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SOME NEW TRAPEZOIDAL TYPE INEQUALITIES FOR STRONGLY GEOMETRIC-ARITHMETICALLY CONVEX FUNCTIONS

A. K. DEMÍREL[1], §

Abstract. This paper considers some preliminary conclusions of Fejér's integral inequality relevant to strongly geometric arithmetic convex functions that is a type of the class of convex functions and also a mapping to produce a novel trapezoidal form. This mapping is used to derive new theorems and results. By utilization these, some applications were given.

Keywords: Trapezoidal type inequalities, strongly convex functions, GA-convex functions, integral inequalities.

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1. Introduction

Convexity theory provides an active research area in both pure and practical sciences. The major of these research areas is the theory of inequality. The convexity theory was also essential in broadening and generalizing the inequalities theory. It established various fundamental integral inequalities for convex functions and their variant forms, utilizing diverse concepts and approaches (see [2, 4, 12, 19]).

In usual analysis, a trapezoidal type inequality is an inequality that ensures upper and/or lower bounds for the cardinality

$$\frac{\phi(\sigma)+\phi(\varsigma)}{2}(\varsigma-\sigma)-\int_{\sigma}^{\varsigma}\phi(\eta)d\eta,$$

in other words this is the error in approaching the integral by a trapezoidal rule for diverse types of integrable functions described on the compact interval $[\sigma,\varsigma]$ (see [6]).

There are a lot of studies on trapezoidal type inequalities that have a considerable place in inequality theory. One of the trapezoidal type inequalities studies in the literature belongs to Dragomir and Sofo [8]. They obtained some novel results relevant to the trapezoidal type inequality. Usta et al. [22] suggested refinements inequalities for

[1] Ordu University, Central Research Laboratory, Turkey.
e-mail: aysekubrademirel@gmail.com; ORCID: https://orcid.org/0000-0002-2389-8699.

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the Hermite-Hadamard's type inequality and they specify explicit bounds for the trapezoid inequalities in terms of s-convexity. Mohammed [14] offered some novel trapezoidal type inequalities for h-convex functions via generalized fractional integrals. Delavar and Dragomir [7], using a $M_g(t)$ mapping, obtained some preliminary results and a new trapezoidal form of the Fejér inequality for h-convex functions. Turhan [21] demonstrated a novel trapezoidal form of Fejér inequality relevant to the harmonically arithmetic convex functions. The author also obtained the new theorems and corollaries by usage a mapping $M(t)$. Baleanu et al. [1] demonstrated certain novel generalized fractional inequalities of the trapezoidal type for λ_φ–preinvex functions. They also investigated the novel estimates on trapezoidal type inequalities for usual integral and Riemann–Liouville fractional integrals. Dragomir [5] established some Ostrowski and trapezoidal type inequalities for the $k-g$-fractional integrals of functions of bounded variation. Kalsoom et al. [11] novel $(p,q)_{\kappa_1}$-integral and $(p,q)^{\kappa_2}$-integral identities. By utilizing these identities, they obtained $(p,q)_{\kappa_1}$ and $(p,q)^{\kappa_2}$- trapezoidal type inequalities for strongly convex and quasi-convex functions. Budak et al. [3] demonstrated an identity for double partially differentiable mappings. By employing this identity, they presented certain generalized inequalities for differentiable coordinated convex functions. Sitthiwirattham et al. [18] proved several trapezoidal and Ostrowski type inequalities via generalized fractional integrals via functions of bounded variations with two variables.

In the next section, we primarily represent the definitions and theorems that will form the basis of the paper.

2. Preliminaries

In 1906, Fejér [9] introduced the following integral inequalities called that the Fejér inequality:

$$\phi\left(\frac{\sigma\varsigma}{2}\right) \leq \int_\sigma^\varsigma \varphi(\lambda)\, d\lambda \leq \int_\sigma^\varsigma \phi(\lambda)\varphi(\lambda)\, d\lambda \leq \frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_\sigma^\varsigma \varphi(\lambda)\, d\lambda, \tag{1}$$

where $\phi : [\sigma,\varsigma] \to \mathbb{R}$ is convex and $\varphi : [\sigma,\varsigma] \to \mathbb{R}^+ = [0,\infty)$ is integrable and symmetric to $\lambda = \sqrt{\sigma\varsigma}\left(\varphi(\lambda) = \varphi\left(\frac{\sigma\varsigma}{\lambda}\right),\ \forall\lambda \in [\sigma,\varsigma]\right)$.

To see some other inequalities relating to Fejér's inequalities see [13, 15, 17, 20, 21] and the references.

Definition 2.1. *Let Ω be a interval, $\phi : \Omega \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be strongly GA-convex function with modulus $c > 0$, if*

$$\phi\left(\lambda^{1-\eta}\kappa^\eta\right) \leq (1-\eta)\phi(\lambda) + \eta\phi(\kappa) - c\eta(1-\eta)\|\ln\kappa - \ln\lambda\|^2, \tag{2}$$

for all $\lambda, \kappa \in \Omega$ and $\eta \in [0,1]$.

Definition 2.2. *A function $\phi : [\sigma,\varsigma] \to \mathbb{R}$ is said to satisfy Lipschitz condition (see [16]) on $[\sigma,\varsigma]$ if there is a constant M so that for any two points $\lambda, \kappa \in [\sigma,\varsigma]$,*

$$|\phi(\lambda) - \phi(\kappa)| \leq M|\lambda - \kappa|. \tag{3}$$

In [10], The Fejér trapezoid inequality for convex functions was obtained by Hwang as follows:

Theorem 2.1. *Let $\phi : \Omega \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable on Ω^0, where $\sigma, \varsigma \in \Omega$ with $\sigma < \varsigma$, and let $\varphi : [\sigma,\varsigma] \to [0,\infty)$ be continuous, positive and symmetric to $\frac{\sigma+\varsigma}{2}$. If $|\phi'|$ is convex* *on* $[\sigma,\varsigma]$, *then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\varphi(\lambda)\,d\lambda-\int_{\sigma}^{\varsigma}\phi(\lambda)\varphi(\lambda)\,d\lambda\right|\\&\leq\frac{\varsigma-\sigma}{4}\left[\left|\phi'(\sigma)\right|+\left|\phi'(\varsigma)\right|\right]\int_0^1\int_{B(\sigma,\varsigma,\eta)}^{T(\sigma,\varsigma,\eta)}\varphi(\lambda)\,d\lambda d\eta,\end{aligned}\tag{4}$$

where $B(\sigma,\varsigma,\eta)=\frac{1+\eta}{2}\sigma+\frac{1-\eta}{2}\varsigma$ *and* $T(\sigma,\varsigma,\eta)=\frac{1-\eta}{2}\sigma+\frac{1+\eta}{2}\varsigma$.

In [7], The Fejér trapezoid inequality for h-convex functions was obtained by Delavar and Dragomir as follows:

Theorem 2.2. *Suppose that* $\phi:\Omega\to\mathbb{R}$ *be differentiable on* Ω^0, *where* $\sigma,\varsigma\in\Omega^0$ *with* $\sigma<\varsigma$ *and let* $\varphi:[\sigma,\varsigma]\to\mathbb{R}^+$ *is differentiable and symmetric to* $\frac{\sigma+\varsigma}{2}$. *If* $|\phi'|$ *is a* h*-convex on* $[\sigma,\varsigma]$, *then*

$$\begin{aligned}&\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\varphi(\lambda)\,d\lambda-\int_{\sigma}^{\varsigma}\phi(\lambda)\varphi(\lambda)\,d\lambda\right|\\&\leq(\varsigma-\sigma)\left[\left|\phi'(\sigma)\right|+\left|\phi'(\varsigma)\right|\right]\int_{\sigma}^{\frac{\sigma+\varsigma}{2}}\int_0^{\frac{\lambda-\sigma}{\varsigma-\sigma}}\varphi(\lambda)\left[h(\eta)+h(1-\eta)\right]d\eta d\lambda.\end{aligned}\tag{5}$$

In [21], The Fejér trapezoid inequality for harmonically-convex functions was obtained by Turhan as follows:

Theorem 2.3. *Suppose that* $\phi:\Omega\to\mathbb{R}\backslash 0$ *is a differentiable on* Ω^0, *where* $\sigma,\varsigma\in\Omega^0$ *with* $\sigma<\varsigma$, *and let* $\varphi:[\sigma,\varsigma]\to\mathbb{R}$ *is differentiable and symmetric to* $\frac{2\sigma\varsigma}{\sigma+\varsigma}$. *If* $|\phi'|$ *is a harmonically-convex on* $[\sigma,\varsigma]$, *then*

$$\begin{aligned}&\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda^2}d\lambda-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda^2}d\lambda\right|\\&\leq(\varsigma-\sigma)\sigma\varsigma\int_{\sigma}^{\frac{2\sigma\varsigma}{\varsigma-\sigma}}\frac{\varphi(\lambda)}{\lambda^2}\left[A_1(\lambda)\left|\phi'(\sigma)\right|+A_2(\lambda)\left|\phi'(\varsigma)\right|\right]d\lambda,\end{aligned}\tag{6}$$

where

$$A_1(\lambda)=\int_0^{\frac{\frac{1}{\sigma}-\frac{1}{\lambda}}{\frac{1}{\sigma}-\frac{1}{\varsigma}}}\frac{\eta}{(\eta\varsigma+(1-\eta)\sigma)^2}d\eta+\int_0^{\frac{\frac{1}{\sigma}-\frac{1}{\lambda}}{\frac{1}{\sigma}-\frac{1}{\varsigma}}}\frac{(1-\eta)}{((1-\eta)\varsigma+\eta\sigma)^2}d\eta$$

and

$$A_2(\lambda)=\int_0^{\frac{\frac{1}{\sigma}-\frac{1}{\lambda}}{\frac{1}{\sigma}-\frac{1}{\varsigma}}}\frac{(1-\eta)}{(\eta\varsigma+(1-\eta)\sigma)^2}d\eta+\int_0^{\frac{\frac{1}{\sigma}-\frac{1}{\lambda}}{\frac{1}{\sigma}-\frac{1}{\varsigma}}}\frac{\eta}{((1-\eta)\varsigma+\eta\sigma)^2}d\eta$$

.

3. Main Results

With respect to a function $\varphi:[\sigma,\varsigma]\to\mathbb{R}$ contemplate the mapping $\chi:[0,1]\to\mathbb{R}$ as follows:

$$\chi(\eta)=\int_{\eta}^{1}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)ds-\int_0^{\eta}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)ds.\tag{7}$$

There are some features for $\chi(\eta)$, gathered in the next lemma.

Lemma 3.1. *Suppose that* $\Omega\subseteq\mathbb{R}^+$ *and* $\sigma,\varsigma\in\Omega^0$ *with* $\sigma<\varsigma$ *and* $\varphi:[\sigma,\varsigma]\subseteq\mathbb{R}^+\to\mathbb{R}$ *integrable on* $[\sigma,\varsigma]$.

(i) If φ is symmetric to $\sqrt{\sigma\varsigma}$, then

$$\chi(\eta)=\begin{cases} 2\int_{\eta}^{1/2}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)ds, & 0\leq\eta\leq 1/2,\\ -2\int_{1/2}^{\eta}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)ds, & 1/2\leq\eta\leq 1.\end{cases}$$

(ii) For $\eta\in[0,1]$, $\chi(\eta)+\chi(1-\eta)=0$.
(iii) If φ is a nonnegative function, then

$$\begin{cases}\chi(\eta)\geq 0, & 0\leq\eta\leq 1/2,\\ \chi(\eta)\leq 0, & 1/2\leq\eta\leq 1.\end{cases}$$

(iv) This inequalities hold:

$$\int_0^1|\chi(\eta)|\,d\eta\leq\frac{1}{2}\|\varphi\|_{\infty}$$

and

$$\int_0^1|\chi(\eta)|\,d\eta\leq 2\|\varphi\|_q\int_0^1\left|\eta-\frac{1}{2}\right|^{1/p}d\eta.$$

(v) Let $\Omega\subseteq\mathbb{R}^{+}$ and $\phi:\Omega^0\to\mathbb{R}$ is a differentiable on Ω^0, Ω^0 is an interior of Ω and φ is a differentiable. If $\phi'\in L[\sigma,\varsigma]$, then this equality holds:

$$\frac{1}{\ln\varsigma-\ln\sigma}\left[\left(\frac{\phi(\sigma)+\phi(\varsigma)}{2}\right)\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right]$$
$$=\frac{\ln\varsigma-\ln\sigma}{2}\int_0^1\chi(\eta)\,\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\sigma^{1-\eta}\varsigma^{\eta}d\eta.$$

Proof.
(i) Using the replace of variable $\lambda=\sigma^{1-s}\varsigma^{s}$ in the description of $\chi(\eta)$, for $0\leq\eta\leq 1/2$, we obtain

$$\chi(\eta)=\frac{1}{\ln\varsigma-\ln\sigma}\left[\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right] \tag{8}$$

where $\sqrt{\sigma\varsigma}\leq\sigma^{1-\eta}\varsigma^{\eta}\leq\varsigma$. Since φ is symmetric to $\sqrt{\sigma\varsigma}$,

$$\int_{\sqrt{\sigma\varsigma}}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda=\int_{\sigma}^{\sqrt{\sigma\varsigma}}\frac{\varphi(\lambda)}{\lambda}d\lambda$$

and so

$$\begin{aligned}\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda&=\int_{\sigma}^{\sqrt{\sigma\varsigma}}\frac{\varphi(\lambda)}{\lambda}d\lambda+\int_{\sqrt{\sigma\varsigma}}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda\\ &=\int_{\sqrt{\sigma\varsigma}}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda+\int_{\sqrt{\sigma\varsigma}}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda\end{aligned} \tag{9}$$

On the other hand,

$$\int_{\sqrt{\sigma\varsigma}}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda=\int_{\sqrt{\sigma\varsigma}}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda+\int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda \tag{10}$$

If we use (9) and (10) to (8), we get

$$\frac{1}{\ln\varsigma-\ln\sigma}\left[\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right]=\frac{2}{\ln\varsigma-\ln\sigma}\int_{\sqrt{\sigma\varsigma}}^{\sigma^{1-\eta}\varsigma^{\eta}}\frac{\varphi(\lambda)}{\lambda}d\lambda.$$

Here $\chi(\eta) = 2\int_{\eta}^{1/2} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds$, where $1/2 \leq \eta \leq 1$.
(ii) For $\eta \in [0,1]$,

$$\begin{aligned}
\chi(\eta) &= \frac{1}{\ln\varsigma - \ln\sigma}\left[\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}} \frac{\varphi(\lambda)}{\lambda} d\lambda - \int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma} \frac{\varphi(\lambda)}{\lambda} d\lambda\right] \\
&= \frac{1}{\ln\varsigma - \ln\sigma}\left[\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}} \frac{\varphi\left(\frac{\sigma\varsigma}{\lambda}\right)}{\lambda} d\lambda - \int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma} \frac{\varphi\left(\frac{\sigma\varsigma}{\lambda}\right)}{\lambda} d\lambda\right] \\
&= \frac{1}{\ln\varsigma - \ln\sigma}\left[\int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma} \frac{\varphi\left(\frac{\sigma\varsigma}{\lambda}\right)}{\lambda} d\lambda - \int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}} \frac{\varphi\left(\frac{\sigma\varsigma}{\eta}\right)}{\eta} d\eta\right] = -\chi(1-\eta).
\end{aligned}$$

(iii) It is an easy result of the (i) claim.
(iv) With the claim (iii), we have

$$\begin{aligned}
&\int_0^1 |\chi(\eta)|\, d\eta = \int_0^{1/2} \chi(\eta)\, d\eta - \int_{1/2}^{1} \chi(\eta)\, d\eta \\
&= 2\int_0^{1/2}\int_{\eta}^{1/2} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds d\eta + 2\int_{1/2}^{1}\int_{1/2}^{\eta} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds d\eta \\
&\leq 2\int_0^{1/2}\int_{\eta}^{1/2} \sup_{s\in[\eta,1/2]} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds d\eta + 2\int_{1/2}^{1}\int_{1/2}^{\eta} \sup_{s\in[1/2,\eta]} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds d\eta \\
&\leq 2\int_0^{1/2} \left(\frac{1}{2} - \eta\right) \|\varphi\|_{\infty} d\eta + 2\int_{1/2}^{1} \left(\eta - \frac{1}{2}\right) \|\varphi\|_{\infty} d\eta \\
&\leq 2\|\varphi\|_{\infty} \int_0^1 \left|\eta - \frac{1}{2}\right| d\eta = \frac{1}{2}\|\varphi\|_{\infty}.
\end{aligned}$$

For the second part of (iv),

$$\int_0^1 |\chi(\eta)|\, d\eta = 2\int_0^1 \left|\int_{\eta}^{1/2} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds\right| d\eta. \tag{11}$$

Using Hölder's inequality to the last inequality, we have

$$\begin{aligned}
&\int_{\eta}^{1/2} \varphi\left(\sigma^{1-s}\varsigma^{s}\right) ds \leq \left|\int_{\eta}^{1/2} ds\right|^{1/p} \left|\int_{\eta}^{1/2} \left(\varphi\left(\sigma^{1-s}\varsigma^{s}\right)\right)^{q} ds\right|^{1/q} \\
&\leq \left|\eta - \frac{1}{2}\right|^{1/p} \|\varphi\|_{q}.
\end{aligned} \tag{12}$$

Now applying (14) in (13), we get

$$\int_0^1 |\chi(\eta)|\, d\eta \leq 2\|\varphi\|_{q} \int_0^1 \left|\eta - \frac{1}{2}\right|^{1/p} d\eta.$$

(v) First, the equality is calculated as follows:

$$\begin{aligned}
&\int_0^1 \chi(\eta)\, d\left(\phi\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\right) \\
&= \frac{1}{\ln\varsigma-\ln\sigma}\int_0^1 \left[\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}} \frac{\varphi(\lambda)}{\lambda}d\lambda - \int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma} \frac{\varphi(\lambda)}{\lambda}d\lambda\right] d\left(\phi\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\right) \\
&= \frac{1}{\ln\varsigma-\ln\sigma}\left[\int_0^1 \left(\int_{\sigma}^{\sigma^{1-\eta}\varsigma^{\eta}} \frac{\varphi(\lambda)}{\lambda}d\lambda\right) d\left(\phi\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\right)\right. \\
&\quad \left. - \int_0^1 \left(\int_{\sigma^{1-\eta}\varsigma^{\eta}}^{\varsigma} \frac{\varphi(\lambda)}{\lambda}d\lambda\right) d\left(\phi\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\right)\right] \\
&= \frac{1}{\ln\varsigma-\ln\sigma}\left[\left(\phi(\sigma)+\phi(\varsigma)\right)\int_{\sigma}^{\varsigma} \frac{\varphi(\lambda)}{\lambda}d\lambda - 2\int_{\sigma}^{\varsigma} \frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right] \\
&= (\ln\varsigma-\ln\sigma)\int_0^1 \chi(\eta)\,\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\sigma^{1-\eta}\varsigma^{\eta} d\eta.
\end{aligned}$$

If we multiply both sides of the above equality by $1/2$, the proof is complete. □

Theorem 3.1. *Assume that $\phi : \Omega \subseteq \mathbb{R}^{+} \to \mathbb{R}$ is a differentiable on Ω^0, $\sigma, \varsigma \in \Omega^0$ with $\sigma < \varsigma$and $\varphi : [\sigma, \varsigma] \to \mathbb{R}$ is a differentiable. Suppose that ϕ' is an integrable on $[\sigma, \varsigma]$ and there are constants $r < R$ such that*

$$-\infty < r \leq \lambda\phi'(\lambda) \leq R < \infty \tag{13}$$

for all $\lambda \in [\ln\sigma, \ln\varsigma]$. Then

$$\begin{aligned}
&\left|\frac{1}{\ln\varsigma-\ln\sigma}\left[\frac{\phi(\sigma)+\phi(\varsigma)}{2}\left(\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right)\right.\right. \\
&\qquad \left.\left. - \int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}\right] - \frac{(\ln\varsigma-\ln\sigma)(r+R)}{8}\int_0^1 \chi(\eta)\,d\eta\right| \\
&\leq \frac{(\ln\varsigma-\ln\sigma)(R-r)}{8}\int_0^1 |\chi(\eta)|\,d\eta.
\end{aligned} \tag{14}$$

Proof. From (v) of Lemma 3.1, we obtain

$$\begin{aligned}
&\frac{1}{\ln\varsigma-\ln\sigma}\left[\frac{\phi(\sigma)+\phi(\varsigma)}{2}\left(\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right) - \int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right] \\
&= \frac{\ln\varsigma-\ln\sigma}{2}\int_0^1 \chi(\eta)\left(\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right) - \frac{r+R}{4} + \frac{r+R}{4}\right)d\eta \\
&= \frac{(\ln\varsigma-\ln\sigma)(r+R)}{8}\int_0^1 \chi(\eta)\,d\eta \\
&\quad + \frac{\ln\varsigma-\ln\sigma}{2}\int_0^1 \chi(\eta)\left(\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right) - \frac{r+R}{4}\right)d\eta.
\end{aligned}$$

As a result of

$$J=\frac{1}{\ln\varsigma-\ln\sigma}\left[\frac{\phi(\sigma)+\phi(\varsigma)}{2}\left(\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right)\right.$$
$$\left.-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right]-\frac{(\ln\varsigma-\ln\sigma)(r+R)}{8}\int_{0}^{1}\chi(\eta)\,d\eta$$
$$=\frac{\ln\varsigma-\ln\sigma}{2}\int_{0}^{1}\chi(\eta)\left(\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)-\frac{r+R}{4}\right)d\eta.$$

If the absolute value is used on both sides of the last equation, we obtain

$$|J|\leq\frac{\ln\varsigma-\ln\sigma}{2}\int_{0}^{1}|\chi(\eta)|\left|\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)-\frac{r+R}{4}\right|d\eta,$$

since from the inequality $r\leq\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\leq R$, we have

$$r-\frac{r+R}{4}\leq\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)-\frac{r+R}{4}\leq R-\frac{r+R}{4}\tag{15}$$

which implies that

$$\left|\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)-\frac{r+R}{4}\right|\leq\frac{R-r}{4}.\tag{16}$$

□

Remark 3.1. *If we take g is symmetric to $\sqrt{\sigma\varsigma}$, the from Lemma (3.1), we have*

$$\left|\frac{1}{\ln\varsigma-\ln\sigma}\left[\frac{\phi(\sigma)+\phi(\varsigma)}{2}\left(\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right)\right.\right.$$
$$\left.\left.-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}\right]-\frac{(\ln\varsigma-\ln\sigma)(r+R)}{8}\int_{0}^{1}\chi(\eta)\,d\eta\right|$$
$$\leq\frac{(\ln\varsigma-\ln\sigma)(R-r)}{8}\|\varphi\|\infty,$$

and

$$\left|\frac{1}{\ln\varsigma-\ln\sigma}\left[\frac{\phi(\sigma)+\phi(\varsigma)}{2}\left(\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda\right)\right.\right.$$
$$\left.\left.-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}\right]-\frac{(\ln\varsigma-\ln\sigma)(r+R)}{8}\int_{0}^{1}\chi(\eta)\,d\eta\right|$$
$$\leq\frac{(\ln\varsigma-\ln\sigma)(R-r)}{4}\|\varphi\|_{q}\int_{0}^{1}\left|\eta-\frac{1}{2}\right|^{\frac{1}{p}}d\eta.$$

Theorem 3.2. *Assume that $\phi:\Omega\subseteq\mathbb{R}^{+}\to\mathbb{R}$ is a differentiable on Ω^{0}, $\sigma,\varsigma\in\Omega^{0}$ with $\sigma<\varsigma$ and $\varphi:[\sigma,\varsigma]\to\mathbb{R}$ is a differentiable and symmetric to $\sqrt{\sigma\varsigma}$. If $|\phi'|$ is a strongly GA-convex (geometric-arithmetic convex) on $[\sigma,\varsigma]$, then*

$$\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right|$$
$$\leq(\ln\varsigma-\ln\sigma)\tag{17}$$
$$\cdot\int_{\sigma}^{\sqrt{\sigma\varsigma}}\frac{\varphi(\lambda)}{\lambda}\left[C_{1}(\lambda)\left|\phi'(\sigma)\right|+C_{2}(\lambda)\left|\phi'(\varsigma)\right|-c\left\|\ln\varsigma-\ln\sigma\right\|^{2}C_{3}(\lambda)\right]d\lambda$$

where

$$C_1(\lambda)=\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}\eta\sigma^{\eta}\varsigma^{1-\eta}d\eta+\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}(1-\eta)\sigma^{1-\eta}\varsigma^{\eta}d\eta,$$

$$C_2(\lambda)=\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}(1-\eta)\sigma^{\eta}\varsigma^{1-\eta}d\eta+\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}\eta\sigma^{1-\eta}\varsigma^{\eta}d\eta,$$

$$C_3(\lambda)=\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}\eta(1-\eta)\sigma^{\eta}\varsigma^{1-\eta}d\eta+\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}\eta(1-\eta)\sigma^{1-\eta}\varsigma^{\eta}d\eta.$$

Proof. Using the description of $\chi(\eta)$, Lemma 3.1 and $|\phi'|$ being a strongly GA-convex function, we obtain

$$\begin{aligned}
&\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right|\\
&=\frac{(\ln\varsigma-\ln\sigma)^2}{2}\left|\int_0^1\chi(\eta)\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\sigma^{1-\eta}\varsigma^{\eta}d\eta\right|\\
&\le\frac{(\ln\varsigma-\ln\sigma)^2}{2}\int_0^1|\chi(\eta)|\left|\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right)\right|\sigma^{1-\eta}\varsigma^{\eta}d\eta\\
&\le(\ln\varsigma-\ln\sigma)^2\left[\int_0^{\frac{1}{2}}\left(\int_{\eta}^{\frac{1}{2}}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)ds\right)\right.\\
&\times\left[(1-\eta)\left|\phi'(\sigma)\right|+\eta\left|\phi'(\varsigma)\right|-c\eta(1-\eta)\|\ln\varsigma-\ln\sigma\|^2\right]\sigma^{1-\eta}\varsigma^{\eta}d\eta\\
&+\int_{\frac{1}{2}}^{1}\left(\int_{\frac{1}{2}}^{1}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)ds\right)\left[(1-\eta)\left|\phi'(\sigma)\right|+\eta\left|\phi'(\varsigma)\right|\right.\\
&\left.\left.-c\eta(1-\eta)\|\ln\varsigma-\ln\sigma\|^2\right]\sigma^{1-\eta}\varsigma^{\eta}d\eta\right].
\end{aligned}\tag{18}$$

If the order of integration is changed, we get

$$\begin{aligned}
&\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right|\\
&\le(\ln\varsigma-\ln\sigma)^2\left[\int_0^{\frac{1}{2}}\int_0^{s}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)\left[(1-\eta)\left|\phi'(\sigma)\right|+\eta\left|\phi'(\varsigma)\right|\right.\right.\\
&\left.-c\eta(1-\eta)\|\ln\varsigma-\ln\sigma\|^2\right]\sigma^{1-\eta}\varsigma^{\eta}d\eta ds+\int_{\frac{1}{2}}^{1}\int_{s}^{1}\varphi\left(\sigma^{1-s}\varsigma^{s}\right)\\
&\left.\times\left[(1-\eta)\left|\phi'(\sigma)\right|+\eta\left|\phi'(\varsigma)\right|-c\eta(1-\eta)\|\ln\varsigma-\ln\sigma\|^2\right]\sigma^{1-\eta}\varsigma^{\eta}d\eta ds\right].
\end{aligned}$$

Using the alter of variable $\lambda=\sigma^{1-s}\varsigma^{s}$, we obtain

$$\begin{aligned}
&\left|\frac{\phi(\sigma)+\phi(\varsigma)}{2}\int_{\sigma}^{\varsigma}\frac{\varphi(\lambda)}{\lambda}d\lambda-\int_{\sigma}^{\varsigma}\frac{\phi(\lambda)\varphi(\lambda)}{\lambda}d\lambda\right|\\
&=(\ln\varsigma-\ln\sigma)\left[\int_{\sqrt{\sigma\varsigma}}^{\varsigma}\int_0^{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}\frac{\varphi(\lambda)}{\lambda}\left[(1-\eta)\left|\phi'(\sigma)\right|+\eta\left|\phi'(\varsigma)\right|\right.\right.\\
&\left.-c\eta(1-\eta)\|\ln\varsigma-\ln\sigma\|^2\right]\sigma^{1-\eta}\varsigma^{\eta}d\eta dx+\int_{\sigma}^{\sqrt{\sigma\varsigma}}\int_{\frac{\ln\lambda-\ln\sigma}{\ln\varsigma-\ln\sigma}}^{1}\frac{\varphi(\lambda)}{\lambda}
\end{aligned}$$

$$\times \left[(1-\eta) \left|\phi'(\sigma)\right| + \eta \left|\phi'(\varsigma)\right| - c\eta (1-\eta) \left\|\ln\varsigma - \ln\sigma\right\|^2 \right] \sigma^{1-\eta}\varsigma^{\eta} d\eta dx \Big]. \tag{19}$$

Since $\varphi(\lambda)$ function is symmetric to $\sqrt{\sigma\varsigma}$, then

$$\int_{\sqrt{\sigma\varsigma}}^{\varsigma} \int_0^{\frac{\ln\lambda - \ln\sigma}{\ln\varsigma - \ln\sigma}} \frac{\varphi(\lambda)}{\lambda} \left[(1-\eta) \left|\phi'(\sigma)\right| + \eta \left|\phi'(\varsigma)\right| \right.$$

$$\left. -c\eta (1-\eta) \left\|\ln\varsigma - \ln\sigma\right\|^2 \right] \sigma^{1-\eta}\varsigma^{\eta} d\eta dx = \int_{\sigma}^{\sqrt{\sigma\varsigma}} \int_0^{\frac{\ln\varsigma - \ln\lambda}{\ln\varsigma - \ln\lambda}} \frac{\varphi(\lambda)}{\lambda}$$

$$\times \left[(1-\eta) \left|\phi'(\sigma)\right| + \eta \left|\phi'(\varsigma)\right| - c\eta (1-\eta) \left\|\ln\varsigma - \ln\sigma\right\|^2 \right] \sigma^{1-\eta}\varsigma^{\eta} d\eta dx. \tag{20}$$

Additionaly it is not difficult to see that,

$$\int_0^{\frac{\ln\varsigma - \ln\lambda}{\ln\varsigma - \ln\sigma}} (1-\eta) \sigma^{1-\eta}\varsigma^{\eta} d\eta = \int_{\frac{\ln\lambda - \ln\sigma}{\ln\varsigma - \ln\sigma}}^{1} \eta \sigma^{\eta}\varsigma^{1-\eta} d\eta,$$

$$\int_0^{\frac{\ln\varsigma - \ln\lambda}{\ln\varsigma - \ln\sigma}} \eta \sigma^{1-\eta}\varsigma^{\eta} d\eta = \int_{\frac{\ln\lambda - \ln\sigma}{\ln\varsigma - \ln\sigma}}^{1} (1-\eta) \sigma^{\eta}\varsigma^{1-\eta} d\eta,$$

$$\int_0^{\frac{\ln\varsigma - \ln\lambda}{\ln\varsigma - \ln\sigma}} \eta (1-\eta) \sigma^{1-\eta}\varsigma^{\eta} d\eta = \int_{\frac{\ln\lambda - \ln\sigma}{\ln\varsigma - \ln\sigma}}^{1} \eta (1-\eta) \sigma^{\eta}\varsigma^{1-\eta} d\eta.$$

By using (20), (19) and (18) to (17), then we complete this proof. □

Theorem 3.3. *Let $\phi : \Omega \subseteq \mathbb{R}^+ \to \mathbb{R}$ be differentiable on Ω^0, $\sigma, \varsigma \in \Omega^0$ with $\sigma < \varsigma$ and let $\varphi : [\sigma, \varsigma] \to \mathbb{R}$ nonnegative and integrable that is differentiable on (σ, ς). Supposing that ϕ' is an integrable on $[\sigma, \varsigma]$ and satisfies a Lipschitz condition for some $M > 0$. Then*

$$\left| \frac{1}{\ln\varsigma - \ln\sigma} \left(\frac{\phi(\sigma) + \phi(\varsigma)}{2} \int_{\sigma}^{\varsigma} \frac{\varphi(\lambda)}{\lambda} d\lambda - \int_{\sigma}^{\varsigma} \frac{\phi(\lambda)\varphi(\lambda)}{\lambda} d\lambda \right) \right.$$

$$\left. - \frac{\ln\varsigma - \ln\sigma}{2} \int_0^1 \chi(\eta) \sigma^{1-\eta}\varsigma^{\eta} \phi'\left(\sqrt{\sigma\varsigma}\right) d\eta \right|$$

$$\leq \frac{M(\ln\varsigma - \ln\sigma)\sqrt{\sigma\varsigma}}{2} \int_0^1 |\chi(\eta)| \left| \sigma^{1-\eta}\varsigma^{\eta} \left(\sigma^{\frac{1}{2}-\eta}\varsigma^{\eta-\frac{1}{2}} - 1 \right) \right| d\eta.$$

Proof. By using (v) of Lemma 3.1, we get

$$\frac{1}{\ln\varsigma - \ln\sigma} \left(\frac{\phi(\sigma) + \phi(\varsigma)}{2} \int_{\sigma}^{\varsigma} \frac{\varphi(\lambda)}{\lambda} d\lambda - \int_{\sigma}^{\varsigma} \frac{\phi(\lambda)\varphi(\lambda)}{\lambda} d\lambda \right)$$

$$= \frac{\ln\varsigma - \ln\sigma}{2} \int_0^1 \chi(\eta) \sigma^{1-\eta}\varsigma^{\eta} \left[\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right) - \phi'\left(\sqrt{\sigma\varsigma}\right) + \phi'\left(\sqrt{\sigma\varsigma}\right) \right] d\eta$$

$$= \frac{\ln\varsigma - \ln\sigma}{2} \int_0^1 \chi(\eta) \sigma^{1-\eta}\varsigma^{\eta} \left[\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right) - \phi'\left(\sqrt{\sigma\varsigma}\right) \right] d\eta$$

$$+ \frac{\ln\varsigma - \ln\sigma}{2} \int_0^1 \chi(\eta) \sigma^{1-\eta}\varsigma^{\eta} \phi'\left(\sqrt{\sigma\varsigma}\right) d\eta. \tag{21}$$

From the last equality, we obtain

$$\left| \frac{1}{\ln\varsigma - \ln\sigma} \left(\frac{\phi(\sigma) + \phi(\varsigma)}{2} \int_{\sigma}^{\varsigma} \frac{\varphi(\lambda)}{\lambda} d\lambda - \int_{\sigma}^{\varsigma} \frac{\phi(\lambda)\varphi(\lambda)}{\lambda} d\lambda \right) \right.$$

$$\left. - \frac{\ln\varsigma - \ln\sigma}{2} \int_0^1 \chi(\eta) \sigma^{1-\eta}\varsigma^{\eta} \phi'\left(\sqrt{\sigma\varsigma}\right) d\eta \right|$$

$$= \frac{\ln \varsigma - \ln \sigma}{2} \int_0^1 |\chi(\eta)| \sigma^{1-\eta} \varsigma^{\eta} \left|\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right) - \phi'\left(\sqrt{\sigma\varsigma}\right)\right| d\eta.$$

If ϕ' satisfies a Lipschitz condition as (3) for some $M > 0$, then

$$\left|\phi'\left(\sigma^{1-\eta}\varsigma^{\eta}\right) - \phi'\left(\sqrt{\sigma\varsigma}\right)\right| \leq M \left|\sigma^{1-\eta}\varsigma^{\eta} - \sigma^{\frac{1}{2}}\varsigma^{\frac{1}{2}}\right|$$
$$= M\left(\sqrt{\sigma\varsigma}\right)\left|\sigma^{\frac{1}{2}-\eta}\varsigma^{\eta-\frac{1}{2}} - 1\right|. \tag{22}$$

Because of this inequality, the proof is completed. □

Remark 3.2. *In Theorem 3.3, assume that φ is symmetric to $\sqrt{\sigma\varsigma}$. If we use (i) of Lemma 3.1, we obtain*

$$\left|\frac{1}{\ln \varsigma - \ln \sigma}\left(\frac{\phi(\sigma) + \phi(\varsigma)}{2}\int_\sigma^\varsigma \frac{\varphi(\lambda)}{\lambda} d\lambda - \int_\sigma^\varsigma \frac{\phi(\lambda)\varphi(\lambda)}{\lambda} d\lambda\right)\right.$$
$$\left. - \frac{\ln \varsigma - \ln \sigma}{2}\int_0^1 \chi(\eta)\sigma^{1-\eta}\varsigma^{\eta}\phi'\left(\sqrt{\sigma\varsigma}\right) d\eta\right|$$
$$\leq M(\ln \varsigma - \ln \sigma)\sqrt{\sigma\varsigma}\int_0^1 \left|\int_\eta^{1/2} \varphi\left(\varsigma^{1-s}\varsigma^{s}\right)\right| \left|\sigma^{1-\eta}\varsigma^{\eta}\left(\sigma^{\frac{1}{2}-\eta}\varsigma^{\eta-\frac{1}{2}} - 1\right)\right| ds d\eta$$
$$\leq M(\ln \varsigma - \ln \sigma)\sqrt{\sigma\varsigma}\int_0^1 \sup_{s\in[\eta,1/2]} \left|\varphi\left(\sigma^{1-s}\varsigma^{s}\right)\right|$$
$$\cdot \left|\frac{1}{2} - \eta\right| \inf_{\eta\in[0,1]} \left|\sigma^{1-\eta}\varsigma^{\eta}\left(\sigma^{\frac{1}{2}-\eta}\varsigma^{\eta-\frac{1}{2}} - 1\right)\right| ds d\eta$$
$$\leq M(\ln \varsigma - \ln \sigma)\sqrt{\sigma\varsigma}\left(\sigma^{3/2}\varsigma^{-1/2} - \sigma\right)\|\varphi\|_\infty \int_0^1 \left(\frac{1}{2} - \eta\right)^2 d\eta$$
$$= \frac{M(\ln \varsigma - \ln \sigma)\left(\sigma^2 - \sigma\sqrt{\sigma\varsigma}\right)\|\varphi\|_\infty}{12}. \tag{23}$$

Corollary 3.1. *In Theorem 3.3, if $\varphi(\lambda) = 1$ is taken for all $\lambda \in [\sigma, \varsigma]$, then*

$$\left|\frac{\phi(\sigma) + \phi(\varsigma)}{2} - \frac{1}{\ln \varsigma - \ln \sigma}\int_\sigma^\varsigma \frac{\phi(\lambda)}{\lambda} d\lambda\right|$$
$$\leq \frac{M(\ln \varsigma - \ln \sigma)\left(\sigma^2 - \sigma\sqrt{\sigma\varsigma}\right)\|\varphi\|_\infty}{12} + \frac{\sigma(\ln \varsigma - \ln \sigma)}{2}\left|\phi'\left(\sqrt{\sigma\varsigma}\right)\right|. \tag{24}$$

4. Application

In the literature, the following means for real numbers $\sigma, \varsigma \in \mathbb{R}$ are well known:

(1) The arithmetic mean:

$$A = A(\sigma, \varsigma) = \frac{\sigma + \varsigma}{2}; \quad \sigma, \varsigma \in \mathbb{R}$$

(2) The geometric mean:

$$G = G(\sigma, \varsigma) = \sqrt{\sigma\varsigma}; \quad \sigma, \varsigma \in [0, \infty)$$

(3) The harmonic mean:

$$H = H(\sigma, \varsigma) = \frac{2\sigma\varsigma}{\sigma + \varsigma}; \quad \sigma, \varsigma \in \mathbb{R}$$

(4) The generalized logaritmic mean:

$$L_n(\sigma,\varsigma)=\left[\frac{\varsigma^{n+1}-\sigma^{n+1}}{(\varsigma-\sigma)(n+1)}\right]^{\frac{1}{n}};\quad n\in\mathbb{Z}\backslash\{-1,0\},\ \sigma,\varsigma\in\mathbb{R},\ \sigma\neq\varsigma$$

Proposition 4.1. *Let $\sigma,\varsigma\in(0,\infty),\sigma<\varsigma$ and $r>2$. Then*

$$\left|A(\sigma^r,\varsigma^r)(\ln\varsigma-\ln\sigma)-(\varsigma-\sigma)L_{r-1}^{r-1}(\sigma,\varsigma)\right|$$

$$\leq(\ln\varsigma-\ln\sigma)\int_{\sigma}^{\sqrt{\sigma\varsigma}}\frac{1}{\lambda}\left[C_1(\lambda)\left|\phi'(\sigma)\right|+C_2(\lambda)\left|\phi'(\varsigma)\right|-c\left\|\ln\varsigma-\ln\sigma\right\|^2C_3(\lambda)\right]d\lambda. \quad (25)$$

Proof. In Theorem 3.2, $\phi(\lambda)=\lambda^r$ and $\varphi(\lambda)=1,\ \ \lambda\in(0,\infty),\ \ r>2$.

$$\left|\frac{\sigma^r+\varsigma^r}{2}\int_{\sigma}^{\varsigma}\frac{1}{\lambda}d\lambda-\int_{\sigma}^{\varsigma}\frac{\lambda^r}{\lambda}d\lambda\right|=\left|A(\sigma^r,\varsigma^r)(\ln\varsigma-\ln\sigma)-\frac{\varsigma^r-\sigma^r}{r}\right|$$

$$=\left|A(\sigma^r,\varsigma^r)(\ln\varsigma-\ln\sigma)-(\varsigma-\sigma)L_{r-1}^{r-1}(\sigma,\varsigma)\right|$$

$$\left|A(\sigma^r,\varsigma^r)(\ln\varsigma-\ln\sigma)-(\varsigma-\sigma)L_{r-1}^{r-1}(\sigma,\varsigma)\right|$$

$$\leq(\ln\varsigma-\ln\sigma)\int_{\sigma}^{\sqrt{\sigma\varsigma}}\frac{1}{\lambda}\left[C_1(\lambda)\left|\phi'(\sigma)\right|+C_2(\lambda)\left|\phi'(\varsigma)\right|-c\left\|\ln\varsigma-\ln\sigma\right\|^2C_3(\lambda)\right]d\lambda. \quad (26)$$

□

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Ayşe Kübra Demirel graduated from the Faculty of Science of the Van Yüzüncü Yıl University in 2012. She received her Ph.D. degree on Mathematics from the Institute of Science of Ordu University in 2019. Presently, she is a lecturer in the Central Research Laboratory of Ordu University. Her research interests include analysis and inequalities.