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ON THE CAUCHY PROBLEM FOR SYSTEMS OF LINEAR EQUATIONS OF ELLIPTIC TYPE OF THE FIRST ORDER IN THE SPACE $\mathbb{R}^m$

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Abstract. In the present paper, on the basis of the Carleman matrix, approximate solutions of the Cauchy problem for matrix factorizations of the Helmholtz equation are found in explicit form.

Keywords: Ill-Posed Cauchy Problems, regularized solution, approximate solution, matrix factorization, elliptical system.

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1. Introduction

One of the fundamental problems in the theory of (ordinary and partial) differential equations: To find a solution (an integral) of a differential equation satisfying what are known as initial conditions (initial data). The Cauchy problem usually appears in the analysis of processes defined by a differential law and an initial state, formulated mathematically in terms of a differential equation and an initial condition (hence the terminology and the choice of notation: The initial data are specified for and the solution is required for). Cauchy problems differ from boundary value problems in that the domain in which the desired solution must be defined is not specified in advance. Nevertheless, Cauchy

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problems, like boundary value problems, are defined by the imposition of limiting conditions for the solution on (part of) the boundary of the domain of definition.

The main questions connected with Cauchy problems are as follows:

1) Does there exist (albeit only locally) a solution?

2) If the solution exists, to what space does it belong? In particular, what is its domain of existence?

3) Is the solution unique?

4) If the solution is unique, is the problem well-posed, i.e. is the solution in some sense a continuous function of the initial data?

The theory of ill-posed problems is a direction of mathematics which has developed intensively in the last two decades and is connected with the most varied applied problems: interpretation of readings of many physical instruments and of geophysical, geological, and astronomical observations, optimization of control, management and planning, synthesis of automatic systems, etc. Development of the theory of ill-posed problems was occasioned by the advent of modern computing technology. Various areas of the theory of ill-posed problems can be included in traditional areas of mathematics such as function theory, functional analysis, differential equations, and linear algebra. The concept of a well-posed problem is connected with investigations by the famous French mathematician Hadamard of various boundary value problems for the equations of mathematical physics. Hadamard expressed the opinion that boundary value problems whose solutions do not satisfy certain continuity conditions are not physically meaningful, and he presented examples of such problems. It was subsequently found that Hadamard's opinion was erroneous. It turned out that many problems of mathematical physics which are ill-posed in the sense of Hadamard and, in particular, problems noted by Hadamard himself have real physical content. It also turned out that ill-posed problems arise in many other areas of mathematics which are connected with applications. Such a classical problem of mathematical analysis as the problem of differentiation is ill-posed if it is connected with processing experimental data (see, for instance [23], [28]. For ill-posed problems of the question arises: What is meant by an approximate solution? Clearly, it should be so defined that it is stable under small changes of the original information. A second question is: What algorithms are there for the construction of such solutions? Answers to these basic questions were given by A.N. Tikhonov (see [2]).

It is known that the Cauchy problem for elliptic equations and for systems of elliptic equations belongs to the class of ill-posed problems (see, for example, [2], [25], [26]-[27], [36]-[37]). Boundary value problems, as well as numerical solutions of some problems, are considered in [3]-[7], [21]-[22], [24], [29], [38]-[39].

The concept of conditional correctness first appeared in the work of Tikhonov [2], and then in the studies of Lavrent'ev [26]-[27]. In a theoretical study of the conditional correctness (correctness according to Tikhonov) of an ill-posed problem of the existence of a solution and its belonging to the correctness set, it is postulated in the very formulation of the problem. The study of uniqueness issues in a conditionally well-posed formulation does not essentially differ from the study in a classically well-posed formulation, and the stability of the solution from the data of the problem is required only from those variations of the data that do not deduce solutions from the well-posedness set. After establishing the uniqueness and stability theorems in the study of the conditional correctness of ill-posed problems, the question arises of constructing effective solution methods, i.e. construction of regularizing operators.

Based on the results of previous works [8]-[18] we have constructed the Carleman matrix and based on it the approximate solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In this article, we find an explicit formula for an approximate solution of the Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain of an odd-dimensional space $\mathbb{R}^m$. The case of an even-dimensional space will be considered in other scientific studies of the authors. Our approximate solution formula also includes the construction of a family of fundamental solutions of the Helmholtz operator in space. This family is parametrized by some entire function $K(z)$, the choice of which depends on the dimension of the space. In this work, relying on the results of previous works [8]-[18], we similarly obtain better results with approximate estimates due to a special selection of the function $K(z)$. In many well-posed problems for systems of equations of elliptic type of the first order with constant coefficients that factorize the Helmholtz operator, it is not possible to calculate the values of the vector function on the entire boundary. Therefore, the problem of reconstructing the solution of systems of equations of first order elliptic type with constant coefficients, factorizing the Helmholtz operator (see, for instance [8]-[18]), is one of the topical problems in the theory of differential equations.

For the last decades, interest in classical ill-posed problems of mathematical physics has remained. This direction in the study of the properties of solutions of the Cauchy problem for the Laplace equation was started in [26]-[28], [36]-[37] and subsequently developed in [19]-[20], [30]-[33], [8]-[18].

Let $\mathbb{R}^m$, $(m = 2k,\ k \geq 1)$ be a $m-$dimensional real Euclidean space,

$$\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{R}^m, \quad \eta = (\eta_1, \dots, \eta_m) \in \mathbb{R}^m,$$

$$\zeta' = (\zeta_1, \dots, \zeta_{m-1}) \in \mathbb{R}^{m-1} \quad \eta' = (\eta_1, \dots, \eta_{m-1}) \in \mathbb{R}^{m-1}.$$

We introduce the following notation:

$$r = |\eta - \zeta|, \quad \alpha = |\eta' - \zeta'|, \quad z = i\sqrt{a^2 + \alpha^2} + \eta_m, \quad a \geq 0,$$

$$\partial_\zeta = (\partial_{\zeta_1}, \dots, \partial_{\zeta_m})^T, \quad \partial_\zeta = \chi^T, \quad \chi^T = \begin{pmatrix} \chi_1 \\ ... \\ \chi_m \end{pmatrix} \text{ - transposed vector } \chi,$$

$$W(\zeta) = (W_1(\zeta), \dots, W_n(\zeta))^T, \quad v^0 = (1, \dots, 1) \in \mathbb{R}^n, \quad n = 2^m, \quad m \geq 2,$$

$$E(w) = \begin{Vmatrix} w_1 & 0 & \cdots & 0 \\ 0 & w_2 & \cdots & 0 \\ \dots & \dots & \ddots & \dots \\ 0 & 0 & 0 & w_n \end{Vmatrix} - \text{diagonal matrix}, w = (w_1, \dots, w_n) \in \mathbb{R}^n.$$

We also consider a bounded simply-connected domain $\Omega \subset \mathbb{R}^m$, having a piecewise smooth boundary $\partial\Omega = \Sigma \bigcup D$, where Σ is a smooth surface lying in the half-space $\eta_m > 0$ and D is the plane $\eta_m = 0$.

$P(\chi^T)$ is an $(n \times n)-$dimensional matrix satisfying:

$$P^*(\chi^T)P(\chi^T) = E((|\chi|^2 + \lambda^2)v^0),$$

where $P^*(\chi^T)$ is the Hermitian conjugate matrix of $P(\chi^T)$, $\lambda \in \mathbb{R}$, the elements of the matrix $P(\chi^T)$ consist of a set of linear functions with constant coefficients from the complex plane $\mathbb{C}$.

Let us consider the following first order systems of linear partial differential equations with constant coefficients

$$P\left(\partial_\zeta\right) W(\zeta)=0, \tag{1}$$

in the domain Ω, where $P\left(\partial_\zeta\right)$ is the matrix differential operator of the first-order.

Also consider the set

$$S(\Omega)=\left\{W: \overline{\Omega} \longrightarrow \mathbb{R}^n \mid W \text{ is continuous on } \overline{\Omega}=\Omega \cup \partial \Omega \text{ and } W \text{ satisfies the system (1)}\right\}.$$

2. Statement of the Cauchy problem

Formulation of the problem. Suppose $W(\eta) \in S(\Omega)$ and

$$\left.W(\eta)\right|_{\Sigma}=f(\eta), \quad \eta \in \Sigma. \tag{2}$$

Here, $f(\eta)$ a given continuous vector-function on Σ. It is required to restore the vector function $W(\eta)$ in the domain Ω, based on it's values $f(\eta)$ on Σ.

If $W(\eta) \in S(\Omega)$, then the following integral formula of Cauchy type is valid

$$W(\zeta)=\int\limits_{\partial \Omega} L(\eta, \zeta ; \lambda) W(\eta) d s_\eta, \quad \zeta \in \Omega, \tag{3}$$

where

$$L(\eta, \zeta ; \lambda)=\left(E\left(\Gamma_m(\lambda r) v^0\right) P^*\left(\partial_\zeta\right)\right) P(t^T).$$

Here $t=\left(t_1, \ldots, t_m\right)$–is the unit exterior normal, drawn at a point η, the surface $\partial \Omega$, $\Gamma_m(\lambda r)-$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^m$, $(m=2k,\ k \geq 1)$, where $\Gamma_m(\lambda r)$ defined by the following formula:

$$\begin{gathered}
\Gamma_m(\lambda r)=B_m \lambda^{(m-2)/2} \frac{H_{(m-2)/2}^{(1)}(\lambda r)}{r^{(m-2)/2}}, \\
B_m=\frac{1}{2 i(2 \pi)^{(m-2)/2}}, \quad m=2k, \quad k \geq 1.
\end{gathered} \tag{4}$$

Here $H_{(m-2)/2}^{(1)}(\lambda r)-$ is the Hankel function of the first kind of $(m-2)/2-$ th order (see for instance [34]).

Let $K(z)$ be an entire function taking real values for real z, $(z=a+i b,\ a, b \in \mathbb{R})$ such that

$$\begin{gathered}
K(a) \neq 0, \quad \sup_{b \geq 1}\left|b^p K^{(p)}(z)\right|=N(a, p)<\infty, \\
-\infty<a<\infty, \quad p=0, \ldots, m.
\end{gathered} \tag{5}$$

We define the function $\Psi(\eta, \zeta ; \lambda)$ at $\eta \neq \zeta$ by the following equality

$$\begin{gathered}
\Psi(\eta, \zeta ; \lambda)=\frac{1}{c_m K\left(\zeta_m\right)} \frac{\partial^{k-1}}{\partial s^{k-1}} \int\limits_0^{\infty} \operatorname{Im}\left[\frac{K(z)}{z-\zeta_m}\right] \frac{a I_0(\lambda a)}{\sqrt{a^2+\alpha^2}} d a, \\
m=2k,\ k \geq 1,
\end{gathered} \tag{6}$$

where $c_2=2 \pi$, $c_m=(-1)^{k-1}(k-1)!(m-2) \omega_m$; $I_0(\lambda a)=J_0(i \lambda a)$–is the Bessel function of the first kind of zero order [19], ω_m- area of a unit sphere in space $\mathbb{R}^m$.

In the formula (6), choosing

$$K(z)=\exp (\sigma z), \quad K\left(\zeta_m\right)=\exp \left(\sigma \zeta_m\right), \quad \sigma>0, \tag{7}$$

we get

$$\Psi_\sigma(\eta,\zeta;\lambda) = \frac{e^{-\sigma\zeta_m}}{c_m}\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty \mathrm{Im}\left[\frac{\exp(\sigma z)}{z-\zeta_m}\right]\frac{a\,I_0(\lambda a)}{\sqrt{a^2+\alpha^2}}da. \tag{8}$$

The formula (3) is true if instead $\Gamma_m(\lambda r)$ of substituting the function

$$\Psi_\sigma(\eta,\zeta;\lambda) = \Gamma_m(\lambda r) + G_\sigma(\eta,\zeta;\lambda), \tag{9}$$

where $G_\sigma(\eta,\zeta;\lambda)-$ is the regular solution of the Helmholtz equation with respect to the variable η, including the point $\eta=\zeta$.

Then the integral formula has the form:

$$W(\zeta) = \int\limits_{\partial\Omega} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta\in\Omega, \tag{10}$$

where

$$L_\sigma(\eta,\zeta;\lambda) = \left(E\left(\Psi_\sigma(\eta,\zeta;\lambda)v^0\right)P^*\left(\partial_\zeta\right)\right)P(t^T).$$

3. Solution of the Cauchy problem (1)-(2)

Theorem 3.1. *Let $W(\eta)\in S(\Omega)$ it satisfy the inequality*

$$|W(\eta)|\le M, \quad \eta\in D. \tag{11}$$

If

$$W_\sigma(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_y, \quad \zeta\in\Omega, \tag{12}$$

then the following estimates are true

$$|W(\zeta)-W_\sigma(\zeta)| \le MK(\lambda,\zeta)\sigma^k e^{-\sigma\zeta_m}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{13}$$

$$\left|\frac{\partial W(\zeta)}{\partial\zeta_j}-\frac{\partial W_\sigma(\zeta)}{\partial\zeta_j}\right| \le MK(\lambda,\zeta)\sigma^k e^{-\sigma\zeta_m}, \quad \sigma>1, \quad \zeta\in\Omega, \quad j=1,\dots,m. \tag{14}$$

Here and below functions bounded on compact subsets of the domain Ω, we denote by $K(\lambda,\zeta)$.

Proof. Let us first estimate inequality (13). Using the integral formula (10) and the equality (12), we obtain

$$W(\zeta) = \int\limits_{\Sigma} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta + \int\limits_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta =$$

$$= W_\sigma(\zeta) + \int\limits_{D} L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta, \quad \zeta\in\Omega.$$

Taking into account the inequality (11), we estimate the following

$$\begin{gathered}|W(\zeta)-W_\sigma(\zeta)| \le \left|\int\limits_{D} N_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta\right| \le \\ \le \int\limits_{D}|L_\sigma(\eta,\zeta;\lambda)|\,|W(\eta)|\,ds_\eta \le M\int\limits_{D}|L_\sigma(\eta,\zeta;\lambda)|\,ds_\eta, \quad \zeta\in\Omega.\end{gathered} \tag{15}$$

To do this, we estimate the integrals $\int\limits_D |\Psi_\sigma(\eta,\zeta;\lambda)|\, ds_\eta$, $\int\limits_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j}\right| ds_\eta$, $(j = 1, 2, \ldots, m-1)$ and $\int\limits_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_m}\right| ds_\eta$ on the part D of the plane $\eta_m = 0$.

Separating the imaginary part of (8), we obtain

$$\Psi_\sigma(\eta,\zeta;\lambda) = \frac{e^{\sigma(\eta_m-\zeta_m)}}{c_m}\left[\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty \frac{\cos\sigma\sqrt{a^2+\alpha^2}}{a^2+r^2}\, aI_0(\lambda a)\, da - \right.$$
$$\left. -\frac{\partial^{k-1}}{\partial s^{k-1}}\int\limits_0^\infty \frac{(\eta_m-\zeta_m)\sin\sigma\sqrt{a^2+\alpha^2}}{a^2+r^2}\,\frac{aI_0(\lambda a)}{\sqrt{a^2+\alpha^2}}da\right], \quad \zeta_m > 0. \tag{16}$$

From (16) and the inequality

$$I_0(\lambda a) \le \sqrt{\frac{2}{\lambda\pi a}}, \tag{17}$$

we have

$$\int\limits_D |\Psi_\sigma(\eta,\zeta;\lambda)|\, ds_\eta \le K(\lambda,\zeta)\sigma^k M e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad x \in G, \tag{18}$$

To estimate the second integral, we use the equality

$$\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j} = \frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial\eta_j} = 2(\eta_j-\zeta_j)\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial s},$$
$$s = \alpha^2, \quad j = 1, 2, \ldots, m-1. \tag{19}$$

Considering equality (16), inequality (17) and equality (19), we obtain

$$\int\limits_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_j}\right| ds_y \le K(\lambda,\zeta)\sigma^k M e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad \zeta \in \Omega,$$
$$j = 1, 2, \ldots, m-1. \tag{20}$$

Now, we estimate the integral $\int\limits_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_m}\right| ds_\eta$.

Taking into account equality (16) and inequality (17), we obtain

$$\int\limits_D \left|\frac{\partial\Psi_\sigma(\eta,\zeta;\lambda)}{\partial\eta_m}\right| ds_\eta \le K(\lambda,\eta)\sigma^k M e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad \zeta \in \Omega, \tag{21}$$

From inequalities (18), (20) and (21), bearing in mind (15), we get an estimate (13).

Now let us prove inequality (14). To do this, we take the derivatives from equalities (10) and (12) with respect to ζ_j, $j = 1, \ldots, m$, then we obtain the following:

$$\frac{\partial W(\zeta)}{\partial\zeta_j} = \int\limits_\Sigma \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta + \int\limits_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta,$$
$$\frac{\partial W_\sigma(\zeta)}{\partial\zeta_j} = \int\limits_\Sigma \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta, \quad \zeta \in \Omega, \quad j = 1, \ldots, m. \tag{22}$$

Taking into account the (22) and inequality (11), we estimate the following

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial_\sigma W(\zeta)}{\partial \zeta_j}\right| \leq \left|\int\limits_D \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq$$
$$\leq \int\limits_D \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| |W(\eta)|\, ds_\eta \leq M \int\limits_D \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \tag{23}$$
$$\zeta \in \Omega, \quad j = 1,\dots,m.$$

To do this, we estimate the integrals $\int\limits_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \quad (j = 1,2,\dots,m-1)$ and $\int\limits_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_m}\right| ds_\eta$ on the part D of the plane $\eta_m = 0$.

To estimate the first integrals, we use the equality

$$\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} = \frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}\frac{\partial s}{\partial \zeta_j} = -2(\eta_j - \zeta_j)\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial s}, \tag{24}$$
$$s = \alpha^2, \quad j = 1,2,\dots,m-1.$$

Given equality (16), inequality (17) and equality (24), we obtain

$$\int\limits_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_y \leq K(\lambda,\zeta)\sigma^k M e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad \zeta \in \Omega, \tag{25}$$
$$j = 1,2,\dots,m-1.$$

Now, we estimate the integral $\int\limits_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_m}\right| ds_\eta$.

Taking into account equality (16) and inequality (17), we obtain

$$\int\limits_D \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_m}\right| ds_\eta \leq K(\lambda,\zeta)\sigma^k M e^{-\sigma\zeta_m}, \quad \zeta \in \Omega, \tag{26}$$

From inequalities (23), (25) and (26), we obtain an estimate (14).
Theorem 3.1 is proved. □

Corollary 3.1. *For each $\zeta \in \Omega$, the equalities are true*

$$\lim_{\sigma\to\infty} W_\sigma(\zeta) = W(\zeta), \quad \lim_{\sigma\to\infty} \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} = \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = 1,\dots,m.$$

We denote by $\overline{\Omega}_\varepsilon$ the set

$$\overline{\Omega}_\varepsilon = \left\{(\zeta_1,\dots,\zeta_m) \in \zeta, \quad q > \zeta_m \geq \varepsilon, \quad q = \max_D \psi(\zeta'), \quad 0 < \varepsilon < q\right\}.$$

Here, at $m = 2$, $\underline{\psi(\zeta_1)}$ - is a curve, and at $m = 2k, \quad k > 1$, $\psi(\zeta')$ - is a surface. It is easy to see that the set $\overline{\Omega}_\varepsilon \subset \Omega$ is compact.

Corollary 3.2. *If $\zeta \in \overline{\Omega}_\varepsilon$, then the families of functions $\{W_\sigma(\zeta)\}$ and $\left\{\dfrac{\partial W_\sigma(\zeta)}{\partial \zeta_j}\right\}$ converge uniformly for $\sigma \to \infty$, i.e.:*

$$W_\sigma(\zeta) \rightrightarrows W(\zeta), \quad \frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} \rightrightarrows \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = 1, \dots, m.$$

It should be noted that the set $E_\varepsilon = \Omega \backslash \overline{\Omega}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

4. Regularized solution of the problem (1)-(2)

Suppose that the surface Σ (or the curve at $m = 2$) is given by the equation

$$\eta_m = \psi(\eta'), \quad \eta' \in \mathbb{R}^{m-1},$$

where $\psi(\eta')$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$q = \max_D \psi(\eta'), \quad l = \max_D \sqrt{1 + \psi'^2(\eta')}.$$

Theorem 4.1. *Let $W(\eta) \in S(\Omega)$ satisfy condition (11), and on a smooth surface Σ the inequality*

$$|W(\eta)| \leq \delta, \quad 0 < \delta < 1. \tag{27}$$

Then the following estimates are true

$$|W(\zeta)| \leq K(\lambda, \zeta)\sigma^k M^{1-\frac{\zeta_m}{q}} \delta^{\frac{\zeta_m}{q}}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{28}$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \leq K(\lambda, \zeta)\sigma^k M^{1-\frac{\zeta_m}{q}} \delta^{\frac{\zeta_m}{q}}, \quad \sigma > 1, \quad \zeta \in \Omega, \qquad j = 1, \dots, m. \tag{29}$$

Proof. Let us first estimate inequality (28). Using the integral formula (10), we have

$$W(\zeta) = \int_\Sigma L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta + \int_D N_\sigma(\eta, \zeta; \lambda)) W(\eta) ds_\eta, \quad \zeta \in \Omega. \tag{30}$$

We estimate the following

$$|W(\zeta)| \leq \left|\int_\Sigma L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta\right| + \left|\int_D L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta\right|, \quad \zeta \in \Omega. \tag{31}$$

Given inequality (27), we estimate the first integral of inequality (31).

$$\begin{aligned} \left|\int_\Sigma L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta\right| &\leq \int_\Sigma |L_\sigma(\eta, \zeta; \lambda)| \, |W(\eta)| \, ds_\eta \leq \\ &\leq \delta \int_\Sigma |L_\sigma(\eta, \zeta; \lambda)| \, ds_\eta, \quad \zeta \in \Omega. \end{aligned} \tag{32}$$

To do this, we estimate the integrals $\int_\Sigma |\Psi_\sigma(\eta, \zeta; \lambda)| \, ds_\eta$, $\int_\Sigma \left|\dfrac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \eta_j}\right| ds_\eta$, $(j = 1, 2, \dots, m-1)$ and $\int_\Sigma \left|\dfrac{\partial \Psi_\sigma(\eta, \zeta; \lambda)}{\partial \eta_m}\right| ds_\eta$ on a smooth surface Σ.

Given equality (16) and the inequality (17), we have

$$\int_{\Sigma} |\Psi_\sigma(\eta,\zeta;\lambda)|\, ds_\eta \le K(\lambda,\zeta)\sigma^k e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{33}$$

To estimate the second integral, using equalities (16) and (19) as well as inequality (17), we obtain

$$\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_j}\right| ds_\eta \le K(\lambda,\zeta)\sigma^k e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega,$$
$$j = 1, \quad , m-1. \tag{34}$$

To estimate the integral $\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_m}\right| ds_\eta$, using equality (16) and inequality (17), we obtain

$$\int_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \eta_m}\right| ds_\eta \le K(\lambda,\zeta)\sigma^k e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{35}$$

From (33)-(35), we obtain

$$\left|\int_{\Sigma} L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta\right| \le K(\lambda,\zeta)\sigma^k \delta\, e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{36}$$

The following is known

$$\left|\int_{D} L_\sigma(\eta,\zeta;\lambda) W(\eta) ds_\eta\right| \le K(\lambda,\zeta)\sigma^k M e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{37}$$

Now taking into account (36)-(37), we have

$$|W(\zeta)| \le \frac{K(\lambda,\zeta)\sigma^k}{2}(\delta e^{\sigma q} + M)e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad \zeta \in \Omega. \tag{38}$$

Choosing σ from the equality

$$\sigma = \frac{1}{q}\ln\frac{M}{\delta}, \tag{39}$$

we obtain an estimate (28).

Now let us prove inequality (29). To do this, we find the partial derivative from the integral formula (10) with respect to the variable $\zeta_j, \quad j = 1, \ldots, m-1$:

$$\frac{\partial W(\zeta)}{\partial \zeta_j} = \int_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta + \int_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta +$$
$$+\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} + \int_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta, \quad \zeta \in \Omega, \quad j = 1, \ldots, m. \tag{40}$$

Here

$$\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j} = \int_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta. \tag{41}$$

We estimate the following

$$\begin{aligned}&\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \leq \left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| + \left|\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq \\ &\leq \left|\frac{\partial W_\sigma(\zeta)}{\partial \zeta_j}\right| + \left|\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda))}{\partial \zeta_j} W(\eta) ds_\eta\right|, \quad \zeta \in \Omega, \quad j = 1,\ldots,m.\end{aligned} \tag{42}$$

Given inequality (27), we estimate the first integral of inequality (42).

$$\begin{aligned}&\left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| |W(\eta)|\, ds_\eta \leq \\ &\leq \delta \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \quad \zeta \in \Omega, \quad j = 1,\ldots,m.\end{aligned} \tag{43}$$

To do this, we estimate the integrals $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta, \quad (j = 1, 2, \ldots, m-1)$ and $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_m}\right| ds_\eta$ on a smooth surface Σ.

Given equality (16), inequality (17) and equality (24), we obtain

$$\begin{aligned}&\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j}\right| ds_\eta \leq K(\lambda,\zeta)\sigma^k e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega, \\ &j = 1, 2, \ldots, m-1.\end{aligned} \tag{44}$$

Now, we estimate the integral $\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_m}\right| ds_\eta$.

Taking into account equality (16) and inequality (17), we obtain

$$\int\limits_{\Sigma} \left|\frac{\partial \Psi_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_m}\right| ds_\eta \leq K(\lambda,\zeta)\sigma^k e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega, \tag{45}$$

From (44)-(45), we obtain

$$\begin{aligned}&\left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta)\right| \leq K(\lambda,\zeta)\sigma^k \delta e^{\sigma(q-\zeta_m)}, \quad \sigma > 1, \quad \zeta \in \Omega, \\ &j = 1,\ldots,m.\end{aligned} \tag{46}$$

The following is known

$$\begin{aligned}&\left|\int\limits_{D} \frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq K(\lambda,\zeta)\sigma^k M e^{-\sigma\zeta_m}, \quad \sigma > 1, \quad \zeta \in \Omega, \\ &j = 1,\ldots,m.\end{aligned} \tag{47}$$

Now taking into account (46)-(47), bearing in mind (42), we have

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j}\right| \leq \frac{K(\lambda,\zeta)\sigma^k}{2}(\delta e^{\sigma q}+M)e^{-\sigma\zeta_m}, \quad \sigma>1, \quad \zeta\in\Omega, \quad j=1,\dots,m. \tag{48}$$

Choosing σ from the equality (39), we obtain an estimate (29).

Theorem 4.1 is proved. □

Let $W(\eta)\in S(\Omega)$ and instead $W(\eta)$ on Σ with its approximation $f_\delta(\eta)$ are given, respectively, with an error $0<\delta<1$,

$$\max_{\Sigma}|W(\eta)-f_\delta(\eta)|\leq\delta. \tag{49}$$

We put

$$W_{\sigma(\delta)}(\zeta)=\int_{\Sigma} L_\sigma(\eta,\zeta;\lambda)f_\delta(\eta)ds_y, \quad \zeta\in\Omega. \tag{50}$$

Theorem 4.2. *Let $W(\eta)\in S(\Omega)$ on the part of the plane $\eta_m=0$ satisfy condition (11). Then the following estimates is true*

$$\left|W(\zeta)-W_{\sigma(\delta)}(\zeta)\right|\leq K(\lambda,\zeta)\sigma^k M^{1-\frac{\zeta_m}{q}}\delta^{\frac{\zeta_m}{q}}, \quad \sigma>1, \quad \zeta\in\Omega. \tag{51}$$

$$\left|\frac{\partial W(\zeta)}{\partial\zeta_j}-\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial\zeta_j}\right|\leq K(\lambda,\zeta)\sigma^k M^{1-\frac{\zeta_m}{q}}\delta^{\frac{\zeta_m}{q}}, \quad \sigma>1, \quad \zeta\in\Omega, \quad j=1,\dots,m. \tag{52}$$

Proof. From the integral formulas (10) and (50), we have

$$W(\zeta)-W_{\sigma(\delta)}(\zeta)=\int_{\partial\Omega}L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta-\int_{\Sigma}L_\sigma(\eta,\zeta;\lambda)f_\delta(\eta)ds_\eta=$$

$$=\int_{\Sigma}L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta+\int_{D}L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta-\int_{\Sigma}L_\sigma(\eta,\zeta;\lambda)f_\delta(\eta)ds_\eta=$$

$$=\int_{\Sigma}L_\sigma(\eta,\zeta;\lambda)\left\{W(\eta)-f_\delta(\eta)\right\}ds_\eta+\int_{D}L_\sigma(\eta,\zeta;\lambda)W(\eta)ds_\eta.$$

and

$$\frac{\partial W(\zeta)}{\partial\zeta_j}-\frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial\zeta_j}=\int_{\partial\Omega}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta-\int_{\Sigma}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}f_\delta(\eta)ds_\eta=$$

$$=\int_{\Sigma}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta+\int_{D}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta-\int_{\Sigma}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}f_\delta(\eta)ds_\eta=$$

$$=\int_{\Sigma}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}\left\{W(\eta)-f_\delta(\eta)\right\}ds_\eta+\int_{D}\frac{\partial L_\sigma(\eta,\zeta;\lambda)}{\partial\zeta_j}W(\eta)ds_\eta, \quad j=1,\dots,m.$$

Using conditions (11) and (49), we estimate the following:

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| = \left|\int\limits_{\Sigma} L_\sigma(\eta, \zeta; \lambda)\left\{W(\eta) - f_\delta(\eta)\right\} ds_\eta\right| +$$

$$+\left|\int\limits_{D} L_\sigma(\eta, \zeta; \lambda) W(\eta) ds_\eta\right| \leq \int\limits_{\Sigma} |L_\sigma(\eta, \zeta; \lambda)| \left|\{W(\eta) - f_\delta(\eta)\}\right| ds_\eta +$$

$$+\int\limits_{D} |L_\sigma(\eta, \zeta; \lambda)| \, |W(\eta)| \, ds_\eta \leq \delta \int\limits_{\Sigma} |L_\sigma(\eta, \zeta; \lambda)| \, ds_\eta + M \int\limits_{D} |L_\sigma(\eta, \zeta; \lambda)| \, ds_\eta.$$

and

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right| = \left|\int\limits_{\Sigma} \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j}\left\{W(\eta) - f_\delta(\eta)\right\} ds_y\right| +$$

$$+\left|\int\limits_{D} \frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j} W(\eta) ds_\eta\right| \leq \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j}\right| \left|\{W(\eta) - f_\delta(\eta)\}\right| ds_\eta +$$

$$+\int\limits_{D} \left|\frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j}\right| |W(\eta)| \, ds_\eta \leq \delta \int\limits_{\Sigma} \left|\frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j}\right| ds_\eta +$$

$$+M \int\limits_{D} \left|\frac{\partial L_\sigma(\eta, \zeta; \lambda)}{\partial \zeta_j}\right| ds_\eta, \quad j = 1, \ldots, m.$$

Now, repeating the proof of Theorems 3.1 and 4.1, we obtain

$$\left|W(\zeta) - W_{\sigma(\delta)}(\zeta)\right| \leq \frac{K(\lambda, \zeta)\sigma^k}{2}(\delta \, e^{\sigma q} + M)e^{-\sigma \zeta_m}.$$

$$\left|\frac{\partial W(\zeta)}{\partial \zeta_j} - \frac{W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right| \leq \frac{K(\lambda, \zeta)\sigma^k}{2}(\delta e^{\sigma q} + M)e^{-\sigma \zeta_m}, \quad j = 1, \ldots, m.$$

From here, choosing σ from equality (39), we have an estimates (51) and (52).
Theorem 4.2 is proved. □

Corollary 4.1. *For each $\zeta \in \Omega$, the equalities are true*

$$\lim_{\delta \to 0} W_{\sigma(\delta)}(\zeta) = W(\zeta), \quad \lim_{\delta \to 0} \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j} = \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = 1, \ldots, m.$$

Corollary 4.2. *If $x \in \overline{\Omega}_\varepsilon$, then the families of functions $\left\{W_{\sigma(\delta)}(\zeta)\right\}$ and $\left\{\dfrac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}\right\}$ converge uniformly for $\delta \to 0$, i.e.:*

$$W_{\sigma(\delta)}(\zeta) \rightrightarrows W(\zeta), \quad \frac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j} \rightrightarrows \frac{\partial W(\zeta)}{\partial \zeta_j}, \quad j = 1, \ldots, m.$$

5. Conclusion

In this paper, we have explicitly found a regularized solution to the ill-posed Cauchy problem for matrix factorizations of the Helmholtz equation in a multidimensional bounded domain. It is assumed that a solution to the problem exists and is continuously differentiable in a closed domain with exactly given Cauchy data. For this case, an explicit formula for the continuation of the solution is established, as well as a regularization formula for the case when, under the indicated conditions, instead of the Cauchy data, their continuous approximations with a given error in the uniform metric are given. We have obtained a stability estimate for the solution of the Cauchy problem in the classical sense.

An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem is considered in which instead of the exact data of the Cauchy problem; their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part D of the boundary of the domain Ω; an explicit regularization formula is obtained.

We note that when solving applied problems, one should find the approximate values of $W(\zeta)$ and $\dfrac{\partial W(\zeta)}{\partial \zeta_j}$, $\zeta \in \Omega$, $j = 1, \dots, m$.

In this paper, we construct a family of vector-functions $W(\zeta, f_\delta) = W_{\sigma(\delta)}(\zeta)$ and $\dfrac{\partial W(\zeta, f_\delta)}{\partial \zeta_j} = \dfrac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$, $j = 1, \dots, m$ depending on a parameter σ, and prove that under certain conditions and a special choice of the parameter $\sigma = \sigma(\delta)$, at $\delta \to 0$, the family $W_{\sigma(\delta)}(\zeta)$ and $\dfrac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ converges in the usual sense to a solution $W(\zeta)$ and its derivative $\dfrac{\partial W(\zeta)}{\partial \zeta_j}$ at a point $\zeta \in \Omega$. Following A.N. Tikhonov (see [2]), a family of vector-valued functions $W_{\sigma(\delta)}(\zeta)$ and $\dfrac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Thus, functionals $W_{\sigma(\delta)}(\zeta)$ and $\dfrac{\partial W_{\sigma(\delta)}(\zeta)}{\partial \zeta_j}$ determines the regularization of the solution of problem (1)-(2).

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