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ON QUASI STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

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ABSTRACT. In this paper, we introduce and investigate the concept of quasi statistical convergence of complex uncertain sequences. We also establish the relationship of the notion with statistical convergence of complex uncertain sequences. Furthermore, we investigate some interrelationships between quasi statistical convergence almost surely, quasi statistical convergence in measure, quasi statistical convergence in mean, and quasi statistical convergence in distribution.

Keywords: Uncertainty theory, complex uncertain variable, quasi-density, quasi statistical convergence.

AMS Subject Classification (2020): 60B10, 40A35, 40A05.

1. INTRODUCTION AND BACKGROUND

Liu [16] was the first to introduce the uncertainty theory based on an uncertain measure that satisfies normality, duality, subadditivity, and product axioms. Nowadays uncertainty theory has become one of the most active areas of research due to its wide applicability in various domains such as uncertain programming, uncertain optimal control, uncertain risk analysis, uncertain differential equation, etc. For more details, one may refer to [17, 18]. In order to define complex uncertain sequences, the notion of uncertain variables was defined over the uncertain space. Complex uncertain sequences are measurable functions from an uncertain space to the set of all complex numbers $\mathbb{C}$. Over the last few years, the study of convergence of sequences in the complex uncertain space has drawn the attention of the researchers. In 2016, Chen et. al. [2] investigated various types of convergence of sequences such as convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution in the complex uncertain space. He mainly studied the interrelationship between the notions. Later on, several works have been carried out in this direction which can be found from [4, 5, 19, 22, 23, 24, 25, 26, 33].

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On the other hand, in 1951 Fast [11] and Steinhaus [29] introduced the idea of statistical convergence independently in connection with summability. Later on, it was further investigated from the sequence space point of view by Fridy [12, 13], Salat [28], Connor [3], Tripathy [31], and many mathematicians across the globe. Statistical convergence has become one of the most active areas of research due to its wide applicability in various branches of mathematics such as number theory, mathematical analysis, probability theory, etc.

In an attempt to generalize the notion of statistical convergence, in 2012 Ozguc and Yurdakadim [21] introduced the concept of quasi-statistical convergence in terms of quasi-density.

In [21], the notion of natural density was extended to quasi density by involving a sequence $c = (c_n)$ satisfying the following properties:

$$c_n > 0\, \forall n \in \mathbb{N}, \lim_{n\to\infty} c_n = \infty \text{ and } \limsup_{n} \frac{c_n}{n} < \infty. \tag{1}$$

The quasi-density of a set $E \subseteq \mathbb{N}$ is defined by $\delta_c(E) = \lim_{n\to\infty} \frac{|E_n|}{c_n}$, provided the limit exists. Here, E_n indicates the set $\{k \in E : k \leq n\}$. It should be noted that if $c_n = n$, then the above definition turns to the definition of natural density. Throughout the paper, we will use $c = (c_n)$ to denote sequences that satisfy (1).

In [21], Ozguc and Yurdakadim introduced the notion of quasi statistical convergence of real-valued sequences as follows:

A sequence (x_k) is said to be quasi statistical convergent to x_0 if for each $\varepsilon > 0$,

$$\delta_c(\{k \in \mathbb{N} :\mid x_k - x_0 \mid\geq \varepsilon\}) = 0.$$

In this case, x_0 is called the quasi statistical limit of the sequence (x_k) and symbolically it is expressed as $x_k \xrightarrow{st_q} x_0$. They mainly studied the relationship of quasi statistical convergence and statistical convergence and show that the condition $\inf_n \frac{c_n}{n} > 0$ along with (1), plays a significant role for the equivalence of the concepts.

It should be noted that, if we choose $c_n = n$, $\forall n \in \mathbb{N}$, then the definition of quasi statistical convergence turns to the definition of statistical convergence.

They investigated the relationship of the newly introduced notion with statistical convergence. Later on, in 2012, Ganguly and Dafadar [14] extended it for double sequences. Further in [32], Turan et. al. investigated this notion in cone metric settings. Very recently Ozguc [20] have introduced the notion of quasi statistical limit and cluster points and investigated a few properties.

The notion of statistical convergence was first developed in terms of complex uncertain sequences by Tripathy and Nath [30]. Later on, a lot of work has been carried out in this direction till date. In [6, 7], Debnath and Das have introduced two generalizations namely statistical convergence of order α (for $0 < \alpha \leq 1$) and deferred statistical convergence for statistical convergence. In [1], Baliarsingh introduced and investigated statistical deferred $A-$convergence of complex uncertain sequences. Furthermore, Dowari and Tripathy [8, 9, 10] studied the statistical convergence of complex uncertain sequences by involving lacunary sequences. For more details, [15, 27] can be addressed where one can find many more references.

2. Definitions and Preliminaries

Definition 2.1. [16] *Let $\mathfrak{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;

Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any* $\Lambda \in \mathfrak{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathfrak{L}$, *we have*

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathfrak{L}, \mathcal{M})$ *is called an uncertainty space and each element* Λ *in* $\mathfrak{L}$ *is called an event. In order to obtain an uncertain measure of compound event, a product uncertain measure is defined by Liu* [16] *as:*

$$\mathcal{M}\{\prod_{k=1}^{\infty} \Lambda_k\} = \bigwedge_{k=1}^{\infty} \mathcal{M}\{\Lambda_k\}.$$

Definition 2.2. [16] *An uncertain variable is a function* ζ *from an uncertainty space* $(\Gamma, \mathfrak{L}, \mathcal{M})$ *to the set of real numbers such that* $\{\zeta \in \mathfrak{B}\} = \{\gamma \in \Gamma : \zeta(\gamma) \in \mathfrak{B}\}$ *is an event for any Borel set* $\mathfrak{B}$ *of real numbers.*
An uncertainty distribution Φ *of an uncertain variable* ζ *is defined by*

$$\Phi(x) = \mathcal{M}\{\zeta \leq x\}, \quad \forall x \in \mathbb{R}.$$

Definition 2.3. [16] *Let* ζ *is an uncertain variable. The expected value of* ζ*is defined by*

$$E[\zeta] = \int_0^{+\infty} \mathcal{M}\{\zeta \geq y\}dy - \int_{-\infty}^{0} \mathcal{M}\{\zeta \leq y\}dy.$$

provided that at least one of the above two integrals is finite.

Definition 2.4. [30] *A complex uncertain sequence* $\{\zeta_n\}$ *is said to be statistically convergent almost surely to* ζ *if for every* $\varepsilon > 0$ *there exists an event* Λ *with* $\mathcal{M}\{\Lambda\} = 1$ *such that*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :\parallel \zeta_k(\gamma) - \zeta(\gamma) \parallel \geq \varepsilon\} \mid = 0,$$

for every $\gamma \in \Lambda$. *In that case, we write* $\zeta_n \to \zeta$, *a.s.*

Definition 2.5. [30] *A complex uncertain sequence* $\{\zeta_n\}$ *is said to be statistically convergent in measure to* ζ *if*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : \mathcal{M}(\parallel \zeta_k - \zeta \parallel \geq \varepsilon) \geq \delta\} \mid = 0,$$

for every $\varepsilon, \delta > 0$.

Definition 2.6. [30] *A complex uncertain sequence* $\{\zeta_n\}$ *is said to be statistically convergent in mean to* ζ *if*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : E(\parallel \zeta_k - \zeta \parallel) \geq \varepsilon\} \mid = 0.$$

Definition 2.7. [30] *Let* $\Phi, \Phi_1, \Phi_2, ...$ *be the complex uncertainty distributions of complex uncertain variables* $\zeta, \zeta_1, \zeta_2, ...$, *respectively. Then the sequence* $\{\zeta_n\}$ *statistically converges in distribution to* ζ *if*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :\parallel \Phi_k(x) - \Phi(x) \parallel \geq \varepsilon\} \mid = 0,$$

for all x *at which* $\Phi(x)$ *is continuous.*

Definition 2.8. [30] *A complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent uniformly almost surely (u.a.s.) to ζ if for every $\varepsilon > 0, \exists \delta > 0$ and a sequence of events (X_k) such that*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :\mid \mathcal{M}(X_k) - 0 \mid \geq \varepsilon\} \mid = 0,$$
$$\Longrightarrow \lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :\mid \zeta_k(\gamma) - \zeta(\gamma) \mid \geq \delta\} \mid = 0.$$

Definition 2.9. [30] *A complex uncertain sequence $\{\zeta_n\}$ is said to be statistically convergent to ζ if for every $\varepsilon > 0$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n :|| \zeta_k(\gamma) - \zeta(\gamma) || \geq \varepsilon\} \mid = 0,$$

for every $\gamma \in \Gamma$.

3. Main Results

Definition 3.1. *The complex uncertain sequence (ζ_n) is said to be quasi statistically convergent almost surely (q.s.a.s.) to ζ if for any $\varepsilon > 0$, there exists an event Λ with $M\{\Lambda\} = 1$ such that for any $\gamma \in \Lambda$,*

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : ||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\} \mid = 0.$$

In this case, we write $\zeta_n \xrightarrow{q.s.a.s.} \zeta$.

Remark 3.1. *(i) If $c_n = n$, then the above definition coincides with statistical convergence of complex uncertain sequences* [30].
(ii) If $c_n = n^\alpha$ $(0 < \alpha \leq 1)$, then the above definition coincides with statistical convergence of order α for complex uncertain sequences [6].

Definition 3.2. *The complex uncertain sequence (ζ_n) is said to be quasi statistically convergent in measure (q.s.M.) to ζ if for any $\varepsilon, \delta > 0$,*

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\} \mid = 0.$$

In this case, we write $\zeta_n \xrightarrow{q.s.M.} \zeta$.

Definition 3.3. *The complex uncertain sequence (ζ_n) is said to be quasi statistically convergent in mean (q.s.E.) to ζ if for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : E[||\zeta_k - \zeta||] \geq \varepsilon\} \mid = 0.$$

In this case, we write $\zeta_n \xrightarrow{q.s.E.} \zeta$.

Definition 3.4. *Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively. Then the complex uncertain sequence (ζ_n) is said to be quasi statistically convergent in distribution (q.s.d.) to ζ if for any $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : ||\Phi_k(s) - \Phi(s)|| \geq \varepsilon\} \mid = 0$$

for all s at which $\Phi(s)$ is continuous.

In this case, we write $\zeta_n \xrightarrow{q.s.d.} \zeta$.

Definition 3.5. *The complex uncertain sequence* (ζ_n) *is said to be quasi statistically convergent uniformly almost surely (q.s.u.a.s.) to* ζ *if for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *and a sequence of events* $\{E'_k\}$ *such that*

$$\lim_{n\to\infty}\frac{1}{c_n} \mid \{k \leq n :\mid \mathcal{M}(E'_k) - 0 \mid\geq \varepsilon\} \mid= 0 \Rightarrow \lim_{n\to\infty}\frac{1}{c_n} \mid \{k \leq n :\mid\mid \zeta_k(x) - \zeta(x) \mid\mid\geq \delta\} \mid= 0.$$

Definition 3.6. *The complex uncertain sequence* (ζ_n) *is said to be quasi statistically convergent to* ζ *if for any* $\varepsilon > 0$ *and for any* $\gamma \in \Lambda$,

$$\lim_{n\to\infty}\frac{1}{c_n} \mid \{k \leq n : ||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon\} \mid= 0.$$

Theorem 3.1. *If a complex uncertain sequence* (ζ_n) *is quasi statistically convergent to* ζ, *then it is statistically convergent to* ζ.

Proof. Let a complex uncertain sequence (ζ_n) is quasi statistically convergent to ζ and $H := \sup_n \frac{c_n}{n}$. Since

$$\frac{1}{n} \mid \{k \leq n :\mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\} \mid\leq \frac{H}{c_n} \mid \{k \leq n :\mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\} \mid,$$

the proof follows immediately. □

But the converse isn't true in general.

Example 3.1. *let us consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, ...\}$ *with* $\mathcal{M}\{\gamma_n\} = \sum_{n=1}^{\infty} 2^{-n}$. *Take* $\zeta(\gamma) \equiv 0$. *Let* (c_n) *be the sequence of positive real numbers such that* $\lim_n c_n = \infty$, *and* $\lim_n \frac{\sqrt{n}}{c_n} = \infty$. *We can choose a subsequence* (c_{n_p}) *such that* $c_{n_p} > 1$ *for each* $p \in \mathbb{N}$. *Consider the uncertain variable* (ζ_n) *defined by:*

$$\zeta_n(\gamma) = \begin{cases} ri, \ if\, n\, is\, a\, square\, and\ \gamma = \gamma_r,\ r \in \{c_{n_p} : p \in \mathbb{N}\}; \\ 2i, \ \ if\, n\, is\, a\, square\, and\ \gamma = \gamma_r,\ r \notin \{c_{n_p} : p \in \mathbb{N}\}; \\ 0,\ otherwise. \end{cases}$$

Then it is easy to verify that (ζ_n) *is statistically convergent to* ζ.

But for $\varepsilon = 1$,

$$\tfrac{1}{c_n} \mid \{k \leq n :\mid\mid \zeta_k - \zeta \mid\mid\geq 1\} \mid= \tfrac{1}{c_n}[\mid \sqrt{n} \mid] = \tfrac{1}{c_n}(\sqrt{n} - b_n),$$

where $0 \leq b_n < 1$ *for each* $n \in \mathbb{N}$. *From the above inequality it is easy to verify that* ζ_n *isn't quasi statistically convergent to* ζ.

Lemma 3.1. *Let* (c_n) *be the sequence of positive real numbers that satisfies* (1) *and* $t := \inf_n \frac{c_n}{n} > 0$. *If a complex uncertain sequence* (ζ_n) *is statistically convergent to* ζ, *then it is quasi statistically convergent to* ζ.

Proof. We have for any $\varepsilon > 0$,

$$\frac{1}{n} \mid \{k \leq n :\mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\} \mid\geq t\frac{1}{c_n} \mid \{k \leq n :\mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\} \mid.$$

From the above inequality, the result follows immediately. □

Theorem 3.2. *If the complex uncertain sequence* (ζ_n) *is quasi statistically convergent in mean to* ζ, *then* (ζ_n) *is quasi statistically convergent in measure to* ζ.

Proof. For any $\varepsilon, \delta > 0$, using Markov inequality, we obtain

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\} \mid \leq \lim_{n\to\infty} \frac{1}{c_n} \left| \left\{ k \leq n : \left(\frac{E(||\zeta_k - \zeta||)}{\varepsilon} \right) \geq \delta \right\} \right|.$$

By our assumption, the right-hand side of the above inequation vanishes. Consequently, the left-hand side also vanishes and the desired result is obtained. □

The converse of the above theorem is not necessarily true. The following counterexample justifies the fact.

Example 3.2. *Consider the uncertainty space (Γ, L, M) to be $\gamma_1, \gamma_2, ...$ with*

$$\mathcal{M}(\Lambda) = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{1}{n+1}, & if \sup\limits_{\gamma_n \in \Lambda} \frac{1}{n+1} < \frac{1}{3} \\ 1 - \sup\limits_{\gamma_n \in \Gamma \backslash \Lambda} \frac{1}{n+1}, & if \sup\limits_{\gamma_n \in \Gamma \backslash \Lambda} \frac{1}{n+1} < \frac{1}{3} \\ 0.5, & otherwise \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, ...\ and\ \zeta = 0.$$

Take $c_n = n$. Then, for very small $\varepsilon, \delta > 0$ and $n \geq 2$ we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\} \mid \\ &= \lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(||\zeta_k(\gamma) - \zeta(\gamma)|| \geq \varepsilon) \geq \delta\} \mid \\ &= \lim_{n\to\infty} \frac{1}{c_n} \mid \{k \in \mathbb{N} : \mathcal{M}\{\gamma_n\} \geq \delta\} \mid \\ &= 0. \end{aligned}$$

Therefore, the sequence (ζ_n) is quasi statistically convergent to ζ in measure. But for any $n \geq 2$, the uncertainty distribution of uncertain variable $||\zeta_n - \zeta|| = ||\zeta_n||$ is given by

$$\Phi_n(x) = \begin{cases} 0, & x < 0 \\ 1 - \frac{1}{n+1}, & 0 \leq x < n+1 \\ 1, & x \geq n+1 \end{cases}.$$

So for $n \geq 2$, we have

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : E[||\zeta_k - \zeta|| - 1]\} \mid = \left(\int_0^{n+1} 1 - \left(1 - \frac{1}{n+1}\right) dx \right) - 1 = 0.$$

In other words, the sequence (ζ_n) does not quasi statistically convergent in mean to ζ.

Theorem 3.3. *Assume complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n), respectively for $n = 1, 2, ...$ If uncertain sequences (ξ_n) and (η_n) are quasi statistical convergent in measure to ξ and η respectively, then the complex uncertain sequence (ζ_n) is quasi statistical convergent in measure to $\zeta = \xi + \eta$.*

Proof. From the definition of quasi statistically convergent in measure of uncertain sequence, it follows that for any small $\varepsilon, \delta > 0$,

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(||\ \xi_k - \xi\ || \geq \frac{\varepsilon}{\sqrt{2}}) \geq \delta\} \mid = 0,$$

and

$$\lim_{n\to\infty}\frac{1}{c_n}\mid\{k\le n:\mathcal{M}(\|\eta_k-\eta\|\ge\frac{\varepsilon}{\sqrt{2}})\ge\delta\}\mid=0.$$

We have, $\|\zeta_n-\zeta\|=\sqrt{|\xi_n-\xi|^2+|\eta_n-\eta|^2}$. Also, we have

$$\{\|\zeta_n-\zeta\|\ge\varepsilon\}\subset\left\{\|\xi_n-\xi\|\ge\frac{\varepsilon}{\sqrt{2}}\right\}\cup\left\{\|\eta_n-\eta\|\ge\frac{\varepsilon}{\sqrt{2}}\right\}.$$

Using subadditivity axiom of uncertain measure, we obtain

$$\mathcal{M}\{\|\zeta_n-\zeta\|\ge\varepsilon\}\le\mathcal{M}\left\{\|\xi_n-\xi\|\ge\frac{\varepsilon}{\sqrt{2}}\right\}+\mathcal{M}\left\{\|\eta_n-\eta\|\ge\frac{\varepsilon}{\sqrt{2}}\right\}.$$

Then,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{c_n}&\mid\{k\le n:\mathcal{M}(\|\xi_k-\xi\|\ge\varepsilon)\ge\delta\}\\&\le\lim_{n\to\infty}\frac{1}{c_n}\mid\{k\le n:\mathcal{M}(\|\xi_k-\xi\|\ge\frac{\varepsilon}{\sqrt{2}})\ge\delta\}\mid\\&\quad+\lim_{n\to\infty}\frac{1}{c_n}\mid\{k\le n:\mathcal{M}(\|\eta_k-\eta\|\ge\frac{\varepsilon}{\sqrt{2}})\ge\delta\}\mid=0.\end{aligned}$$

□

Theorem 3.4. *Assume complex uncertain sequence (ζ_n) with real part (ξ_n) and imaginary part (η_n) are quasi statistically convergent in measure to ξ and η, respectively. Then the complex uncertain sequence (ζ_n) is quasi statistically convergent in distribution to $\zeta=\xi+i\eta$.*

Proof. Let $z=x+iy$ be a given continuity point of the complex uncertainty distribution Φ. On the other hand, for any $\alpha>x,\beta>y$, we have

$$\begin{aligned}\{\xi_n\le x,\eta_n\le y\}&=\{\xi_n\le x,\eta_n\le y,\xi\le\alpha,\eta\le\beta\}\cup\{\xi_n\le x,\eta_n\le y,\xi>\alpha,\eta>\beta\}\\&\cup\{\xi_n\le x,\eta_n\le y,\xi\le\alpha,\eta>\beta\}\cup\{\xi_n\le x,\eta_n\le y,\xi>\alpha,\eta\le\beta\}\\&\subset\{\xi\le\alpha,\eta\le\beta\}\cup\{\|\xi_n-\xi\|\ge\alpha-x\}\cup\{\|\eta_n-\eta\|\ge\beta-y\}.\end{aligned}$$

It follows from the subadditivity axiom that

$$\Phi_n(z)=\Phi_n(x+iy)\le\Phi(\alpha+i\beta)+\mathcal{M}\{\|\xi_n-\xi\|\ge\alpha-x\}+\mathcal{M}\{\|\eta_n-\eta\|\ge\beta-y\}.$$

Since (ξ_n) and (η_n) are quasi statistically convergent in measure to ξ and η, respectively. So for any small $\varepsilon>0$ we have

$$\lim_{n\to\infty}\frac{1}{c_n}\mid\{k\le n:\mathcal{M}(\|\xi_k-\xi\|\ge\alpha-x)\ge\varepsilon\}\mid=0$$

and

$$\lim_{n\to\infty}\frac{1}{c_n}\mid\{k\le n:\mathcal{M}(\|\eta_k-\eta\|\ge\beta-y)\ge\varepsilon\}\mid=0.$$

Thus we obtain $\limsup_{n\to\infty}\Phi_n(z)\le\Phi(\alpha+i\beta)$ for any $\alpha>x,\beta>y$.

Letting $\alpha+i\beta\to x+iy$, we get

$$\limsup_{n\to\infty}\Phi_n(z)\le\Phi(z).\tag{2}$$

On the other hand, for any $\gamma < x, \delta < y$ we have,
$$\{\xi \leq \gamma, \eta \leq \delta\} = \{\xi_n \leq x, \eta_n \leq y, \xi \leq \gamma, \eta \leq \delta\} \cup \{\xi_n > x, \eta_n > y, \xi \leq \gamma, \eta \leq \delta\}$$
$$\cup \{\xi_n > x, \eta_n \leq y, \xi \leq \gamma, \eta \leq \delta\} \cup \{\xi_n \leq x, \eta_n > y, \xi \leq \gamma, \eta \leq \delta\}$$
$$\subset \{\xi_n \leq x, \eta_n \leq y\} \cup \{\| \xi_n - \xi \| \geq x - \gamma\} \cup \{\| \eta_n - \eta \| \geq y - \delta\}.$$
It follows from the subadditivity axiom that
$$\Phi(\gamma + i\delta) \leq \Phi_n(x + iy) + \mathcal{M}\{\| \xi_n - \xi \| \geq x - \gamma\} + \mathcal{M}\{\| \eta_n - \eta \| \geq y - \delta\}.$$
Since (ξ_n) and (η_n) are statistically convergent in measure to ξ and η, respectively. So for any small $\varepsilon > 0$ we have
$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(\| \xi_k - \xi \| \geq x - \gamma) \geq \varepsilon\} \mid = 0$$
and
$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(\| \eta_k - \eta \| \geq y - \delta) \geq \varepsilon\} \mid = 0.$$
Thus we obtain $\Phi(\gamma + i\delta) \leq \liminf_{n\to\infty} \Phi_n(x + iy)$ for any $\gamma < x, \delta < y$. Letting $\gamma + i\delta \to x + iy$, we get
$$\Phi(z) \leq \liminf_{n\to\infty} \Phi_n(z). \tag{3}$$
It follows from (2) and (3) that $\Phi_n(z) \to \Phi(z)$ as $n \to \infty$ .i.e., the complex uncertain sequence (ζ_n) is quasi statistically convergent in distribution to $\zeta = \xi + i\eta$. □

Remark 3.2. *Quasi statistical converges in distribution does not imply quasi statistical convergence in measure. Following example illustrates this.*

Example 3.3. *Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = \frac{1}{2}$. and the complex uncertain variables be defined by*
$$\zeta_n(\gamma) = \begin{cases} i, \ if \ \gamma = \gamma_1; \\ -i, \ if \ \gamma = \gamma_2; \end{cases}$$
for $n \in \mathbb{N}$ and $\zeta = -\zeta_n$. Then (ζ_n) and ζ have the same distribution
$$\Phi_n(c) = \Phi_n(x + iy) = \begin{cases} 0, \ if \ x < 0, \ -\infty < y < +\infty; \\ 0, \ if \ x \geq 0, \ y < -1; \\ \frac{1}{2}, \ if \ x \geq 0, \ -1 \leq y < 1; \\ 1, \ \ if \ x \geq 0, \ y \geq 1. \end{cases}$$
Let $c_n = n$. Then (ζ_n) is quasi statistical converges in distribution to ζ. But for small number $\varepsilon > 0$ and $\delta = \frac{1}{2}$, we have
$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(\gamma : \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \delta\} \mid \neq 0.$$
Therefore the complex uncertain sequence (ζ_n) does not quasi statistical converges in measure to ζ. In addition, (ζ_n) does not quasi statistical converges in a.s. to ζ.

Theorem 3.5. *Quasi statistically convergence a.s. does not imply quasi statistical convergence in measure.*

Example 3.4. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, ...\}$ *with*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(3n+1)}, \ if \ \sup\limits_{\gamma_n \in \Lambda} \frac{n}{(3n+1)} < 0.5 \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(3n+1)}, \ \ if \ \sup\limits_{\gamma_n \in \Lambda^c} \frac{n}{(3n+1)} < 0.5 \\ 0.5, \ \ otherwise, \end{cases}$$

and the complex uncertain variables be defined by

$$\zeta_n(\gamma) = \begin{cases} n^2 i, \ if \ \gamma = \gamma_n; \\ 0, \ \ otherwise; \end{cases}$$

for $n = 1, 2, ...$ *and* $\zeta \equiv 0$. *Take* $c_n = n$. *Then the sequence* (ζ_n) *quasi statistically converges a.s. to* ζ. *However for small* $\varepsilon > 0$ *and taking* $\delta = \frac{1}{4}$, *we have*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{c_n} \mid \{k : \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \frac{1}{4}\} \mid \\ &= \lim_{n\to\infty} \frac{1}{c_n} \mid \{k : \mathcal{M}(\gamma : \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \frac{1}{4}\} \mid \\ &= \lim_{n\to\infty} \frac{1}{n} \mid \{n \in \mathbb{N} : \mathcal{M}\{\gamma_n\} \geq \frac{1}{4}\} \mid = 1 (\neq 0), \end{aligned}$$

as $n \to \infty$. *Thus, the sequence* (ζ_n) *does not quasi statistically converges in measure to* ζ.

Remark 3.3. *The converse of the above theorem is not true i.e., quasi statistically convergence in measure does not imply quasi statistically convergence a.s.*

Example 3.5. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $[0, 1]$ *with Borel algebra and Lebesgue measure. For any positive integer* n, *there exists an integer* r *such that* $n = 2^r + k$ *where* k *is an integer between* 0 *and* $2^r - 1$. *Then we define a complex uncertain variable by*

$$\zeta_n(\gamma) = \begin{cases} i, \ if \ \frac{k}{2^r} \leq \gamma \leq \frac{k+1}{2^r}; \\ 0, \ \ otherwise; \end{cases}$$

for $n = 1, 2, ...$ *and* $\zeta \equiv 0$. *Take* $c_n = n$. *However for small* $\varepsilon, \delta > 0$ *and* $n \geq 2$, *we have*

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(\| \zeta_k - \zeta \| \geq \varepsilon) \geq \delta\} \mid \\ &= \lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mathcal{M}(\gamma : \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon) \geq \delta\} \mid \\ &= \lim_{n\to\infty} \frac{1}{n} \mid \{n \in \mathbb{N} : \mathcal{M}\{\gamma_n\} \geq \delta\} \mid = 0, \ as \ n \to \infty. \end{aligned}$$

Thus, the sequence (ζ_n) *quasi statistically converges in measure to* ζ.
But, for any $\gamma \in [0, 1]$, *there is an infinite number of intervals of the form* $[\frac{k}{2^r}, \frac{k+1}{2^r}]$ *containing* γ. *Thus* $(\zeta_n(\gamma))$ *does not quasi statistically converges to* 0 *i.e., the sequence* (ζ_n) *does not quasi statistically converges a.s. to* ζ.

Theorem 3.6. *Quasi statistically convergence a.s. does not imply quasi statistical convergence in mean.*

Example 3.6. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, ...\}$ *with* $\mathcal{M}\{\Lambda\} = 2\sum_{\gamma_n \in \Lambda} 3^{-n}$. *We define a complex uncertain variable by*

$$\zeta_n(\gamma) = \begin{cases} i3^n, & if\ \gamma = \gamma_n; \\ 0, & otherwise; \end{cases}$$

for $n = 1, 2, ...$ *and* $\zeta \equiv 0$. *Take* $c_n = n$. *Then, the sequence* (ζ_n) *is quasi statistically converges a.s. to* ζ.
But, the uncertainty distribution of $\{\zeta_n\}$ *are*

$$\Phi_n(x) = \begin{cases} 0, & if\ x < 0; \\ 1 - \frac{2}{3^n}, & if\ 0 \leq x < 3^n; \\ 1, & otherwise; \end{cases}$$

for $n = 1, 2, ...,$ *respectively. Then, we have*

$$\lim_{n\to\infty} \frac{1}{n} \mid \{k \leq n : E(\| \zeta_k - \zeta \| > \frac{1}{2})\} \mid = 1 (\neq 0).$$

So the sequence (ζ_n) *does not quasi statisticaly converge in mean to* ζ.
From Example 3.5, we can obtain that quasi statistically convergence in mean does not imply quasi statistically convergence a.s.

Proposition 3.1. *Let* $\zeta, \zeta_1, \zeta_2, ...$ *be complex uncertain variables. Then,* (ζ_n) *is quasi statistically converges a.s. to* ζ *if and only if for any* $\varepsilon, \delta > 0$, *we have*

$$\lim_{n\to\infty} \frac{1}{c_n} \left| \left\{ k : \mathcal{M}\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

Proof. From the definition of quasi statistical convergence a.s., we have there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that, for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k : \| \zeta_k(\gamma) - \zeta(\gamma) \| \geq \varepsilon\} \mid = 0.$$

Then for any $\varepsilon > 0$, there exists k such that $\| \zeta_n - \zeta \| < \varepsilon$ where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\lim_{n\to\infty} \frac{1}{c_n} \left| \left\{ k : \mathcal{M}\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| < \varepsilon \right) \geq 1 \right\} \right| = 0.$$

It follows from the duality axiom of uncertain measure that

$$\lim_{n\to\infty} \frac{1}{c_n} \left| \left\{ k : \mathcal{M}\left(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \| \zeta_k - \zeta \| \geq \varepsilon \right) \geq \delta \right\} \right| = 0.$$

□

Definition 3.7. *A complex uncertain sequence* (ζ_n) *is said to be quasi statistically convergent uniformly almost surely (d.u.a.s.) to* ζ *if for every* $\varepsilon, \exists\, \delta > 0$ *and a sequence* (X_k) *of events such that*

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k \leq n : \mid \mathcal{M}(X_k) - 0 \mid \geq \varepsilon\} \mid = 0.$$

Equivalently,

$$\lim_{n\to\infty} \frac{1}{c_n} \mid \{k :\mid \zeta_k(\gamma) - \zeta(\gamma) \mid\geq \delta\} \mid= 0.$$

Proposition 3.2. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then, (ζ_n) is quasi statistically converges uniformly a.s. to ζ if and only if for any $\varepsilon, \delta > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{c_n}\left|\left\{k : \mathcal{M}\left(\bigcup_{n=k}^{\infty} \mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\right) \geq \delta\right\}\right| = 0.$$

Proof. If (ζ_n) is quasi statistically converges uniformly a.s. to ζ, then for any $\delta > 0$, there exists B such that $\mathcal{M}\{B\} < \delta$ and (ζ_n) quasi statistically uniformly converges to ζ on $\Gamma - B$. Thus, for any $\varepsilon > 0$, there exists $k > 0$ such that $\mid\mid \zeta_n - \zeta \mid\mid< \varepsilon$ where $n \geq k$ and $\gamma \in \Gamma - B$. That is

$$\bigcup_{n=k}^{\infty}\{\mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\} \subset B.$$

It follows from the subadditivity axiom that

$$\lim_{n\to\infty} \frac{1}{c_n}\left|\left\{k : \mathcal{M}\left(\bigcup_{n=k}^{\infty} \mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\right)\right\}\right| \leq \delta(\mathcal{M}\{B\}) < \delta.$$

Then,

$$\lim_{n\to\infty} \frac{1}{c_n}\left|\left\{k : \lim_{n\to\infty} \mathcal{M}\left(\bigcup_{n=k}^{\infty} \mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\right) \geq \delta\right\}\right| = 0.$$

Conversely,
if $\lim\limits_{n\to\infty} \frac{1}{c_n}\left|\left\{k : \mathcal{M}\left(\bigcup\limits_{n=k}^{\infty} \mid\mid \zeta_k - \zeta \mid\mid\geq \varepsilon\right) \geq \delta\right\}\right| = 0$, then, for any $\varepsilon, \delta > 0$ and $m \geq 1$, there exists m_k such that

$$\delta\left(\mathcal{M}\left(\bigcup_{n=m_k}^{\infty}\left\{\mid\mid \zeta_n - \zeta \mid\mid\geq \frac{1}{m}\right\}\right)\right) < \frac{\delta}{2^m}.$$

Let $B = \bigcup\limits_{m=1}^{\infty} \bigcup\limits_{n=m_k}^{\infty} \{\mid\mid \zeta_k - \zeta \mid\mid\geq \frac{1}{m}\}$. Then we have,

$$\delta(\mathcal{M}\{B\}) \leq \sum_{m=1}^{\infty} \delta\left(\mathcal{M}\left(\bigcup_{n=m_k}^{\infty}\left\{\mid\mid \zeta_n - \zeta \mid\mid\geq \frac{1}{m}\right\}\right)\right) \leq \sum_{m=1}^{\infty} \frac{\delta}{2^m}.$$

Furthermore, we have

$$\sup_{\gamma\in\Gamma-B} \mid\mid \zeta_n - \zeta \mid\mid< \frac{1}{m}$$

for any $m = 1, 2, 3, ...$ and $n > m_k$. The proposition is thus proved. □

Proposition 3.3. *If the complex uncertain sequence (ζ_n) is quasi statistically converges uniformly a.s. to ζ, then (ζ_n) is quasi statistically converges a.s. to ζ.*

Proof. We know from the above proposition that if (ζ_n) quasi statistically converges uniformly a.s. to ζ, then

$$\lim_{n\to\infty}\frac{1}{c_n}\left|\left\{k:\mathcal{M}\left(\bigcup_{n=k}^{\infty}\|\,\zeta_k-\zeta\,\|\geq\varepsilon\right)\geq\delta\right\}\right|=0.$$

Since

$$\delta\Bigg(\mathcal{M}\Bigg(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Big\{\|\,\zeta_n-\zeta\,\|\geq\varepsilon\Big\}\Bigg)\Bigg)\leq\delta\Bigg(\mathcal{M}\Bigg(\bigcup_{n=k}^{\infty}\Big\{\|\,\zeta_n-\zeta\,\|\geq\varepsilon\Big\}\Bigg)\Bigg),$$

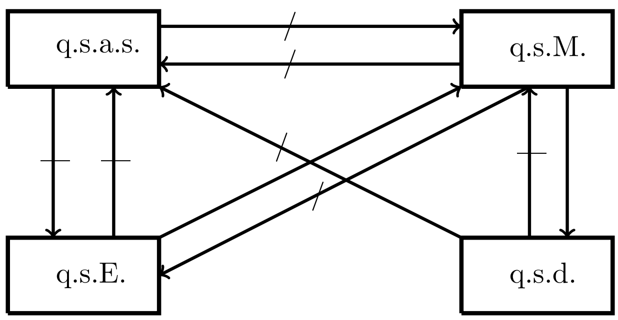
taking limit as $n\to\infty$ on both sides of the above inequality, we have

$$\delta\Bigg(\mathcal{M}\Bigg(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\Big\{\|\,\zeta_n-\zeta\,\|\geq\varepsilon\Big\}\Bigg)\Bigg)=0.$$

By Proposition 3.1, (ζ_n) quasi statistically converges a.s. to ζ. □

4. Conclusions

In this paper, we have investigated the notion of quasi statistical convergence of complex uncertain sequences. The following diagram represents the obtained interrelationships of various convergence concepts, namely convergence in mean, convergence in measure, convergence almost surely and convergence in distribution:



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