TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 579-596

EXISTENCE AND UNIQUENESS RESULTS FOR FUZZY BOUNDARY VALUE PROBLEMS OF NONLINEAR DIFFERENTIAL EQUATIONS INVOLVING ATANGANA-BALEANU FRACTIONAL DERIVATIVES

F. ZAMTAIN[1], M. ELOMARI[1*], S. MELLIANI[1], A. EL MFADEL[1], §

Abstract. This manuscript is devoted to the investigation of the existence and uniqueness results for fuzzy fractional boundary value problems of some nonlinear differential equations involving fuzzy Atangana-Baleanu fractional derivatives of order $\alpha \in (1, 2)$. By applying Banach fixed point theorem, some new results and properties of Atangana-Baleanu fractional derivatives and generalized Hukuhara difference, we establish our main existence theorems. As application, a nontrivial example is given to illustrate our theoretical results.

Keywords: fuzzy number, fuzzy fractional integral, fuzzy Atangana-Baleanu derivative, fuzzy Fractional differential equations, Banach fixed point theorem.

AMS Subject Classification: 34A08, 26A33.

1. Introduction

Physical models of real-world phenomena frequently contain some uncertainty, which can be attributed to a variety of factors. Fuzzy sets also appear to be an excellent tool for modelling the uncertainty highlighted by impreciseness and ambiguity. Indeed, the fuzzy set theory has become widely known due to its presence in various areas of life related to mathematics, as it can be said that all sciences are related to mathematics, and therefore the fuzzy theory is related to all sciences. Moreover, it has a number of qualities that make it ideal for formalizing the ambiguous information that underpins life's occurrences. In [2], Klaus-Peter Adlassnig worked on medical Diagnosis probelm by using this theory . There are many works which are an extension of several previously studied concepts in the ordinary theory, like Bede et al. [12], who established the concept of strongly generalized differentiability . The generalized Hukuhara differentiability was presented by Stefanini in [12] for solving fuzzy fractional diffential equations. The fuzzy differential equations were first proposed by Kandel and Byatt [21] in 1987. D. Sindu and K. Ganesan in [15] showed the existence of a solution to the fuzzy equation in the field of physics regarding the second-order linear fuzzy differential equation for modeling electric circuit problems. Many researchers have obtained some interesting results on the existence and uniqueness of solutions of boundary

[1] Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco.
e-mail: fouziyazzamtainz@gmail.com; ORCID: https://orcid.org/0000-0002-7446-0056.
e-mail: m.elomari@usms.ma; ORCID: https://orcid.org/0000-0002-4256-1434.
* Corresponding author.
e-mail: s.melliani@usms.ma; ORCID: https://orcid.org/0000-0002-5150-1185.
e-mail: elmfadelali@gmail.com; ORCID: https://orcid.org/0000-0002-2479-1762.
§ Manuscript received: April 19, 2022; accepted: May 28, 2022.

value problems for fractional differential equations involving different fractional derivatives such as Riemann-Liouville [23], Hilfer [20], Erdelyi-Kober [24] and Hadamard [4]. There are many works that proved the use of fractional calculus in many sciences as Kashkynbayev and A. Rihan [22] showed the global stability of the steady-states of a fractional-order epidemic model by using Caputo's fractional derivative. Inés Tejado et all [26] in economics, and finance with Enrico Scalas et al. [25]. Agarwal et al. in [3] were the first who introduced the fuzzy Riemann-Liouville fractional differential equations . In [6], the authors introduced the fuzzy Caputo fractional differential equation under the Generalized Hukuhara differentiability. In [7], Allahviranlo et al. proposed the fuzzy fractional differential equation with an interval Atangana-Baleanu derivative. In [18] El Mfadel et al. established some new existence and uniqueness results for fuzzy linear and semilinear fractional evolution equations involving Caputo fractional derivative. The existence theorems are proved by using fuzzy fractional calculus, Picard's iteration method, and Banach contraction principle. The reader can also consul articles [17, 18] and the references therein for more details.
Developing an appropriate fractional differential equation in the context of mathematical modeling is not an easy process. It necessitates a thorough understanding of the underlying physical phenomena. However, actual physical phenomena are always tinged with uncertainty. When working with "living" things such as soil, water, and microbial populations, this is clear. When a real physical phenomena is modelled by a classical fractional differential equation as follows:

$$\begin{cases} D^{\alpha}x(t) = k(t, x(t)), \ t \in [0, T], \\ \\ x(0) = x_0. \end{cases} \tag{1}$$

We can't always be certain that the model is accurate. The initial value in (1), for example, may not be known precisely. It can be anything between "less than x_0," "around x_0," and "greater than x_0." Classical mathematics, on the other hand, is incapable of dealing with this circumstance. As a result, other theories are required to address this problem. There are several theories for describing this circumstance, the most popular of which is the fuzzy set theory. See[18, 19].

Motivated by the above works, the purpose of this paper is to to establish some new definitions and prove some properties of fuzzy Atangana-Baleanu fractional derivative for a higher order $\alpha \in (n, n+1)$ where n is an arbitrary integer number. In addition, by utilizing some new proprieties of fuzzy Atangna-Baleanu fractional derivatives and the Green function, we establish the existence and uniqueness of solutions for the following boundary value problem:

$$\begin{cases} {}^{\mathrm{ABC}}D^{\alpha}y(t) = g(t, y(t)), \ t \in [a, b], \\ \\ y(a) = A, \quad y(b) = B \qquad\quad A, B \in \mathbb{E}^1. \end{cases} \tag{2}$$

Where ${}^{\mathrm{ABC}}D^{\alpha}$ is the fuzzy Atangna-Baleanu fractional derivative of y at order $\alpha \in (1, 2)$ and $g : [a, b] \subset \mathbb{R}^+ \times \mathbb{E}^1 \longrightarrow \mathbb{E}^1$ is fuzzy continuous function.
Our paper is organized as follows. In Section 2, we give some basic definitions and properties of fuzzy Atangna-Baleanu fractional fractional integral and fuzzy Atangna-Baleanu fractional derivative of order $\alpha \in (0, 1)$. In Section 3, we define some new basic concept of Atangana Baleanu fractional derivative in the sense of Caputo (ABC for short form) and in the sense of Riemann-Liouville (ABR for short form) for order $\alpha \in (n, n+1)$ where n is an arbitrary integer number. In Section 4, we prove the existence and uniqueness of solutions for fractional boundary value problem (2) by using Banach fixed point theorem . As application, an illustrative example is presented in Section 5 followed by conclusion in Section 6.

2. Preliminaries

In this section, we give some definitions and proprieties of fuzzy theory and of the Atangana-Baleanu of order $\alpha \in (0, 1)$.

Definition 2.1. [14] *A fuzzy number is a function $u : \mathbb{R} \to [0, 1]$ satisfying the following properties:*

(1) *u is normal, i. e. $\exists t_0 \in \mathbb{R}$ with $u(t_0) = 1$,*

(2) *u is a convex fuzzy set i. e. $u((1-\lambda)x+\lambda y) \geq \min\{u(x), u(y)\}$, $\forall x, y \in \mathbb{R}, \lambda \in [0,1]$,*
(3) *u is upper semi-continuous on $\mathbb{R}$,*
(4) *$cl\{t \in \mathbb{R} : u(t) > 0\}$ is compact, where cl denotes the closure of a subset.*

We denote by $\mathbb{E}^1$ the space of all fuzzy numbers.

Let $r \in [0,1]$ and $u \in \mathbb{E}^1$, we define the $r-cut$ of u by

$$[u]^r = \{x \in \mathbb{R} : u(x) \geq r\}.$$

Definition 2.2. [14] *The metric structure is given by the Hausdorff distance*

$$D : \mathbb{E}^1 \times \mathbb{E}^1 \longrightarrow \mathbb{R}^+ \cup \{0\}$$

by

$$D(u,v) = \sup_{0 \leq r \leq 1} \max\left\{\left|\underline{u}_r - \underline{v}_r\right|, \left|\bar{u}_r - \bar{v}_r\right|\right\}.$$

The space $\left(\mathbb{E}^1, D\right)$ is a complete metric space and the following properties of the metric D hold.

$$\begin{array}{ll} D(u+w, v+w) = D(u,v), & \forall u, v, w \in \mathbb{E}^1, \\ D(ku, kv) = |k| D(u,v), & \forall k \in \mathbb{R}, u, v \in \mathbb{E}^1, \\ D(u+v, w+z) \leq D(u,w) + D(v,z) & \forall u, v, w, z \in \mathbb{E}^1. \end{array}$$

Definition 2.3. [12] *The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathbb{E}^1$ is defined as follows*

$$u \ominus_{gH} v = w \Longleftrightarrow \begin{cases} (i) \quad u = v + w, \\ or(ii) \;\; v = u + (-1)w. \end{cases}$$

Definition 2.4. [12] *The generalized Hukuhara derivative of a fuzzy-valued function $f : (a,b) \longrightarrow \mathbb{E}^1$ at t_0 is defined as follows:*

$$f'_{gH}\left(t_0\right) = \lim_{h \to 0} \frac{f\left(t_0+h\right) \ominus_{gH} f\left(t_0\right)}{h},$$

if $f'_{gH}\left(t_0\right) \in \mathbb{E}^1$, we say that f is generalized Hukuhara differentiable (g H-differentiable) at t_0. Also we say that f is (i)-gH-differentiable at t_0 if

$$\left[f'_{i.gH}\left(t_0\right)\right]^r = \left[\underline{f}'\left(t_0, r\right), \bar{f}'\left(t_0, r\right)\right], \quad 0 \leq r \leq 1,$$

and that f is (ii)- gH-differentiable at t_0 if

$$\left[f'_{i.gH}\left(t_0\right)\right]^r = \left[\bar{f}\left(t_0; r\right), \underline{f}'\left(t_0; r\right)\right], \quad 0 \leq r \leq 1.$$

Definition 2.5. [7] *The* ABC *fractional derivative in the sense of Caputo is defined in two cases as follow,*

$$\begin{aligned} \left[{}_0^{ABC}D_t^{i,\alpha} y(t)\right]^r &= \left[{}_0^{ABC}D_t^{i,\alpha} \underline{y}(t;r), {}_0^{ABC}D_t^{i,\alpha} \overline{y}(t;r)\right], \quad \text{Case } (1), \\ \left[{}_0^{ABC}D_t^{ii,\alpha} y(t)\right]^r &= \left[{}_0^{ABC}D_t^{ii,\alpha} \overline{y}(t;r), {}_0^{ABC}D_t^{ii,\alpha} \underline{y}(t;r)\right], \quad \text{Case } (2), \end{aligned}$$

where

$$
\begin{aligned}
{}_{0}^{ABC}D_t^{i,\alpha}y(t) =& \frac{B(\alpha)}{1-\alpha}\int_0^t i-gHy'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)d\tau,\\
=& \frac{B(\alpha)}{1-\alpha}\int_0^t i-gHy'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\Big|_{E_\alpha\geq 0}d\tau\\
&+\frac{B(\alpha)}{1-\alpha}\int_0^t ii-gHy'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\Big|_{E_\alpha< 0}d\tau,\\
{}_{0}^{ABC}D_t^{ii,\alpha}y(t) =& \frac{B(\alpha)}{1-\alpha}\int_0^t if-gHy'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)d\tau,\\
=& \frac{B(\alpha)}{1-\alpha}\int_0^t ii-gHy'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\Big|_{E_\alpha\geq 0}d\tau\\
&+\frac{B(\alpha)}{1-\alpha}\int_0^t i-gHy'(\tau)E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\Big|_{E_\alpha< 0}d\tau.
\end{aligned}
$$

Lemma 2.1. [7] *For $0<\alpha<1$, we have*

$$\left({}_{a}^{AB}I^{\alpha}{}_{a}^{ABC}D^\alpha f\right)(x) = \left(f(x)\ominus_{gH} f(a)\right)_{E_\alpha\geq 0}\oplus\left(f(x)\ominus_{gH} f(a)\right)_{E_\alpha<0},$$

and

$$\left({}_{b}^{AB}I^{\alpha}{}_{b}^{ABC}D^\alpha f\right)(x) = \left(f(x)\ominus_{gH} f(b)\right)_{E_\alpha\geq 0}\oplus\left(f(x)\ominus_{gH} f(b)\right)_{E_\alpha<0}.$$

Proposition 2.1. *Let $f:[a,b]\to\mathbb{E}$ and $\alpha\in(0,1)$, then*

- *If f is i-differentiable at t_0 then ${}^{ABC}D^\alpha f$ is $[(i)-gH]$-differentiable at t_0.*
- *If f is ii-differentiable at t_0 then ${}^{ABC}D^\alpha f$ is $[(ii)-gH]$-differentiable at t_0.*

Proof.

- Suppose that f is i-differentiable at s then we have

$$[f'_{gH}(s)]^r = [\underline{f'}(s),\overline{f'}(s)].$$

Moreover, we use the proprieties of r-cut, we find

$$
\begin{aligned}
E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)\Big[f'_{gH}(s)\Big]^r =& \Big(E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0}\\
&+E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0}\Big)\Big[f'_{gH}(s)\Big]^r\\
=&\Big[\underline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0},\overline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0}\Big]\\
&+\Big[\overline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0},\underline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0}\Big]\\
=&\Big[\underline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0}+\overline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0},\\
&\overline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0}+\underline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0}\Big].
\end{aligned}
$$

Then we apply the integral, and since $\frac{B(\alpha)}{1-\alpha}$ is a positive number, then we obtain

$$\begin{aligned}\frac{B(\alpha)}{1-\alpha}\int_0^t E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)\Big[f'_{gH}(s)\Big]^r ds = \Big[&\frac{B(\alpha)}{1-\alpha}\int_0^t \underline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0}\\ &+\frac{B(\alpha)}{1-\alpha}\int_0^t \overline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0}ds,\\ &\frac{B(\alpha)}{1-\alpha}\int_0^t \overline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\geq 0}\\ &+\frac{B(\alpha)}{1-\alpha}\int_0^t \underline{f'}(s)E_\alpha\Big(-\frac{\alpha}{1-\alpha}(t-s)^\alpha\Big)|_{E_\alpha\leq 0}ds\Big].\end{aligned}$$

thus we find

$$\begin{aligned}\Big[\frac{B(\alpha)}{1-\alpha}\int_0^t f'_{gH}(s)E_\alpha(-\frac{\alpha}{1-\alpha}(t-s)^\alpha)ds\Big]^r = \Big[&\frac{B(\alpha)}{1-\alpha}\int_0^t \underline{f'}(s)E_\alpha\Big(\frac{-\alpha}{1-\alpha}(t-s)^\alpha\Big)ds,\\ &\frac{B(\alpha)}{1-\alpha}\int_0^t \overline{f'}(s)E_\alpha\Big(\frac{-\alpha}{1-\alpha}(t-s)^\alpha\Big)ds\Big].\end{aligned}$$

That means ${}^{ABC}D^\alpha f$ is i-differentiable.

- Suppose that f is *ii*-differentiable, then $[f'_{gH}(s)]^r = [\underline{f'}(s), \overline{f'}(s)]$ and by same steps of the first case we get ${}^{ABC}D^\alpha f$ is *ii*-differentiable.

□

Definition 2.6. [10] *The second generalized Hukuhara derivative of a fuzzy-valued function $f:(a,b)\longrightarrow \mathbb{E}^1$ at t_0 is defined as*

$$f''_{gH}(t_0) = \lim_{h\to 0}\frac{f'(t_0+h)\ominus_{gH} f'(t_0)}{h},$$

if $f''_{gH}(t_0)\in\mathbb{E}^1$, we say that f'_{gH} is generalized Hukuhara differentiable at t_0. Also we say that $f'_{gH}(t)$ is (i)-gH-differentiable at t_0 if

$$\Big[f''_{i.gH}(t_0)\Big]^r = \begin{cases}\big[\underline{f}''(t_0;r), \bar{f}''(t_0;r)\big], & \textit{if } f \textit{ be } (\ i\ )-gH- \textit{ differentiable on } (a,b),\\ \big[\bar{f}''(t_0;r), \underline{f}''(t_0;r)\big], & \textit{if } f \textit{ be } (\ ii\ )-gH-\text{differentiable on } (a,b),\end{cases}$$

for all $r\in[0,1]$, and that $f'_{gh}(t)$ is (ii)- gH-differentiable at t_0 if

$$\Big[f''_{ii.gH}(t_0)\Big]^r = \begin{cases}\big[\bar{f}''(t_0;r), f''(t_0;r)\big], & \textit{if } f \textit{ be } (\ i\ )-gH- \textit{ differentiable on } (a,b),\\ \big[\underline{f}''(t_0;r), \bar{f}''(t_0;r)\big], & \textit{if } f \textit{ be } (\ ii\ )-gH- \textit{ differentiable on } (a,b),\end{cases}$$

for all $r\in[0,1]$.

3. Higher-order fuzzy fractional Atangana-Baleanu derivatives

In this part, we define the fuzzy fractional derivative Atangana-Baleanu of the higher order $\alpha\in(n,n+1]$, and we introduce the proprieties of this derivative for the order $\alpha\in(1,2]$.

Definition 3.1. *Let $n<\alpha\leq n+1$ and f be a fuzzy function such that $f^{(n)}\in H^1(a,b)$. Set $\beta=\alpha-n$. Then $\beta\in(0,1]$ and we define the fractional derivative of Atangana-Baleanu in the sense of Caputo by*

$$\begin{aligned}\left({}^{ABC}_{a}D^\alpha f\right)(t) &= \left({}^{ABC}_{a}\ D^\beta f^{(n)}\right)(t)\\ &= \frac{B(\alpha-n)}{1+n-\alpha}\int_a^b f^{(n+1)}(s)E_\beta\left(-(\alpha-n)\frac{(t-s)^{\alpha-n}}{1+n-\alpha}\right)ds.\end{aligned}$$

And in the sense of Riemann-Liouville by

$$\begin{aligned}\left({}_{a}^{ABR}D^{\alpha}f\right)(t) &= \left({}_{a}^{ABR}D^{\beta}f^{(n)}\right)(t)\\ &= \frac{B(\alpha-n)}{1+n-\alpha}\frac{d}{dt}\int_{a}^{t} f^{(n)}(s)E_{\beta}\left(-(\alpha-n)\frac{(t-s)^{\alpha-n}}{1+n-\alpha}\right)ds.\end{aligned}$$

Definition 3.2. *Let $n<\alpha\leq n+1$ and f be a fuzzy function such that $[f(t)]^{r}=[\underline{f}(t,r),\overline{f}(t,r)]$. Set $\beta=\alpha-n$. Then $\beta\in(0,1]$ and we define*

$$\begin{aligned}\left({}_{a}^{AB}\mathbf{I}^{\alpha}f\right)(t) &= \left({}_{a}I^{n}\,{}_{a}^{AB}I^{\beta}f\right)(t)\\ &= \int_{a}^{t}\int_{a}^{t}....\int_{a}^{t}\frac{1+n-\alpha}{B(\alpha-1)}f(\tau)ds\\ &+\frac{\alpha-1}{B(\alpha-n)\Gamma(\alpha-n)}\int_{a}^{t}\int_{a}^{t}....\int_{a}^{t}\int_{a}^{\tau}(\tau-s)^{\alpha-n-1}f(s)dsds.\end{aligned}$$

Remark 3.1. *In this work, we will be concerned with the situation where $\alpha\in(1,2]$ i.e. we have for the previous definitions*

$$\left({}_{a}^{ABC}\mathbf{D}^{\alpha}f\right)(t)=\frac{B(\alpha-1)}{2-\alpha}\int_{a}^{t}f''(s)E_{\alpha-1}\left(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\right)ds.$$

$$\left({}_{a}^{AB}\mathbf{I}^{\alpha}f\right)(t)=\int_{a}^{t}\frac{2-\alpha}{B(\alpha-1)}f(\tau)ds+\frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha-1)}\int_{a}^{t}\left(\int_{a}^{\tau}(\tau-s)^{\alpha-2}f(s)ds\right)dt.$$

Definition 3.3. *Let $f:(a,b)\to\mathbb{E}^{1}$. We say that f is (m,l)-differentiable at $t_0\in(a,b)$ if f is (m)-differentiable on a neighborhood of t_0 as a fuzzy function, and $f^{(n)}$ is (l)-differentiable at t_0. The n-th derivatives of f at t are denoted by $f^{(n)}(t)=D^{n}_{m,l}f(t),\quad m,l\in\{i,ii\}$.*

Lemma 3.1. *Let $f:[a,b]\to\mathbb{E}^{1}$ and $\alpha\in(1,2]$, then*

- *If f is (i,i)-differentiable at t_0 or f is (ii,ii)-differentiable at t_0 then ${}^{ABC}D^{\alpha}f$ is i-differentiable at t_0.*
- *If f is (i,ii)-differentiable at t_0 or f is (ii,i)-differentiable at t_0 then ${}^{ABC}D^{\alpha}f$ is ii-differentiable at t_0.*

Proof. Let us prove only the first case and since the second case is the same as the first, let us suppose that f is (i,i)-differentiable so by using the definition 2.6, we have

$$\left[f''(s)\right]^{r}=\left[\underline{f}''(s),\overline{f}''(s)\right],$$

or

$${}^{ABC}D^{\alpha}f(t)=\frac{B(\alpha-1)}{2-\alpha}\int_{a}^{t}f''(s)E_{\alpha-1}\left(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\right)ds,$$

$$\begin{aligned}
E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big[f''(s)\Big]^r &= \Big(E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\geq 0} \\
&+ E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\leq 0}\Big)\Big[\underline{f}''(s),\overline{f}''(s)\Big] \\
&= \Big[\underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\geq 0}, \\
&\overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha}\Big)\Big|_{E_{\alpha-1}\geq 0}\Big] \\
&+ \Big[\overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha}\Big)\Big|_{E_{\alpha-1}\leq 0}, \\
&\underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\leq 0}\Big] \\
&= \Big[\underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\geq 0} \\
&+ \overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\leq 0}, \\
&\overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\geq 0} \\
&+ \underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\leq 0}\Big].
\end{aligned}$$

Then we apply the fuzzy integral and its proprieties with r-cut, we also multiply by the positive number $\frac{B(\alpha-1)}{2-\alpha}$, we get

$$\begin{aligned}
&\frac{B(\alpha-1)}{2-\alpha}\int_0^t E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big[f''_{gH}(s)\Big]^r ds \\
&= \Big[\frac{B(\alpha-1)}{2-\alpha}\int_0^t \underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\geq 0} ds \\
&+ \frac{B(\alpha-1)}{2-\alpha}\int_0^t \overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha--1}\Big)\Big|_{E_{\alpha-1}\leq 0} ds, \\
&\frac{B(\alpha-1)}{2-\alpha}\int_0^t \overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\geq 0} ds \\
&+ \frac{B(\alpha-1)}{2-\alpha}\underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)\Big|_{E_{\alpha-1}\leq 0} ds\Big].
\end{aligned}$$

Thus we obtain

$$\begin{aligned}\left[{}^{ABC}D^{\alpha}f(t)\right]^r &= \left[\frac{B(\alpha-1)}{2-\alpha}\int_0^t f''_{gH}(s)E_{\alpha-1}(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1})ds\right]^r\\ &= \Bigg[\frac{B(\alpha-1)}{2-\alpha}\int_0^t \underline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)ds,\\ &\quad \frac{B(\alpha-1)}{2-\alpha}\int_0^t \overline{f''}(s)E_{\alpha-1}\Big(-\frac{\alpha-1}{2-\alpha}(t-s)^{\alpha-1}\Big)ds\Bigg]\\ &= \left[{}^{ABC}D^{\alpha}\underline{f}(t), {}^{ABC}D^{\alpha}\overline{f}(t)\right].\end{aligned}$$

Consequently ${}^{ABC}D^{\alpha}f$ is i-differentiable. The same for other case. □

Proposition 3.1. [1] *For $u(t)$ defined on $[a,b]$ and $\alpha \in (n, n+1]$, for some $n \in \mathbb{N}^*$, we have:*

- $\left({}^{ABR}_{a}\mathbf{D}^{\alpha}\,{}^{AB}_{a}\mathbf{I}^{\alpha}u\right)(t) = u(t)$.
- $\left({}^{AB}_{a}\mathbf{I}^{\alpha}_{a}\,{}^{ABR}\mathbf{D}^{\alpha}u\right)(t) = u(t) - \sum_{k=0}^{n-1}\frac{u^{(k)}(a)}{k!}(t-a)^k$.
- $\left({}^{AB}_{a}\mathbf{I}^{\alpha}_{a}\,{}^{ABC}\mathbf{D}^{\alpha}u\right)(t) = u(t) - \sum_{k=0}^{n}\frac{u^{(k)}(a)}{k!}(t-a)^k$.

We denote by $AC^F[a,b]$ the space of all absolutely continuous fuzzy functions.

Theorem 3.1. *Let $\alpha \in (1,2]$ and $f, f' \in AC^F[a,b]$.*

(1) *If f is (i,i)-differentiable, then*

$$\Big({}^{AB}I^{\alpha\,ABC}D^{\alpha}f\Big)(x) = \Big(f(x)\ominus f(a)\ominus f'(a)(x-a)\Big)\Big|_{E_{\alpha-1}\geq 0} \ominus(-1)\Big(f(x)\ominus f(a)\ominus f'(a)(x-a)\Big)\Big|_{E_{\alpha-1}<0}.$$

(2) *If f is (i,ii)-differentiable, then*

$$\begin{aligned}\Big({}^{AB}I^{\alpha\,ABC}D^{\alpha}f\Big)(x) &= \Big(-f(a)+(-f'(a))(x-a)\ominus(-f(x))\Big)\Big|_{E_{\alpha-1}\geq 0}\\ &\quad\ominus(-1)\Big(-f(a)+(-f'(a))(x-a)\ominus(-f(x))\Big)\Big|_{E_{\alpha-1}<0}.\end{aligned}$$

(3) *If f is (ii,i)-differentiable, then*

$$\begin{aligned}\Big({}^{AB}I^{\alpha\,ABC}D^{\alpha}f\Big)(x) &= \Big(-f(a)\ominus f'(a)(x-a)\ominus(-f(x))\Big)\Big|_{E_{\alpha-1}\geq 0}\\ &\quad\ominus(-1)\Big(-f(a)\ominus f'(a)(x-a)\ominus(-f(x))\Big)\Big|_{E_{\alpha-1}<0}.\end{aligned}$$

(4) *If f is (ii,ii)-differentiable, then*

$$\begin{aligned}\Big({}^{AB}I^{\alpha\,ABC}D^{\alpha}f\Big)(x) &= \Big(f(x)\ominus f(a)+(-f'(a))(x-a)\Big)\Big|_{E_{\alpha-1}\geq 0}\\ &\quad\ominus(-1)\Big(f(x)\ominus f(a)+(-f'(a))(x-a)\Big)\Big|_{E_{\alpha-1}<0}.\end{aligned}$$

Proof. Let be f a fuzzy function, we have $[f(t)]^r = \left[\underline{f}(t,r), \overline{f}(t,r)\right]$ for $r \in [0,1]$. Then, we have for the real valued functions $\underline{f}$ and $\overline{f}$,

$$\left(I^{\alpha\,ABC}D^{\alpha}\underline{f}\right)(x) = \Big(\underline{f}(x)-\underline{f}(a)-\underline{f}'(a)(x-a)\Big)\Big|_{E_{\alpha-1}\geq 0} + \Big(\overline{f}(x)-\overline{f}(a)-\overline{f}'(a)(x-a)\Big)\Big|_{E_{\alpha-1}<0},$$

and

$$\left(I^{\alpha\,ABC}D^{\alpha}\overline{f}\right)(x) = \Big(\overline{f}(x)-\overline{f}(a)-\overline{f}'(a)(x-a)\Big)\Big|_{E_{\alpha-1}\geq 0} + \Big(\underline{f}(x)-\underline{f}(a)-\underline{f}'(a)(x-a)\Big)\Big|_{E_{\alpha-1}<0}.$$

Suppose that f is (i,i)-differentiable then by definition 3.2 and lemma 3.1 we have

$$\begin{aligned}
\left[{}^{AB}I^{\alpha\, ABC}D^{\alpha} f(x)\right]^r &= \left[I^{1\, AB}I^{\beta\, ABC}D^{\beta} f'(x)\right]^r \\
&= \Big[I^{1\, AB}I^{\beta}\left({}^{ABC}D^{\beta}\underline{f}'(x)\right)_{E_{\alpha-1}\geq 0} \\
&\quad + I^{1\, AB}I^{\beta}\left({}^{ABC}D^{\beta}\overline{f}'(x)\right)_{E_{\alpha-1}<0}, \\
&\quad I^{1\, AB}I^{\beta}\left({}^{ABC}D^{\beta}\overline{f}'(x)\right)_{E_{\alpha-1}\geq 0} \\
&\quad + I^{1\, AB}I^{\beta}\left({}^{ABC}D^{\beta}\underline{f}'(x)\right)_{E_{\alpha-1}<0}\Big] \\
&= \Big[I^{1}\left(\underline{f}'(x)\right)_{E_{\alpha-1}\geq 0} + I^{1}\left(\overline{f}'(x)\right)_{E_{\alpha-1}<0}, \\
&\quad I^{1}\left(\overline{f}'(x)\right)_{E_{\alpha-1}\geq 0} + I^{1}\left(\underline{f}'(x)\right)_{E_{\alpha-1}<0}\Big] \\
&= \Big[\left(\underline{f}(x) - \underline{f}(a) - \underline{f}'(a)(x-a)\right)\Big|_{E_{\alpha-1}\geq 0} \\
&\quad + \left(\overline{f}(x) - \overline{f}(a) - \overline{f}'(a)(x-a)\right)\Big|_{E_{\alpha-1}<0}, \\
&\quad \left(\overline{f}(x) - \overline{f}(a) - \overline{f}'(a)(x-a)\right)\Big|_{E_{\alpha-1}\geq 0} \\
&\quad + \left(\underline{f}(x) - \underline{f}(a) - \underline{f}'(a)(x-a)\right)\Big|_{E_{\alpha-1}<0}\Big] \\
&= \Big[\left(f(x) \ominus f(a) \ominus f'(a)(x-a)\right)\Big|_{E_{\alpha-1}\geq 0} \\
&\quad \oplus \left(\overline{f}(x) \ominus f(a) \ominus f'(a)(x-a)\right)\Big|_{E_{\alpha-1}<0}\Big].
\end{aligned}$$

Thus we have

$$I^{\alpha\, ABC}D^{\alpha} f(x) = \left(f(x) \ominus f(a) \ominus f'(a)(x-a)\right)\Big|_{E_{\alpha-1}\geq 0} \oplus \left(\overline{f}(x) \ominus f(a) \ominus f'(a)(x-a)\right)\Big|_{E_{\alpha-1}<0}.$$

The same for other cases. □

4. Existence and uniqueness results

In this section, we are concerned with studying the existence and uniqueness of the solutions for the fuzzy fractional boundary value problem (7) under the AB derivative in the sense of Caputo.

For this purpose, we denote by $\mathcal{C}_{[a,b]} = \mathcal{C}([a,b], \mathbb{E}^1)$ the space of continuous fuzzy functions equipped with the metric $d(x,y) = \sup_{t\in[a,b]} D(x(t), y(t))$.

Remark 4.1. *The metric space $\left(\mathcal{C}_{[a,b]}, d\right)$ is complet .*

Remark 4.2. *By Theorem 3.1 and Lemma 3.1 we deduce that the cases (i,i) and (ii,ii) are similarly shaped. The cases (i,ii) and (ii,i) are also similar. Then we reduce the study of problem* (2) *in two principal cases:*
If f is (i,i)-differentiable or (ii, ii)-differentiable, then ***case 1*** *applies.*
If f is (i, ii)-differentiable or (ii, i)-differentiable, then ***case 2*** *applies.*

Theorem 4.1. *The fractional boundary value problem* (2) *with continuous function* $g(t, y(t))$ *has a solution in the following form*

$$\Big(y(t) \ominus_{gH} \varphi(t)\Big)_{E_{\alpha-1}\geq 0} \oplus \Big(y(t) \ominus_{gH} \varphi(t)\Big)_{E_{\alpha-1}<0} = \int_a^b G(t,s)g(s,y(s))ds, \tag{3}$$

where

$$\varphi(t) = \frac{b-t}{b-a}A + \frac{t-a}{b-a}B,$$

$$G(t,s) = \begin{cases} \frac{(\alpha-2)(t-a)}{(b-a)M(\alpha-1)} + \frac{(\alpha-1)(t-s)^{\alpha-1}}{M(\alpha-1)\Gamma(\alpha)} + \frac{2-\alpha}{M(\alpha-1)} - \frac{(\alpha-1)(t-a)(b-s)^{\alpha-1}}{(b-a)M(\alpha-1)\Gamma(\alpha)}, & a \leq s < t \leq b, \\ \\ \frac{(\alpha-2)(t-a)}{(b-a)M(\alpha-1)} - \frac{(\alpha-1)(t-a)(b-s)^{\alpha-1}}{(b-a)M(\alpha-1)\Gamma(\alpha)}, & a \leq t < s \leq b. \end{cases} \tag{4}$$

Proof. By applying the fractional integral on the both sides of equation (2), we get

$$^{AB}I^{\alpha\, ABC}D^{\alpha}y(t) = {}^{AB}I^{\alpha}g(t,y(t))$$

We are going to interested by **case 1** where

$$[y(a)]^r = [\underline{y}(a), \overline{y}(a)] = [\underline{A}, \overline{A}],$$

and

$$[y(b)]^r = [\underline{y}(b), \overline{y}(b)] = [\underline{B}, \overline{B}].$$

Then we deduce the result for **case 2** .

If y is (i,i)-differentiable or (ii,ii)-differentiable (**case1**), we have by using Theorem 3.1 and Lemma 3.1

$$\Big(\underline{y}(t) - \underline{y}(a) - \underline{y}'(a)(t-a)\Big)\Big|_{E_{\alpha-1}\geq 0} + \Big(\overline{y}(t) - \overline{y}(a) - \overline{y}'(a)(t-a)\Big)\Big|_{E_{\alpha-1}<0} = {}^{AB}I^{\alpha}\underline{g}(t,y(t)),$$

$$\Big(\overline{y}(t) - \overline{y}(a) - \overline{y}'(a)(t-a)\Big)\Big|_{E_{\alpha-1}\geq 0} + \Big(\underline{y}(t) - \underline{y}(a) - \underline{y}'(a)(t-a)\Big)\Big|_{E_{\alpha-1}<0} = {}^{AB}I^{\alpha}\overline{g}(t,y(t)),$$

Then we have

$$\begin{aligned} \Big(\underline{y}(t)|_{E_{\alpha-1}\geq 0} + \overline{y}(t)|_{E_{\alpha-1}<0}\Big) &= \Big(\underline{y}(a)|_{E_{\alpha-1}\geq 0} + \overline{y}(a)|_{E_{\alpha-1}<0}\Big) \\ +\Big(\underline{y}'(a)_{E_{\alpha-1}\geq 0}(t-a) &+ \overline{y}'(a)_{E_{\alpha-1}<0}(t-a)\Big) + {}^{AB}I^{\alpha}\underline{g}(t,y(t)), \\ \Big(\overline{y}(t)|_{E_{\alpha-1}\geq 0} + \underline{y}(t)|_{E_{\alpha-1}<0}\Big) &= \Big(\overline{y}(a)|_{E_{\alpha-1}\geq 0} + \underline{y}(a)|_{E_{\alpha-1}<0}\Big) \\ +\Big(\overline{y}'(a)_{E_{\alpha-1}\geq 0}(t-a) &+ \underline{y}'(a)_{E_{\alpha-1}<0}(t-a)\Big) + {}^{AB}I^{\alpha}\overline{g}(t,y(t)), \end{aligned}$$

So for **case 1**, we have

$$\begin{aligned} \Big(\underline{y}(t)|_{E_{\alpha-1}\geq 0} + \overline{y}(t)|_{E_{\alpha-1}<0}\Big) = & \Big(\underline{A}|_{E_{\alpha-1}\geq 0} + \overline{A}|_{E_{\alpha-1}<0}\Big) \\ & + \Big(\underline{y}'(a)(t-a)|_{E_{\alpha-1}\geq 0} + \overline{y}'(a)(t-a)|_{E_{\alpha-1}<0}\Big) \\ & + \int_a^t \frac{2-\alpha}{B(\alpha-1)}\underline{g}(s,y(s))ds \\ & + \frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha-1)}\int_a^t \Big(\int_a^{\tau}(\tau-s)^{\alpha-2}\underline{g}(s,y(s))ds\Big)dt, \end{aligned}$$

$$\begin{aligned}\left(\overline{y}(t)|_{E_{\alpha-1}\geq 0}+\underline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\overline{A}|_{E_{\alpha-1}\geq 0}+\underline{A}|_{E_{\alpha-1}<0}\right)\\ &+\left(\overline{y}'(a)(t-a)|_{E_{\alpha-1}\geq 0}+\underline{y}'(a)(t-a)|_{E_{\alpha-1}<0}\right)\\ &+\int_a^t \frac{2-\alpha}{B(\alpha-1)}\underline{g}(s,y(s))ds\\ &+\frac{\alpha-1}{B(\alpha-1)\Gamma(\alpha-1)}\int_a^t\left(\int_a^\tau(\tau-s)^{\alpha-2}\underline{g}(s,y(s))ds\right)dt.\end{aligned}$$

By using the boundary condition we calculate the value of $y'(a)$, we have

$$\underline{y}(b)=\underline{B}=\underline{A}+\underline{y}'(a)(b-a)+\frac{2-\alpha}{M(\alpha-1)}\int_a^b\underline{g}(s,y(s))ds+\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\underline{g}(s,y(s))ds,$$

$$\overline{y}(b)=\overline{B}=\overline{A}+\overline{y}'(a)(b-a)+\frac{2-\alpha}{M(\alpha-1)}\int_a^b\overline{g}(s,y(s))ds+\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha-1)}\int_a^b(b-s)^{\alpha-1}\overline{g}(s,y(s))ds,$$

Then

$$\underline{y}'(a)=\frac{1}{(b-a)}\left(\underline{B}-\underline{A}-\frac{2-\alpha}{M(\alpha-1)}\int_a^b\underline{g}(s,y(s))ds-\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\underline{g}(s,y(s))ds\right),$$

$$\overline{y}'(a)=\frac{1}{(b-a)}\left(\underline{B}-\overline{A}-\frac{2-\alpha}{M(\alpha-1)}\int_a^b\overline{g}(s,y(s))ds-\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\overline{g}(s,y(s))ds\right).$$

Thus we obtain

$$\begin{aligned}\underline{y}(t)&=\underline{A}+\frac{t-a}{b-a}\underline{B}-\frac{t-a}{b-a}\underline{A}-\frac{t-a}{b-a}\frac{2-\alpha}{M(\alpha-1)}\int_a^b\underline{g}(s,y(s))ds\\ &-\frac{t-a}{b-a}\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\underline{g}(s,y(s))ds\\ &+\frac{2-\alpha}{M(\alpha-1)}\int_a^t\underline{g}(s,y(s))ds+\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\underline{g}(s,y(s))ds,\end{aligned}$$

$$\begin{aligned}\overline{y}(t)&=\overline{A}+\frac{t-a}{b-a}\overline{B}-\frac{t-a}{b-a}\overline{A}-\frac{t-a}{b-a}\frac{2-\alpha}{M(\alpha-1)}\int_a^b\overline{g}(s,y(s))ds\\ &-\frac{t-a}{b-a}\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\overline{g}(s,y(s))ds\\ &+\frac{2-\alpha}{M(\alpha-1)}\int_a^t\overline{g}(s,y(s))ds+\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\overline{g}(s,y(s))ds.\end{aligned}$$

Finally we get

$$\begin{aligned}\underline{y}(t)=&\frac{b-t}{b-a}\underline{A}+\frac{t-a}{b-a}\underline{B}-\frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_a^b\underline{g}(s,y(s))ds+\frac{2-\alpha}{M(\alpha-1)}\int_a^t\underline{g}(s,y(s))ds\\ &-\frac{(\alpha-1)(t-a)}{(b-a)M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\underline{g}(s,y(s))ds+\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\underline{g}(s,y(s))ds,\end{aligned}$$

$$\begin{aligned}\overline{y}(t)=&\frac{b-t}{b-a}\overline{A}+\frac{t-a}{b-a}\overline{B}-\frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_a^b\overline{g}(s,y(s))ds+\frac{2-\alpha}{M(\alpha-1)}\int_a^t\overline{g}(s,y(s))ds\\ &-\frac{(\alpha-1)(t-a)}{(b-a)M(\alpha-1)\Gamma(\alpha)}\int_a^b(b-s)^{\alpha-1}\overline{g}(s,y(s))ds+\frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\overline{g}(s,y(s))ds.\end{aligned}$$

From the previous expression, we obtain

$$\begin{aligned}\underline{y}(t) &= \frac{b-t}{b-a}\underline{A} + \frac{t-a}{b-a}\underline{B} - \frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_a^t \underline{g}(s,y(s))ds \\ &- \frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_t^b \underline{g}(s,y(s))ds + \frac{2-\alpha}{M(\alpha-1)}\int_a^t \underline{g}(s,y(s))ds \\ &- \frac{(\alpha-1)(t-a)}{(b-a)M(\alpha-1)\Gamma(\alpha)}\left(\int_a^t (b-s)^{\alpha-1}\underline{g}(s,y(s))ds + \int_t^b (b-s)^{\alpha-1}\underline{g}(s,y(s))ds\right) \\ &+ \frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\underline{g}(s,y(s))ds,\end{aligned}$$

$$\begin{aligned}\overline{y}(t) &= \frac{b-t}{b-a}\overline{A} + \frac{t-a}{b-a}\overline{B} - \frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_a^t \overline{g}(s,y(s))ds \\ &- \frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_t^b \overline{g}(s,y(s))ds + \frac{2-\alpha}{M(\alpha-1)}\int_a^t \overline{g}(s,y(s))ds \\ &- \frac{(\alpha-1)(t-a)}{(b-a)M(\alpha-1)\Gamma(\alpha)}\left(\int_a^t (b-s)^{\alpha-1}\overline{g}(s,y(s))ds + \int_t^b (b-s)^{\alpha-1}\overline{g}(s,y(s))ds\right) \\ &+ \frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{g}(s,y(s))ds.\end{aligned}$$

If we use the Green function, we have the solution for this case as follows

$$\begin{aligned}\left(\underline{y}(t)|_{E_{\alpha-1}\geq 0} + \overline{y}(t)|_{E_{\alpha-1}<0}\right) &= \frac{b-t}{b-a}\underline{A}|_{E_{\alpha-1}\geq 0} + \frac{b-t}{b-a}\overline{A}|_{E_{\alpha-1}\geq 0} + \frac{t-a}{b-a}\underline{B}|_{E_{\alpha-1}\geq 0} \\ &+ \frac{t-a}{b-a}\overline{B}|_{E_{\alpha-1}\geq 0} + \int_a^b G(t,s)\underline{g}(s,y(s))ds,\end{aligned}$$

$$\begin{aligned}\left(\overline{y}(t)|_{E_{\alpha-1}\geq 0} + \underline{y}(t)|_{E_{\alpha-1}<0}\right) &= \frac{b-t}{b-a}\overline{A}|_{E_{\alpha-1}\geq 0} + \frac{b-t}{b-a}\underline{A}|_{E_{\alpha-1}\geq 0} + \frac{t-a}{b-a}\overline{B}|_{E_{\alpha-1}\geq 0} \\ &+ \frac{t-a}{b-a}\underline{B}|_{E_{\alpha-1}\geq 0} + \int_a^b G(t,s)\overline{g}(s,y(s))ds.\end{aligned}$$

Using the same steps as in the first case, we obtain the solution for another case as follows:

$$\begin{aligned}\left(\underline{y}(t)|_{E_{\alpha-1}\geq 0} + \overline{y}(t)|_{E_{\alpha-1}<0}\right) &= \frac{b-t}{b-a}\underline{A}|_{E_{\alpha-1}\geq 0} + \frac{b-t}{b-a}\overline{A}|_{E_{\alpha-1}\geq 0} + \frac{t-a}{b-a}\underline{B}|_{E_{\alpha-1}\geq 0} \\ &+ \frac{t-a}{b-a}\overline{B}|_{E_{\alpha-1}\geq 0} + \int_a^b G(t,s)\overline{g}(s,y(s))ds,\end{aligned}$$

$$\begin{aligned}\left(\overline{y}(t)|_{E_{\alpha-1}\geq 0} + \underline{y}(t)|_{E_{\alpha-1}<0}\right) &= \frac{b-t}{b-a}\overline{A}|_{E_{\alpha-1}\geq 0} + \frac{b-t}{b-a}\underline{A}|_{E_{\alpha-1}\geq 0} + \frac{t-a}{b-a}\overline{B}|_{E_{\alpha-1}\geq 0} \\ &+ \frac{t-a}{b-a}\underline{B}|_{E_{\alpha-1}\geq 0} + \int_a^b G(t,s)\underline{g}(s,y(s))ds.\end{aligned}$$

In general, we can write

$$\left(y(t)\ominus_{gH}\varphi(t)\right)_{E_{\alpha-1}\geq 0} \oplus \left(y(t)\ominus_{gH}\varphi(t)\right)_{E_{\alpha-1}<0} = \int_a^b G(t,s)g(s,y(s))ds.$$

Now, let y satisfied the equation (3), we should prove that y verified also the problem (2). In **case 1**, we have

$$\begin{cases} \left(\underline{y}(t)|_{E_{\alpha-1}\geq 0}+\overline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}<0} \\ & +\int_a^b G(t,s)\underline{g}(s,y(s))ds, \\ \left(\overline{y}(t)|_{E_{\alpha-1}\geq 0}+\underline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}<0} \\ & +\int_a^b G(t,s)\overline{g}(s,y(s))ds. \end{cases}$$

$$\begin{cases} \left(\underline{y}(t)|_{E_{\alpha-1}\geq 0}+\overline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}<0} \\ & +\int_a^t G(t,s)\underline{g}(s,y(s))ds+\int_t^b G(t,s)\underline{g}(s,y(s))ds, \\ \left(\overline{y}(t)|_{E_{\alpha-1}\geq 0}+\underline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}<0} \\ & +\int_a^t G(t,s)\overline{g}(s,y(s))ds+\int_t^b G(t,s)\overline{g}(s,y(s))ds. \end{cases}$$

Using the expression of G, we get
In **case 1**

$$\begin{aligned} \left(\underline{y}(t)|_{E_{\alpha-1}\geq 0}+\overline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}<0} \\ &- \frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_a^b \underline{g}(s,y(s))ds+\frac{2-\alpha}{M(\alpha-1)}\int_a^t \underline{g}(s,y(s))ds \\ &- \frac{(\alpha-1)(t-a)}{(b-a)M(\alpha-1)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\underline{g}(s,y(s))ds \\ &+ \frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\underline{g}(s,y(s))ds. \end{aligned}$$

$$\begin{aligned} \left(\overline{y}(t)|_{E_{\alpha-1}\geq 0}+\underline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}<0} \\ &- \frac{(t-a)(2-\alpha)}{(b-a)M(\alpha-1)}\int_a^b \overline{g}(s,y(s))ds+\frac{2-\alpha}{M(\alpha-1)}\int_a^t \overline{g}(s,y(s))ds \\ &- \frac{(\alpha-1)(t-a)}{(b-a)M(\alpha-1)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\overline{g}(s,y(s))ds \\ &+ \frac{\alpha-1}{M(\alpha-1)\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{g}(s,y(s))ds. \end{aligned}$$

Then we have

$$\begin{aligned} \left(\underline{y}(t)|_{E_{\alpha-1}\geq 0}+\overline{y}(t)|_{E_{\alpha-1}<0}\right) &= \left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}<0} \\ &+ {}^{AB}I^{\alpha}\underline{g}(t,y(t))-\frac{t-a}{b-a}{}^{AB}I^{\alpha}\underline{g}(b,y(b)), \end{aligned} \tag{5}$$

$$\begin{aligned}\left(\overline{y}(t)|_{E_{\alpha-1}\geq 0}+\underline{y}(t)|_{E_{\alpha-1}<0}\right)=\left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)_{E_{\alpha-1}\geq 0}+\left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)_{E_{\alpha-1}<0}\\ +{}^{AB}I^{\alpha}\overline{g}(t,y(t))-\frac{t-a}{b-a}{}^{AB}I^{\alpha}\overline{g}(b,y(b)).\end{aligned}\tag{6}$$

Then applying the Atangana-Baleanu derivative on both sides of (5) and (6), we find

$$\begin{aligned}{}^{ABC}D^{\alpha}\left(\underline{y}(t)|_{E_{\alpha-1}\geq 0}+\overline{y}(t)|_{E_{\alpha-1}<0}\right)&={}^{ABC}D^{\alpha}\Bigg[\left(\frac{b-t}{b-a}\underline{A}-\frac{t-a}{b-a}\underline{B}\right)\\ &\quad+{}^{AB}I^{\alpha}\underline{g}(t,y(t))-\frac{t-a}{b-a}{}^{AB}I^{\alpha}\underline{g}(b,y(b))\Bigg],\\ {}^{ABC}D^{\alpha}\left(\overline{y}(t)|_{E_{\alpha-1}\geq 0}+\underline{y}(t)|_{E_{\alpha-1}<0}\right)&={}^{ABC}D^{\alpha}\Bigg[\left(\frac{b-t}{b-a}\overline{A}-\frac{t-a}{b-a}\overline{B}\right)\\ &\quad+{}^{AB}I^{\alpha}\overline{g}(t,y(t))-\frac{t-a}{b-a}{}^{AB}I^{\alpha}\overline{g}(b,y(b))\Bigg].\end{aligned}$$

Since we have in general, ${}^{ABC}D^{\alpha}x^{\beta}=0$ for $\alpha>\beta$ then we obtain

$$\begin{aligned}{}^{ABC}D^{\alpha}\underline{y}(t)&={}^{ABC}D^{\alpha}\left[{}^{AB}I^{\alpha}\underline{g}(t,y(t))\right],\\ {}^{ABC}D^{\alpha}\overline{y}(t)&={}^{ABC}D^{\alpha}\left[{}^{AB}I^{\alpha}\overline{g}(t,y(t))\right].\end{aligned}$$

Using the proprieties of composition of integral and derivative of Atangana-Baleanu, we get

$$\begin{aligned}{}^{ABC}D^{\alpha}\underline{y}(t)&=\underline{g}(t,y(t)),\\ {}^{ABC}D^{\alpha}\overline{y}(t)&=\overline{g}(t,y(t)),\end{aligned}$$

following the same steps as the first case, we get

$$\begin{aligned}{}^{ABC}D^{\alpha}\underline{y}(t)&=\overline{g}(t,y(t)),\\ {}^{ABC}D^{\alpha}\overline{y}(t)&=\underline{g}(t,y(t)),\end{aligned}$$

thus we have

$${}^{ABC}D^{\alpha}y(t)=g(t,y(t)).$$

For boundary conditions we have for t=a, then for t=b

$$\begin{aligned}\underline{y}(a)&=\underline{A}, &\quad \overline{y}(a)&=\overline{A},\\ \underline{y}(b)&=\underline{B}, &\quad \overline{y}(b)&=\overline{B}.\end{aligned}$$

□

Particular case In this paragraph we will consider the problem (2) on the interval $[0,1]$ inside of $[a,b]$

$$\begin{cases}{}^{\mathrm{ABC}}D^{\alpha}y(t)=g(t,y(t)) & 1<\alpha\leq 2,\\ y(0)=A \quad y(1)=B & A,B\in\mathbb{E}^1.\end{cases}\tag{7}$$

then the solution is

$$y(t)\ominus_{gH}\Big((1-t)A+tB\Big)=\int_0^1 G(t,s)g(s,y(s))ds.$$

Lemma 4.1. *The function $G(t,s)$ defined by 4 satisfies the following condition*

$$\int_a^b\Big|G(t,s)\Big|ds\leq\frac{3(2-\alpha)(b-a)}{M(\alpha-1)}+\frac{3(\alpha-1)b^{\alpha}}{M(\alpha-1)\Gamma(\alpha+1)}.$$

Proof. We will use the expression of the Green function and the function Beta given by

$$B(p,q)=\int_0^1 x^{p-1}(x-1)^{q-1}dx, \qquad Re(p)>0,\ Re(q)>0,$$

then we have

$$\begin{aligned}
\int_a^b \Big|G(t,s)\Big|ds &= \int_a^t \Big|G(t,s)\Big|ds + \int_t^b \Big|G(t,s)\Big|ds,\\
&\leq \int_a^t \frac{(2-\alpha)(t-a)}{(b-a)M(\alpha-1)}ds + \int_a^t \frac{(\alpha-1)(t-s)^{\alpha-1}}{M(\alpha-1)\Gamma(\alpha)}ds\\
&+ \int_a^t \frac{2-\alpha}{M(\alpha-1)}ds + \int_a^t \frac{(\alpha-1)(t-a)(b-s)^{\alpha-1}}{(b-a)M(\alpha-1)\Gamma(\alpha)}ds\\
&+ \int_t^b \frac{(\alpha-2)(t-a)}{(b-a)M(\alpha-1)} + \int_t^b \frac{(\alpha-1)(t-a)(b-s)^{\alpha-1}}{(b-a)M(\alpha-1)\Gamma(\alpha)}ds,\\
&\leq \frac{3(2-\alpha)(b-a)}{M(\alpha-1)} + 3\frac{(\alpha-1)}{M(\alpha-1)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}ds,\\
&\leq \frac{3(2-\alpha)(b-a)}{M(\alpha-1)} + \frac{3(\alpha-1)}{M(\alpha-1)\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}ds,\\
&\leq \frac{3(2-\alpha)(b-a)}{M(\alpha-1)} + \frac{3b^\alpha(\alpha-1)}{M(\alpha-1)\Gamma(\alpha+1)}.
\end{aligned}$$

□

Theorem 4.2. *Let g given in* (2) *be a fuzzy continuous function and satisfying the Lipschitz condition with constant $K>0$ as follows*

$$D\Big(g(t,y_1(t));g(t,y_2(t))\Big) \leq KD\Big(y_1(t);y_2(t)\Big) \qquad \forall y_1,y_2\in\mathcal{C}_{[a,b]},$$

then the problem (2) *has a unique fuzzy solution provided*

$$K\bigg(\frac{3(2-\alpha)(b-a)}{M(\alpha-1)} + \frac{3(\alpha-1)b^\alpha}{M(\alpha-1)\Gamma(\alpha+1)}\bigg) < 1.$$

Proof. Consider the following operator $T:\mathcal{C}_{[a,b]}\to\mathcal{C}_{[a,b]}$ defined by expressions:
In **case 1**

$$T(y(t)) = \frac{b-t}{b-a}A \oplus \frac{t-a}{b-a}B \oplus \int_a^b G(t,s)g(s,y(s))ds.$$

In **case 2**

$$T(y(t)) = \frac{b-t}{b-a}A \oplus \frac{t-a}{b-a}B \ominus (-1)\int_a^b G(t,s)g(s,y(s))ds.$$

then we have In **case 1**

$$\begin{aligned}
D\Big(T(y_1(t)),T(y_2(t))\Big) =& D\bigg(\frac{b-t}{b-a}A \oplus \frac{t-a}{b-a}B \oplus \int_a^b G(t,s)g(s,y_1(s))ds,\\
&\frac{b-t}{b-a}A \oplus \frac{t-a}{b-a}B \oplus \int_a^b G(t,s)g(s,y_2(s))ds\bigg).
\end{aligned}$$

Using the proprieties of metric given in definition 2.2, we have

$$\begin{aligned}
D\Big(T(y_1(t)),T(y_2(t))\Big) &\leq D\bigg(\int_a^b G(t,s)g(s,y_1(s))ds, \int_a^b G(t,s)g(s,y_2(s))ds\bigg),\\
&\leq \int_a^b \Big|G(t,s)\Big|D\Big(g(s,y_1(s)),g(s,y_2(s))\Big)ds,\\
&\leq K\int_a^b \Big|G(t,s)\Big|D\Big(y_1(s),y_2(s)\Big)ds.
\end{aligned}$$

Since f is a fuzzy function continuous and Lipschitz, then we get

$$\begin{aligned} d\Big(T(y_1), T(y_2)\Big) &= \sup_{t\in[a,b]} D\Big(T(y_1(t)), T(y_2(t))\Big) \\ &\leq K\left(\frac{3(2-\alpha)(b-a)}{M(\alpha-1)} + \frac{3(\alpha-1)b^{\alpha}}{M(\alpha-1)\Gamma(\alpha+1)}\right) d\Big(y_1, y_2\Big). \end{aligned}$$

Since $K\left(\frac{3(2-\alpha)(b-a)}{M(\alpha-1)} + \frac{3(\alpha-1)b^{\alpha}}{M(\alpha-1)\Gamma(\alpha+1)}\right) < 1$, then the operator T is a contraction mapping and by Banach fixed point theorem, then operator T has a unique fixed point which is the solution of problem (2). By the same way, we prove this result for **case 2** which complete the proof. □

5. An illustrative example

In this section, we illustrate the existence and uniqueness of solutions for the problem (2) on the interval $[0,1]$ with $g(t,y(t)) = \dfrac{t}{6}y(t)$ such that g is Lipschitz with constant $K = \frac{1}{6}$ and with the boundary conditions A=(0,0,0), B=(-1,0,1). Indeed, We have

$$\begin{cases} {}^{\mathrm{ABC}}D^{\frac{4}{3}} y(t) = \frac{t}{6} y(t), \\ y(0) = (0,0,0), \quad y(1) = (-1,0,1). \end{cases} \tag{8}$$

Then the solution is given by

$$y(t) \ominus_{gH} \Big((1-t)A \oplus tB\Big) = \int_0^1 G(t,s) s y(s) ds. \tag{9}$$

If y is (1,1)-differentiable or y is (2,2)-differentiable then we have the solution as follows

$$\underline{y}(t) = t(r-1) + \frac{1}{6}\int_0^1 G(t,s).s\underline{y}(s)ds,$$

$$\overline{y}(t) = t(1-r) + \frac{1}{6}\int_0^1 G(t,s).s\overline{y}(s)ds.$$

If y is (1,2)-differentiable or y is (2,1)-differentiable we have the solution as follows

$$\underline{y}(t) = t(r-1) + \frac{1}{6}\int_0^1 G(t,s)s\overline{y}(s)ds,$$

$$\overline{y}(t) = t(r-1) + \frac{1}{6}\int_0^1 G(t,s)s\underline{y}(s)ds.$$

Moreover we have

$$\frac{(2-\frac{4}{3})}{2M(\frac{4}{3}-1)} + \frac{(\frac{4}{3}-1)}{2M(\frac{4}{3}-1)\Gamma(\frac{7}{3})} = 0.6573 < 1,$$

then (8) has a unique solution defined on $[0,1]$.

6. Conclusion

In the current paper , we proved the existence and uniqueness of solutions of fuzzy fractional boundary value problems for some nonlinear differential equations involving fuzzy Atangana-Baleanu fractional derivative of order $\alpha \in (1,2)$. As a first step, we established some new results and properties of Atangana-Baleanu fractional derivatives in the fuzzy case by using generalized Hukuhara difference and we build a general structure of solutions associated with our proposed model. Once the fixed point operator equation is available, the existence result is established by using Banach fixed point theorem. Finally, the investigation of the result has been illustrated by providing a suitable example.

Acknowledgement

The authors are grateful to the anonymous referee(s) for their valuable remarks that led to the improvement of the original manuscript.

References

[1] Abdeljawad, T., (2017), A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, Journal of inequalities and applications;, 2017(1), pp.1-11.

[2] Adlassnig, K.P., (1986), Fuzzy set theory in medical diagnosis. IEEE Transactions on Systems, Man, and Cybernetics., 16(2), pp.260-265.

[3] Agarwal, R.P., Lakshmikantham, V., Nieto, J.J., (2010), On the concept of solution for fractional differential equations with uncertainty, Nonlinear Analysis: Theory, Methods and Applications., (72), pp.2859-2862.

[4] Agarwal, R.P., Ntouyas, S.K., Ahmad, B., Alzahrani,A.K., (2016), Hadamard-type fractional functional differential equations and inclusions with retarded and advanced arguments, Advances in Difference Equations., 1, pp.1-15.

[5] Allahviranloo, T., Gouyandeh, T., Armand, Z., (2014), Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, Journal of Intelligent and Fuzzy Systems., (26),pp. 1481-90.

[6] Allahviranloo, T., Armand, A., Gouyandeh, Z., (2014), Fuzzy fractional differential equations under generalized fuzzy Caputo derivative, Journal of Intelligent and Fuzzy Systems., 26(3), pp.1481-1490.

[7] Allahviranloo, T., Ghanbari, B., (2020), On the fuzzy fractional differential equation with interval Atangana-Baleanu fractional derivative approach, Chaos, Solitons and Fractals., 130, pp.109-397.

[8] Alikhani, R., Bahrami, F., (2013), Global solutions for nonlinear fuzzy fractional integral and integrodifferential equations, Commun Nonlinear Sci Numer Simulat., (18), pp. 2007-2017.

[9] Arshad, S., Lupulescu, S., (2011), On the fractional differential equations with uncertainty,Nonlinear Analysis., (74), pp. 3685-3693.

[10] Armand, A., Gouyandeh, Z., (2013) , Solving two-point fuzzy boundary value problem using the variational iteration method, Communications on Advanced Computational Science with Applications., 2013, pp.1-10.

[11] Bede,B., (2013), Mathematics of Fuzzy Sets and Fuzzy Logic, Springer, London.

[12] Bede, B., Stefanini, L., (2013), Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems., 230, pp.119-141.

[13] Chang, S.L., Zadeh, L.A., (1972), IEEE Trans Systems Man cybernet., 2, pp.30-34.

[14] Diamond, P., Kloeden, P., (1994), Metric Spaces of Fuzzy Sets, World Scientific, Singapore. http://dx.doi.org/10.1142/2326.

[15] Devi, S.S., Ganesan, K., (2019), Modelling electric circuit problem with fuzzy differential equations, In Journal of Physics: Conference Series., 1377(1), pp. 012-024.

[16] Dixit, V., Chaudhuri, A., Srivastava, R.K., (2018), Procurement scheduling in engineer procure construct projects: a comparison of three fuzzy modelling approaches, International Journal of Construction Management., 18(3), pp.189-206.

[17] Dubois, D., Prade, H., (1982), Towards fuzzy differential calculus part 1: Integration of fuzzy mappings. Fuzzy sets and Systems., 8(1), pp.1-17.

[18] El Mfadel, A., Melliani, S., Elomari, M., (2021), A Note on the Stability Analysis of Fuzzy Nonlinear Fractional Differential Equations Involving the Caputo Fractional Derivative, International Journal of Mathematics and Mathematical Sciences., https://doi.org/10.1155/2021/7488524.

[19] El Mfadel, A., Melliani, S., Elomari, M., (2021), Notes on Local and Nonlocal Intuitionistic Fuzzy Fractional Boundary Value Problems with Caputo Fractional Derivatives. Journal of Mathematics., https://doi.org/10.1155/2021/4322841.

[20] Hilfer, R., (2000), Fractional calculus and regular variation in thermodynamics, In Applications of fractional calculus in physics., pp. 429-463.

[21] Kandel, A., Byatt, W. J., (1980), Fuzzy Sets and Systems., 110 , pp. 117-152.

[22] Kashkynbayev, A., Rihan, F.A., (2021), Dynamics of fractional-order epidemic models with general nonlinear incidence rate and time-delay, Mathematics., 9(15), p.1829.

[23] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., (2006), Theory and applications of fractional differential equations., 204, elsevier.

[24] Luchko, Y., Trujillo, J., (2007), Caputo-type modification of the Erdélyi-Kober fractional derivative. Fractional Calculus and Applied Analysis., 10(3), pp.249-267.

[25] Scalas, E., Gorenflo, R., Mainardi, F., (2000), Fractional calculus and continuous-time finance, Physica A: Statistical Mechanics and its Applications., 284(1-4), pp.376-384.

[26] Tejado, I., Pérez, E., Valério, D., (2020), Fractional derivatives for economic growth modelling of the group of twenty: Application to prediction, Mathematics., 8(1), pp.50.

Fouziya Zamtain is currently a PhD student in the Faculty of Science and technics at Sultan Moulay Slimane University (USMS), Morocco. Her research interests are mainly in the field of fuzzy fractional differential equations..

M'hamed Elomari is a professor of mathematics at Sultan Moulay Slimane University, Morocco. His research interest includes fuzzy mathematics, generalized functions and fractional calculus.

Said Melliani is a professor of mathematics and the Dean of faculty of sciences and technics at Sultan Moulay Slimane University, Morocco. His research interest includes fuzzy sets theory, generalized functions, fractional calculus, optimization and functional analysis .

Ali El Mfadel is a doctor of mathematics at Sultan Moulay Sliman University, Morocco. His research interest includes fuzzy mathematics and fractional differential equations.