the conditions of Proposition 4.1 hold. Then, exact formulas for moments* $(E(X^n))$ *of the stationary distribution of* $X(t)$ *can be given as:*

$$\begin{aligned} E(X^{2n}) &= \frac{1}{E(N)}\left[\frac{a^{2n+2}}{(2n+1)(2n+2)} + \frac{a^{2n+1}}{(2n+1)}\right]; \ n = 1, 2, ... \\ E(X^{2n-1}) &= 0; \ n = 1, 2, ..., \end{aligned} \tag{17}$$

where $E(N) = 1 + a + a^2/2$ *(Feller* [5]*).*

Proof. The Taylor series method will be used to obtain exact expressions for the moments of the stationary distribution of $X(t)$. The characteristic function $(\varphi_X(\theta))$ of the stationary distribution of $X(t)$ can be written as follows, using Taylor expansions of $\sin\theta$ and $\cos\theta$ in Eq. (5):

$$\varphi_X(\theta) = 1 + \sum_{k=2}^{\infty} \frac{1}{E(N)} \left[\frac{a^{2k-1}}{(2k-1)!} + \frac{a^{2k}}{(2k)!} \right] (i\theta)^{2k-2} \tag{18}$$

On the other hand, we can write $\varphi_X(\theta)$ as follows:

$$\varphi_X(\theta) \quad = \quad 1 + \sum_{k=1}^{\infty} \frac{E(X^{2k-1})}{(2k-1)!} (i\theta)^{2k-1} + \sum_{k=2}^{\infty} \frac{E(X^{2k-2})}{(2k-2)!} (i\theta)^{2k-2} \tag{19}$$

Comparing Eq.(18) and Eq.(19), the exact formulas for moments ($E(X^n)$) of the stationary distribution of $X(t)$ are derived as Eq.(17).

The proof of Theorem 6.1 is completed. $\square$

The three-term asymptotic expansions for the moments of the stationary distribution of $X(t)$ are given in the following theorem.

Theorem 6.2. *Under the conditions of Proposition 4.1, the three-term asymptotic expansions for the moments of the stationary distribution of $X(t)$ can be written as follows, when $a \to \infty$:*

$$E(X^k) = \begin{cases} \frac{2a^k}{(k+1)(k+2)} + \frac{2ka^{k-1}}{(k+1)(k+2)} - \frac{4a^{k-2}}{(k+2)} + o(a^{k-2}); & if\ k\ is\ even \\ 0; & if\ k\ is\ odd \end{cases}$$

Proof. Substituting asymptotic result for $1/E(N)$ into Eq.(17), the asymptotic expansion for $E(X^{2n})$, $n = 1, 2, 3, ...$ can be given as follows, when $a \to \infty$:

$$E(X^{2n}) = \frac{2a^{2n}}{(2n+1)(2n+2)} + \frac{4na^{2n-1}}{(2n+1)(2n+2)} - \frac{4a^{2n-2}}{(2n+2)} + o(a^{2n-2}) \tag{20}$$

So, the proof of Theorem 6.2 is finalized. $\square$

Now, the stationary moments of $Y(t)$ can be addressed in the below section.

7. Main Results for Moments of Stationary Distribution of $Y(t)$

The main aim of this section is to study the moments of the stationary distribution of $Y(t)$. Therefore, let us state the following theorem regarding the exact results for stationary moments of $Y(t)$.

Theorem 7.1. *Under the assumptions of Proposition 4.1, the exact formulas for the moments of the stationary distribution of $Y(t)$ are as follows:*

$$E(Y^{2n}) = \frac{a^{2n+2}}{E(N)} A_{2n} + \frac{a^{2n+1}}{E(N)} B_{2n} + a^{2n}, \tag{21}$$

$$E(Y^{2n-1}) = \frac{a^{2n+1}}{E(N)} C_{2n-1} + \frac{a^{2n}}{E(N)} D_{2n-1} + a^{2n-1}, \tag{22}$$

where $E(Y^n) \equiv \lim_{t\to\infty} E(Y^n(t))$, $E(N) = \frac{a^2}{2} + a + 1$, $n = 1, 2, ...,$

$$A_{2n} \quad \equiv \quad \sum_{r=1}^{n} \frac{\binom{2n}{2r}}{(2r+1)(2r+2)}; \;\; B_{2n} \equiv \sum_{r=1}^{n} \frac{\binom{2n}{2r}}{(2r+1)};$$

$$C_{2n-1} \quad \equiv \quad \sum_{r=1}^{n-1} \frac{\binom{2n-1}{2r}}{(2r+1)(2r+2)}; \;\; D_{2n-1} \equiv \sum_{r=1}^{n-1} \frac{\binom{2n-1}{2r}}{(2r+1)}.$$

Proof. Using the binomial formula and the definition of $Y(t)$, the following equation can be written:

$$E(Y^n(t)) = E(X(t)+a)^n = \sum_{m=0}^{n} \binom{n}{m} E(X^m(t))a^{n-m} \tag{23}$$

From Eq.(23), the following formula can be written for the stationary moments of $Y(t)$:

$$E(Y^n) = a^n + \binom{n}{1} a^{n-1}E(X) + \binom{n}{2} a^{n-2}E(X^2) + \ldots + \binom{n}{n} E(X^n). \tag{24}$$

On the other hand, in Theorem 6.1, it has been proved that $E(X^k) = 0$ when k is odd. For convenience, let's calculate the even and odd order moments of $Y(t)$ separately:

$$E(Y^{2k}) = \sum_{r=1}^{k} \binom{2k}{2r} a^{2k-2r}E(X^{2r}) + a^{2k}; \tag{25}$$

$$E(Y^{2k-1}) = \sum_{r=1}^{k-1} \binom{2k-1}{2r} a^{2k-2r-1}E(X^{2r}) + a^{2k-1}. \tag{26}$$

Considering the formula (19) in Eq.(25) and Eq.(26), we can rewrite the even and odd stationary moments of $Y(t)$ as follows:

$$\begin{aligned} E(Y^{2n}) &= \frac{1}{E(N)}\sum_{r=1}^{n} \binom{2n}{2r} \frac{a^{2n+2}}{(2r+1)(2r+2)} + \frac{1}{E(N)}\sum_{r=1}^{n} \binom{2n}{2r} \frac{a^{2n+1}}{2r+1} + a^{2n}; \\ E(Y^{2n-1}) &= \frac{1}{E(N)}\sum_{r=1}^{n-1} \binom{2n-1}{2r} \frac{a^{2n+1}}{(2r+1)(2r+2)} \\ &+ \frac{1}{E(N)}\sum_{r=1}^{n-1} \binom{2n-1}{2r} \frac{a^{2n}}{2r+1} + a^{2n-1}. \end{aligned}$$

Using the definition of the coefficients A_{2n}, B_{2n}, C_{2n-1}, D_{2n-1} in the above formulas, the exact formulas for moments of the stationary distribution of $Y(t)$ are derived.

Then, the proof of Theorem 7.1 is completed. □

Now, let us obtain asymptotic expansions for the stationary moments of $Y(t)$.

Theorem 7.2. *Under the conditions of Proposition 4.1, the three-term asymptotic expansions for the moments of the stationary distribution of $Y(t)$ can be written as follows, when $a \to \infty$:*

$$\begin{aligned} E(Y^{2n}) &= (2A_{2n}+1)a^{2n} + 2(B_{2n}-2A_{2n})a^{2n-1} - 4(B_{2n}-A_{2n})a^{2n-2} + o(a^{2n-2}), \\ E(Y^{2n-1}) &= (2C_{2n-1}+1)a^{2n-1} + 2(D_{2n-1}-2C_{2n-1})a^{2n-2} \\ &\quad -4(D_{2n-1}-C_{2n-1})a^{2n-3} + o(a^{2n-3}), \end{aligned}$$

Here, the coefficients $A_{2n}, B_{2n}, C_{2n-1}, D_{2n-1}$ are defined as in Theorem 7.1.

Proof. It is known that $E(N) = a^2/2 + a + 1$ (see, Feller [5], p.601)). Therefore, when $a \to \infty$:

$$\frac{1}{E(N)} = \frac{2}{a^2}\left[1 - \frac{2}{a} + \frac{2}{a^2} + o\left(\frac{1}{a^2}\right)\right] \tag{27}$$

Using the Eq.(21) and asymptotic expansion (27), it can be derived, when $a \to \infty$:

$$E(Y^{2n}) = (2A_{2n}+1)a^{2n} + 2(B_{2n}-A_{2n})a^{2n-1} - 4(B_{2n}-A_{2n})a^{2n-2} + o(a^{2n-2})$$

Similarly, using the Eq.(22) and asymptotic expansion (27), the following expansion can be written, when $a \to \infty$:

$$E(Y^{2n-1}) = (2C_{2n-1}+1)a^{2n-1}+2(D_{2n-1}-C_{2n-1})a^{2n-2}-4(D_{2n-1}-C_{2n-1})a^{2n-3}+o(a^{2n-3})$$

Thus, the proof of Theorem 7.2 is completed. □

The following additional results can be obtained from Theorem 7.2.

Remark 7.1. *The results of this study form a mathematical basis for evaluating the optimal buffer stock capacity required to maintain uninterrupted operation of a production line.*

8. Conclusions

This article presents a novel stochastic approach to the buffer stock problem. The level of buffer stock is represented by a random walk process $(Y(t))$ with two special barriers at levels 0 and $2a$. Next, the exact expression of the one-dimensional distribution of the process $X(t) \equiv Y(t) - a$ is found. Then, the ergodicity of the process $X(t)$ is investigated. Additionally, the exact expression of the characteristic function of the stationary distribution of $X(t)$ is obtained. Furthermore, it is demonstrated that for sufficiently large values of a, the ergodic distributions of $X(t)$ and $Y(t)$ approach to symmetrical triangular distributions in the intervals $[-a, a]$ and $[0, 2a]$, respectively. The capacity of the buffer stock is represented by the number $2a$. The exact and asymptotic formulas for the stationary moments of $X(t)$ and $Y(t)$ are obtained. These mathematical results can be effectively used to determine the optimal capacity of the buffer stock warehouse, which will allow the production line to run with the fewest number of interruptions and at the lowest possible cost while maintaining maximum efficiency. The mathematical techniques used in this study, could potentially be useful for determining optimal capacity for other buffer stock problems in the future.

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Zulfiye Hanalioglu graduated from Mathematical and Mechanical Faculty of Baku State University in 1981. In 2011, she got her Philosophy Doctor degree from the Department of Mathematics at Karadeniz Technical University in the field of stochastic processes. She worked in Technology Faculty of Gazi University between 2012 and 2013. Since 2013, she has been working as and Associated Professor in the Department of Actuary and Risk Management at Karabuk University.

Aynura Poladova graduated from Mathematical and Mechanical Faculty of Baku State University in 2008. Afterwards, she completed her MSc in the Department of Industrial Engineering at TOBB University of Economics and Technology with full scholarship in 2017. She is in a PhD programme in Department of Industrial Engineering at TOBB University of Economics and Technology.

Basak Gever graduated from Karadeniz Technical University, the Department of Statistics and Computer Science in 2008. She completed MSc and PhD programs in the Department of Industrial Engineering at TOBB University of Economics and Technology, respectively in 2011 and 2017. During her PhD, she studied at the University of Edinburgh in the School of Mathematics as a visiting researcher. After working at Defence Technologies Engineering Inc. as an Expert of Artificial Intelligence Engineer, she started to work as an Assist. Prof. Dr. in the Department of Industrial Engineering at the University of Turkish Aeronautical Association in 2019.

Tahir Khaniyev graduated from Mathematical and Mechanical Faculty of Lomonosov Moscow State University in 1980. In 1987, he got his Philosophy Doctor degree from Institute of Mathematics of Ukrainian Academy of Sciences in field of stochastic processes. In 2007, he got his Doctor of Sciences degree from Institute of Mathematics and Mechanics of Azerbaijan Academy of Sciences. He has been a professor of Statistics since 2004. He has made more than 100 scientific publications, including 4 books and 5 book chapters. Since 2007, he has been working in TOBB University of Economics and Technology (Ankara-Turkey).