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ON EDGE IRREGULARITY STRENGTH OF LADDER RELATED GRAPHS

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Abstract. For a simple graph G, a vertex labeling $\phi : V(G) \to \{1, 2, \ldots, k\}$ is called k-labeling. The weight of an edge xy in G, written $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e., $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling is defined to be an edge irregular k-labeling of the graph G if for every two different edges e and f, $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the edge irregularity strength of G, written $es(G)$. In this paper, we investigate the edge irregularity strength of ladder graph, triangular ladder graph, and diagonal ladder graph.

Keywords: Irregularity strength, edge irregularity strength, ladder graphs, triangular ladder graphs, diagonal ladder graphs.

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1. Introduction

Let G be a connected, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. By a labeling we mean any mapping that maps a set of graph elements to a set of numbers (usually positive integers), called *labels*. If the domain is the vertex-set (the edge-set), then the labeling is called *vertex labelings* (*edge labelings*). If the domain is $V(G) \cup E(G)$, then the labeling is called *total labeling*. Thus, for an edge k-labeling $\delta : E(G) \to \{1, 2, \ldots, k\}$ the associated weight of a vertex $x \in V(G)$ is $w_\delta(x) = \sum \delta(xy)$, where the sum is over all vertices y adjacent to x.

Chartrand et al. [5] defined irregular labeling for a graph G as an assignment of labels from the set of natural numbers to the edges of G such that the sums of the labels assigned to the edges of each vertex are different. The minimum value of the largest label of an edge over all existing irregular labelings is known as the *irregularity strength* of G and it is denoted by $s(G)$. Finding the irregularity strength of a graph seems to be hard even for simple graphs [5].

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Motivated by this, Baca et al. [4] investigated two modifications of the irregularity strength of graphs, namely *total edge irregularity strength*, denoted by $tes(G)$; and *total vertex irregularity strength*, denoted by $tvs(G)$. Motivated by the work of Chartrand et al. [5], Ahmad et al. [1] introduced the concept of edge irregular k-labelings of graphs.

A vertex labeling $\phi : V(G) \to \{1, 2, \dots, k\}$ is called k-labeling. The weight of an edge xy in G, written $w_\phi(xy)$, is the sum of the labels of end vertices x and y, i.e., $w_\phi(xy) = \phi(x) + \phi(y)$. A vertex k-labeling of a graph G is defined to be an *edge irregular k-labeling* of the graph G if for every two different edges e and f, $w_\phi(e) \neq w_\phi(f)$. The minimum k for which the graph G has an edge irregular k-labeling is called the *edge irregularity strength* of G, written $es(G)$. Over the last years, $es(G)$ has been investigated for different families of graphs including trees with the help of algorithmic solutions. For the survey of graph labelings, we refer to [3].

2. Preliminary results

In [1], the authors estimated the bounds of the edge irregularity strength and then determined its exact values for several families of graphs namely, paths, stars, double stars, and Cartesian product of two paths. Ahmad et al. [2] determined the edge irregularity strength of Toeplitz graphs. Tarawneh et al. [10] determined the exact value of edge irregularity strength of corona product of graphs with paths. Tarawneh et al. [11] determined the exact value of edge irregularity strength of disjoint union of graphs. Umme Salma et al. [12] determined the edge irregularity strength of sunlet graph. Recently, Nagesh et al. [8] determined the edge irregularity strength of line graph and line cut-graph of comb graph. Motivated by the studies above, in this paper we investigate the edge irregularity strength of ladder graphs, triangular ladder graphs, and diagonal ladder graphs. For the proof technique, we refer to [6, 7, 9].

The following theorem in [1] establishes the lower bound for the edge irregularity strength of a graph G.

Theorem 2.1. *Let $G = (V, E)$ be a simple graph with maximum degree $\Delta(G)$. Then*

$$es(G) \geq max\{\lceil \frac{|E(G)|+1}{2} \rceil, \Delta(G)\}.$$

3. Edge irregularity strength of ladder graph

The *n-ladder graph* is defined as $L_n = P_2 \times P_n$, where P_n is a path on n vertices. The following results in [1] are well known.

Theorem 3.1. *Let P_n be a path on $n \geq 2$ vertices. Then $es(P_n) = \lceil \frac{n}{2} \rceil$.*

Theorem 3.2. *Let C_n be a cycle on $n \geq 3$ vertices. Then* $es(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{for } n \equiv 1 (mod\ 4) \\ \lceil \frac{n}{2} \rceil + 1 & \text{otherwise} \end{cases}$

In the next theorem, we find the exact value of edge irregularity strength of ladder graph.

Theorem 3.3. *Let L_n, $n \geq 1$, be a ladder graph. Then $es(L_n) = \lceil \frac{3n-1}{2} \rceil$.*

Proof. Let $G = L_n$, $n \geq 1$, be a ladder graph. We consider the following three cases.
Case 1: For $G = L_1$, by Theorem 3.1, $es(L_1) = 1$.
Case 2: For $G = L_2$, by Theorem 3.2, $es(L_2) = 3$.
Case 3: For a ladder graph L_n, $n \geq 3$, let us consider the vertex set and the edge set:

$$V(L_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq n\},$$
$$E(L_n) = \{x_i x_{i+1} : 1 \leq i \leq n-1\} \cup \{y_j y_{j+1} : 1 \leq j \leq n-1\} \cup \{x_i y_j : 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Clearly, $|V(L_n)| = 2n$, $|E(L_n)| = 3n-2$, and the maximum degree $\Delta = 3$. According to the Theorem 2.1, $es(L_n) \geq max\{\lceil \frac{3n-1}{2} \rceil, 3\}$. Since $\lceil \frac{3n-1}{2} \rceil > 3$ for $n \geq 3$, $es(L_n) \geq \lceil \frac{3n-1}{2} \rceil$.

To prove the equality, it suffices to prove the existence of an edge irregular $\lceil \frac{3n-1}{2} \rceil$−labeling. Define a labeling on vertex set of L_n as follows:

$$\phi(x_i) = \begin{cases} i & \text{for } 1 \leq i \leq 2 \\ i+j & \text{for } i = 2j+1, j \geq 1 \\ i+2j & \text{for } i = 4j, j \geq 1 \\ i+2j & \text{for } i = 4j+2, j \geq 1 \end{cases}$$

$$\phi(y_j) = \begin{cases} 1 & \text{for } j = 1 \\ j+i & \text{for } j = 2i+1, i \geq 1 \\ j+2i-1 & \text{for } j = 4i-2, i \geq 1 \\ j+2i-1 & \text{for } j = 4i, i \geq 1 \end{cases}$$

The edge weights are as follows:

$$w_\phi(x_i y_j) = \begin{cases} 2 & \text{for } i = j = 1 \\ i+j+l & \text{for } 2 \leq i \leq n, 2 \leq j \leq n, 1 \leq l \leq n-1 \end{cases}$$

For $n \geq 2$ and $n \neq 4l+1, l \geq 1$,

$$w_\phi(x_{4i-3}x_{4i-2}) = 12i-9 \quad \text{for } 1 \leq i \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(x_{4i-2}x_{4i-1}) = 12i-6 \quad \text{for } 1 \leq i \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(x_{4i-1}x_{4i}) = 12i-2 \quad \text{for } 1 \leq i \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(x_{4i}x_{4i+1}) = 12i+1 \quad \text{for } 1 \leq i \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(y_{4j-3}y_{4j-2}) = 12j-8 \quad \text{for } 1 \leq j \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(y_{4j-2}y_{4j-1}) = 12j-5 \quad \text{for } 1 \leq j \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(y_{4j-1}y_{4j}) = 12j-3 \quad \text{for } 1 \leq j \leq \lceil \tfrac{n}{4} \rceil.$$
$$w_\phi(y_{4j}y_{4j+1}) = 12j \quad \text{for } 1 \leq j \leq \lceil \tfrac{n}{4} \rceil.$$

For $n \geq 2$ and $n = 4l+1, l \geq 1$,

$$w_\phi(x_{4i-3}x_{4i-2}) = 12i-9 \quad \text{for } 1 \leq i \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(x_{4i-2}x_{4i-1}) = 12i-6 \quad \text{for } 1 \leq i \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(x_{4i-1}x_{4i}) = 12i-2 \quad \text{for } 1 \leq i \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(x_{4i}x_{4i+1}) = 12i+1 \quad \text{for } 1 \leq i \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(y_{4j-3}y_{4j-2}) = 12j-8 \quad \text{for } 1 \leq j \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(y_{4j-2}y_{4j-1}) = 12j-5 \quad \text{for } 1 \leq j \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(y_{4j-1}y_{4j}) = 12j-3 \quad \text{for } 1 \leq j \leq \lfloor \tfrac{n}{4} \rfloor.$$
$$w_\phi(y_{4j}y_{4j+1}) = 12j \quad \text{for } 1 \leq j \leq \lfloor \tfrac{n}{4} \rfloor.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Hence the vertex labeling ϕ is an edge irregular $\lceil \frac{3n-1}{2} \rceil$−labeling. Therefore, $es(L_n) = \lceil \frac{3n-1}{2} \rceil$. □

4. Edge irregularity strength of triangular ladder graph

A *triangular ladder* graph of order $n \geq 2$, denoted by TL_n, is the graph obtained from a ladder graph by adding single diagonal edge to each rectangle.

In the next theorem, we determine the bounds for the edge irregularity strength of triangular ladder graph.

Theorem 4.1. *Let TL_n, $n \geq 2$, be a triangular ladder graph. Then*

$$\lceil 2n-1 \rceil \leq es(TL_n) \leq 2n.$$

Proof. Let $G = TL_n$, $n \geq 2$, be a triangular ladder graph. We consider the following two cases.

Case 1: For $G = TL_2$, let us consider the vertex set and edge set:

$$V(TL_2) = \{x_i : 1 \leq i \leq 2\} \cup \{y_j : 1 \leq j \leq 2\},$$
$$E(TL_2) = \{x_1x_2, y_1y_2, x_1y_2, x_iy_j : 1 \leq i \leq 2, 1 \leq j \leq 2\}.$$

Clearly, $|V(TL_2)| = 4$, $|E(TL_2)| = 5$, and the maximum degree $\Delta = 3$. According to the Theorem 2.1, $es(TL_2) \geq max\{3, 3\} = 3 = \lceil 2n-1 \rceil$.

For the upper bound, we define a vertex labeling ϕ as follows:

$$\phi(x_1) = 1; \phi(x_2) = 3; \phi(y_1) = 2; \phi(y_2) = 4.$$

The edge weights are as follows:

$$w_\phi(x_1x_2) = 4; w_\phi(y_1y_2) = 6, w_\phi(x_1y_2) = 5,\ w_\phi(x_1y_1) = 3; \text{ and } w_\phi(x_2y_2) = 7.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, the vertex labeling ϕ is an edge irregular $2n-$labeling, i.e., $es(TL_2) \leq 2n$.

Case 2: For a triangular ladder graph TL_n, $n \geq 3$, let us consider the vertex set and the edge set:

$$V(TL_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq n\},$$
$$E(TL_n) = \{x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{y_jy_{j+1} : 1 \leq j \leq n-1\} \cup$$
$$\{x_iy_j : 1 \leq i \leq n-1, 2 \leq j \leq n\} \cup \{x_iy_j : 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Clearly, $|V(TL_n)| = 2n$, $|E(TL_n)| = 4n-3$, and the maximum degree $\Delta = 4$. According to the Theorem 2.1, $es(TL_n) \geq max\{\lceil 2n-1 \rceil, 4\}$. Since $\lceil 2n-1 \rceil > 4$ for $n \geq 3$, $es(TL_n) \geq \lceil 2n-1 \rceil$.

To prove the equality, it suffices to prove the existence of an edge irregular $2n-$labeling. Define a labeling on vertex set of TL_n as follows:

$$\phi(x_i) = 2i-1 \text{ for } 1 \leq i \leq n;\ \phi(y_j) = 2j \text{ for } 1 \leq j \leq n.$$

The edge weights are as follows:

$$w_\phi(x_ix_{i+1}) = 4i \quad \text{for } 1 \leq i \leq n-1.$$
$$w_\phi(y_jy_{j+1}) = 4j+2 \quad \text{for } 1 \leq j \leq n-1.$$
$$w_\phi(x_iy_i) = 4i-1 \quad \text{for } 1 \leq i \leq n.$$
$$w_\phi(x_iy_{i+1}) = 4i+1 \quad \text{for } 1 \leq i \leq n-1.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, the vertex labeling ϕ is an edge irregular $2n-$labeling, i.e., $es(TL_n) \leq 2n$. □

5. Edge irregularity strength of diagonal ladder graph

A *diagonal ladder* graph of order $n \geq 2$, denoted by DL_n, is the graph obtained from ladder by adding two diagonals to each rectangle.

In the next theorem, we determine the bounds for the edge irregularity strength of diagonal ladder graph.

Theorem 5.1. *Let DL_n, $n \geq 2$, be a diagonal ladder graph. Then*
$$\lceil \tfrac{5n-3}{2} \rceil \leq es(DL_n) \leq \lceil \tfrac{5n-1}{2} \rceil.$$

Proof. Let $G = DL_n$, $n \geq 2$, be a diagonal ladder graph. We consider the following two cases.
Case 1: For $G = DL_2$, let us consider the vertex set and edge set:

$$V(DL_2) = \{x_i : 1 \leq i \leq 2\} \cup \{y_j : 1 \leq j \leq 2\},$$
$$E(DL_2) = \{x_1x_2, y_1y_2, x_1y_2, y_1x_2, x_iy_j : 1 \leq i \leq 2, 1 \leq j \leq 2\}.$$

Clearly, $|V(DL_2)| = 4$, $|E(DL_2)| = 6$, and the maximum degree $\Delta = 3$. According to the Theorem 2.1, $es(DL_2) \geq max\{4, 3\} = 4 = \lceil \frac{5n-3}{2} \rceil$.

For the upper bound, we define a vertex labeling ϕ as follows:

$$\phi(x_1) = 1; \phi(x_2) = 5; \phi(y_1) = 2; \phi(y_2) = 3.$$

The edge weights are as follows:

$$w_\phi(x_1x_2) = 6; w_\phi(y_1y_2) = 5, w_\phi(x_iy_i) = 5i - 2, 1 \leq i \leq 2; w_\phi(x_1y_2) = 4, w_\phi(y_1x_2) = 7.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, the vertex labeling ϕ is an edge irregular $\lceil \frac{5n-1}{2} \rceil-$ labeling, i.e., $es(DL_2) \leq \lceil \frac{5n-1}{2} \rceil$.
Case 2: For a diagonal ladder graph DL_n, $n \geq 3$, let us consider the vertex set and the edge set of DL_n:

$$V(DL_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_j : 1 \leq j \leq n\},$$
$$E(DL_n) = \{x_ix_{i+1} : 1 \leq i \leq n-1\} \cup \{y_jy_{j+1} : 1 \leq j \leq n-1\} \cup$$
$$\{x_iy_j : 1 \leq i \leq n-1, 2 \leq j \leq n\} \cup \{y_jx_i : 1 \leq j \leq n-1, 2 \leq i \leq n\} \cup$$
$$\{x_iy_j : 1 \leq i \leq n, 1 \leq j \leq n\}.$$

Clearly, $|V(DL_n)| = 2n$, $|E(DL_n)| = 5n - 4$, and the maximum degree $\Delta = 5$. According to the Theorem 2.1, $es(DL_n) \geq max\{\lceil \frac{5n-3}{2} \rceil, 5\}$. Since $\lceil \frac{5n-3}{2} \rceil > 5$ for $n \geq 3$, $es(DL_n) \geq \lceil \frac{5n-3}{2} \rceil$.

To prove the equality, it suffices to prove the existence of an edge irregular $\lceil \frac{5n-1}{2} \rceil$–labeling. Define a labeling on vertex set of DL_n as follows:

$$\phi(x_1) = 1;\ \phi(x_{2i}) = 5i \text{ for } i \geq 1;\ \phi(x_{2i+1}) = 4i + l + 2 \text{ for } i \geq 1, l \geq 0.$$
$$\phi(y_j) = j + 1 \text{ for } 1 \leq j \leq 2;\ \phi(y_{2j+1}) = 5j + 2 \text{ for } j \geq 1;$$
$$\phi(y_{2(j+1)}) = 4j + l + 4 \text{ for } j \geq 1, l \geq 0.$$

The edge weights are as follows:

$$w_\phi(x_ix_{i+1}) = 5i + 1 \quad \text{for} \quad 1 \leq i \leq n-1.$$
$$w_\phi(y_jy_{j+1}) = 5j \quad \text{for} \quad 1 \leq j \leq n-1.$$
$$w_\phi(x_iy_i) = 5i - 2 \quad \text{for} \quad i \geq 1.$$
$$w_\phi(x_1y_2) = 4; w_\phi(x_{2i}y_{2i+1}) = 10i + 2 \quad \text{for} \quad i \geq 1.$$

$$w_\phi(x_{2i+1}y_{2i+2}) = 10i + 4 \quad \text{for} \quad i \geq 1.$$
$$w_\phi(y_{2i-1}x_{2i}) = 10i - 3 \quad \text{for} \quad i \geq 1.$$
$$w_\phi(y_{2i}x_{2i+1}) = 10i - 1 \quad \text{for} \quad i \geq 1.$$

On the basis of above calculations we see that the edge weights are distinct for all pairs of distinct edges. Therefore, the vertex labeling ϕ is an edge irregular $\lceil \frac{5n-1}{2} \rceil-$ labeling, i.e., $es(DL_n) \leq \lceil \frac{5n-1}{2} \rceil$. □

6. Conclusion

In this paper, we investigated the edge irregularity strength, as a modification of the well-known irregularity strength, total edge irregularity strength and total vertex irregularity strength. We determined the edge irregularity strength of ladder graphs, triangular ladder graphs, and diagonal ladder graphs. However, to determine the edge irregularity strength of different ladder graphs such as circular triangular ladder graphs, mobius ladder graphs, etc, still remain open.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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