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GEODETIC PARAMETERS IN TREE DERIVED ARCHITECTURES

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Abstract. A geodetic cover of a graph G refers to a subset of vertices of G that covers all the vertices of G using the shortest paths between the vertices in the set and the cardinality of a minimum geodetic cover is the geodetic number of the graph G, denoted by $g(G)$. This paper is devoted to the study of the geodetic number, the strong geodetic number, the edge geodetic number and the strong edge geodetic number of certain tree derived architectures.

Keywords: geodetic number, sibling tree, hypertree, slim tree, l-sibling tree, l-complete binary tree.

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1. Introduction

A connected graph which is acyclic is referred to as a tree. A tree with each vertex having at most two children is referred to as the binary tree. Binary trees have a wide range of applications in data structures, because they are easy to store, manipulate, and retrieve. Binary tree derived architectures are considered in this paper. There are numerous problems in literature related to shortest path. The classic geodetic number problem is to find a subset of vertices of minimum order such that all the vertices of a graph are covered using the chosen vertices and the shortest paths between them. In 1993, Harary, Loukakis and Tsouros introduced the geodetic number problem [1]. They found the geodetic number of simple graphs such as complete graphs, stars, cycles, wheel graphs, complete bipartite graphs and meshes. Geodetic number finds its application in location theory, fixed point theory and game theory. In 2007, the edge version of the problem was introduced [2]. Chartrand and Zhang initiated the study of geodetic number of oriented graphs [3]. The computational complexity of the problem has been widely discussed by Harary, Loukakis and Tsouros [1] and Atici [4]. Tree derived architectures have been explored already with respect to many parameters such as decycling number, cycle packing

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number [5] and metric dimension [6]. In this paper certain geodetic parameters of some of the tree derived architectures are studied. The following is a overview of the paper: In section 2, basic concepts are given. Sections 3 to 6 deal with the main results and section 7 gives the conclusion.

2. Basic concepts

In this paper, $G = (V, E)$ denotes a connected graph with $|V| \geq 2$ and M denotes a subset of V. Throughout this paper, the strong version of the geodetic set(edge geodetic set) refers to the strong geodetic set(strong edge geodetic set). The number of edges in the shortest path connecting two vertices u and v determines the *distance* between them [7]. A geodesic or an isometric path refers to a shortest path. The maximum of all the distances from v to any other vertex is the *eccentricity* $e(v)$ of v. The *diameter* $diam(G)$ is the length of any longest geodesic or the maximum eccentricity of the vertices [7]. If $e(v) = diam(G)$, then v is referred to as a *peripheral vertex* and the set of all such vertices is referred to as the *periphery* [7]. The vertices that are adjacent to a vertex v are referred to as its *neighbors. Extreme vertices* are those vertices whose neighbors induce a subgraph which is complete [7]. A graph is *bigeodetic* if there are at most 2 isometric paths connecting every pair of vertices [7]. A tree in which one vertex is distinguished as the root, is called a *rooted tree* and any vertex of degree one other than the root, constitutes a *leaf* in a rooted-tree [8]. The distance from the root to a vertex, determines the *level* of the vertex in a rooted tree [8]. If the root of a rooted tree is of degree 2, and the internal vertices possess degree at most 3, then it is termed as a *binary tree* [8]. A binary tree with all its internal vertices having exactly 3 neighbors is referred to as a *complete binary tree* [8].

Definition 2.1. [7] *If every vertex of G is covered by the geodesics joining some pair of vertices in M then, M is said to be a geodetic cover of G. The geodetic cover of minimum cardinality and the minimum cardinality of its geodetic covers are referred to as the geodetic basis of G and the geodetic number $g(G)$, respectively.*

Definition 2.2. [9] *For a graph G, M is referred to as a strong geodetic set, if all the vertices of G are covered using the shortest paths between the vertices in M, subject to the condition that one path is fixed between every pair of vertices in M. The minimum cardinality of the strong geodetic sets is referred to as the strong geodetic number of G, denoted by $sg(G)$.*

Definition 2.3. [2] *If every edge of G is covered by a geodesic joining some pair of vertices in M, then M is referred to as an edge geodetic cover of G. An edge geodetic cover of minimum cardinality and the minimum cardinality of its edge geodetic covers are referred to as an edge geodetic basis of G and the edge geodetic number $g_e(G)$, respectively.*

Definition 2.4. [10] *For a graph G, the minimum cardinality of its strong edge geodetic covers is referred to as the strong edge geodetic number of G, denoted by $sg_e(G)$. A strong edge geodetic cover refers to a set $M \subseteq V(G)$ such that, for any pair $x, y \in M$ an isometric path (connecting the vertices x and y), P_{xy} can be fixed such that union of the edges in all such paths is equal to the edge set E.*

3. Hypertrees

The geodetic number, the strong geodetic number and their corresponding edge versions of hypertrees are computed in this section.

Definition 3.1. [5] *A binary tree which is complete gives rise to the hypertree $HT(n)$ with n levels when vertices x and y in the ith level of the binary tree are made adjacent if a difference of 2^{i-1} exist between their labels. In a hypertree, the children of the vertex t receive the label $2t$ and $2t+1$.*

Observation 3.1. *In $HT(n)$, the shortest path from the root to the peripheral vertices is unique.*

We use the following corollary to calculate $sg_e(HT(n))$.

Corollary 3.1. [10] *For a graph G, if E is an edge-cut which is convex, then $sg_e(G) \geq \lceil 2\sqrt{\mid E \mid} \rceil$.*

3.1. **Geodetic parameters of hypertrees.**

3.1.1. *Geodetic number of hypertree $HT(n)$.*

Remark 3.1. *It is trivial that $g(HT(1)) = 3$ as $HT(1)$ is K_3.*

Lemma 3.1. $g(HT(2)) = 3$.

Proof. The root is an extreme vertex in a hypertree, and therefore it is a member of any geodetic cover of $HT(2)$ and hence $g(HT(2)) \geq 2$. There is a unique path from the root to all the vertices and hence choosing one more vertex which is diametrically opposite to the root will not form a geodetic cover of $HT(2)$. See Figure 1(a). Hence $g(HT(2)) > 2$. Next we claim that $g(HT(2)) = 3$. Two vertices u and v are chosen such that they are diametrically opposite and u and v are at distance 2 from the root, since the eccentricity of the root is 2. Obviously the root together with these 2 vertices form a geodetic cover of $HT(2)$, as there are 2 shortest paths of length 3 between these 2 vertices and so the 4 other vertices are covered. Hence $g(HT(2)) \leq 3$. Therefore the chosen set is a geodetic basis. □

Theorem 3.1. *For $n \geq 3$, $g(HT(n)) = 2^{n-1} + 1$.*

Proof. Let M denote the geodetic set of $HT(n)$ and r' denote the root of $HT(n)$. In $HT(n)$ there are 2^{n-2} $HT^*(2)$, where $HT^*(2) = HT(2) \setminus r'$. As the root r' is an extreme vertex it should be included in the set M. Take $M = \bigcup_{i=1}^{2^{n-2}} M_i \cup \{r'\}$, where M_i is a geodetic cover of $HT^*(2)$. Then M will be a geodetic cover of $HT(n)$. Hence $g(HT(n)) \leq 2^{n-1} + 1$.
Suppose $g(HT(n)) < 2^{n-1} + 1$, then a vertex which is removed from the geodetic cover M, will be left uncovered which is a contradiction. Hence $g(HT(n)) = 2^{n-1} + 1$. □

Remark 3.2. *The strong geodetic number of $HT(1)$ and $HT(2)$ are 3 and 4, respectively.*
3.1.2. *Algorithm to find the strong geodetic number of $HT(n)$.*

Input: Hypertree $HT(n)$, $n \geq 3$.
Algorithm: Choose 2^{n-1} consecutive peripheral vertices such that they are in each of the $HT^*(2)$, where $HT^*(2) = HT(2) \setminus r'$ and they are in both the levels $n+1$ and $n+2$, on either side of the edge connecting the vertices 2 and 3. i.e., $\frac{2^{n-1}}{2}$ consecutive vertices in level $n+1$ on one side and $\frac{2^{n-1}}{2}$ consecutive vertices in level $n+2$ on the other side.
Output: The strong geodetic number of $HT(n)$.
Proof of correctness:
The root is an extreme vertex and hence it belongs to any strong geodetic set.
The geodesic from the root to the chosen vertices, covers its neighbors and also covers

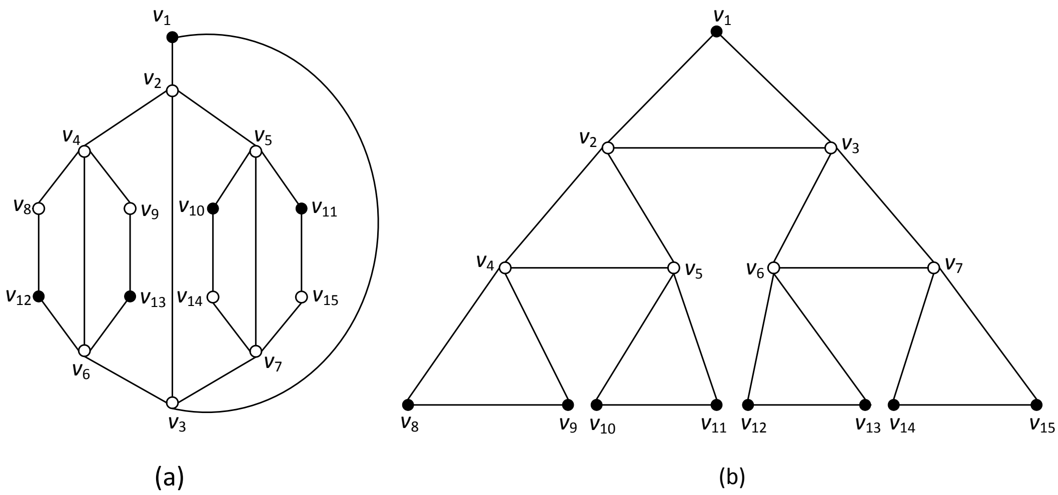

Figure 1: Darkened vertices denote (a) a minimum geodetic set of $HT(3)$ (b) a minimum geodetic set of $ST(3)$

the neighbors of the vertices 2, 3 which lie on the side of the edge connecting the vertices 2 and 3 where the chosen peripheral vertices lie. If we choose a pair of peripheral vertices one on either side of the edge connecting the vertices 2 and 3, there are two disjoint geodesics between each of the chosen peripheral vertices. If vertices in one of the geodesic is considered as covered, then the vertices in the other geodesic will be covered when the geodesics between other pair of vertices are considered. Also only when 2 consecutive peripheral vertices are chosen, their neighbouring peripheral vertices are covered. Hence the set constructed in the above manner is a strong geodetic cover of $HT(n)$ and it is a strong geodetic basis.
As a consequence of the above algorithm we arrive at the following theorem.

Theorem 3.2. *For* $n \geq 3$, $sg(HT(n)) = 2^{n-1} + 1$.

3.1.3. *Edge geodetic parameters of* $HT(n)$.

Theorem 3.3. *For* $n \geq 2$, $g_e(HT(n)) = sg_e(HT(n)) = 2^{n-1} + 1$.

Proof. Let M denote the subset of vertices which covers the edges of $HT(n)$. As the neighbors of the root induce a complete subgraph, any edge geodetic cover of G includes the root. Choose M such that the elements are the peripheral vertices in $HT(n)$ and they are diametrically opposite in each of the $HT^*(2)$. This set forms a geodetic cover which covers the edges of $HT(n)$. Hence $g_e(HT(n)) \leq 2^{n-1} + 1$.

Suppose $g(HT(n)) < 2^{n-1} + 1$, then a vertex which is removed from the edge geodetic cover M, will be left uncovered and thereby the incident edges will be left uncovered, which is a contradiction.

Since the edge-cut E of $HT^*(n)$, (where $HT^*(n) = HT(n) \setminus \text{root}$) is a convex edge-cut, the cardinality of the set consisting of edges in the convex edge-cut including the root gives the upper bound. Root is an extreme vertex and hence it belongs to any strong geodetic set which covers the edges of $HT(n)$. Choose M such that the elements are the peripheral vertices in $HT(n)$ and they are diametrically opposite in each of the $HT^*(2)$. This set forms a strong edge geodetic cover of $HT(n)$ and hence the lower bound. □

4. Sibling trees

Definition 4.1. [5] *A complete binary tree T_n gives rise to the sibling tree ST_n when sibling edges are added between the children of every parent vertex. The label and level of the root vertex are considered to be 1 and 0, respectively. The children of the vertex v receive the label $2v$ and $2v+1$. For $1 \leq i \leq n$, each level i has 2^i vertices.*

4.1. Geodetic parameters of sibling trees.

Theorem 4.1. *For $n \geq 2$, $g(ST_n) = sg(ST_n) = g_e(ST_n) = sge(ST_n) = 2^n + 1$.*

Proof. Let $S^{'}$ denote the set consisting of the root and the vertices in the level n of ST_n. All the vertices in $S^{'}$ are extreme vertices and thereby they are members of any geodetic set, its strong version and any edge geodetic set, its strong version. Since there is a unique geodesic joining every vertex with the other remaining vertices, all the four sets are the same. See Figure 1(b). If M is a geodetic basis, strong geodetic basis, edge geodetic basis, and strong edge geodetic basis, then $|M| \geq |S^{'}|$ where $|S^{'}| = 2^n + 1$.

Next our claim is that $|M| \leq 2^n + 1$. Choose the root and all the 2^n vertices in level n in M. These vertices form a geodetic cover(strong geodetic cover) and edge geodetic cover(strong edge geodetic cover) of ST_n. Hence the claim. Therefore M is a geodetic basis(strong geodetic basis) and edge geodetic basis(strong edge geodetic basis) of ST_n. □

Remark 4.1. *The result holds good for 1-rooted sibling tree.*

4.2. k-rooted sibling tree.

Consider k copies of 1-rooted sibling tree ST_n^1 on 2^n vertices with roots say x_1, x_2,..., x_k. Add the edges $(x_i,\ x_{i+1})$, $1 \leq i \leq k-1$. The resultant graph is a k-rooted sibling tree, ST_n^k. The length of any longest geodesic in ST_n^k is $2n + k - 1$ [5].

4.2.1. *Geodetic parameters of k-rooted sibling tree.*

Theorem 4.2. *For $n \geq 2$, $g(ST_n^k) = sg(ST_n^k) = g_e(ST_n^k) = sg_e(ST_n^k) = k(2^n)$.*

The result is trivial as the set consisting of the extreme vertices in ST_n^k, acts as a geodetic basis, strong geodetic basis, edge geodetic basis and strong edge geodetic basis.

4.3. l-sibling tree.

By adding new edges between vertices in the last level of ST_1^k and the corresponding vertices in the last level of ST_2^k we obtain an l-sibling tree, denoted by l-ST_n^k where ST_1^k, ST_2^k refers to two copies of ST_n^k [5].

4.3.1. *Geodetic parameters of l-sibling tree.*

Theorem 4.3. *For $n \geq 2$, $g(l\text{-}ST_n^k) = g_e(l\text{-}ST_n^k) = 2$.*

Proof. The root vertex in the first copy of the rooted sibling tree ST_1^k and its diametrically opposite root vertex in the other copy of the rooted sibling tree ST_2^k form a geodetic basis and an edge geodetic basis. Hence $g(l\text{-}ST_n^k) = g_e(l\text{-}ST_n^k) = 2$. □

Algorithm to find the strong geodetic number of l-ST_n^k

Input: l-sibling tree l-ST_n^k, $n \geq 2$.

Algorithm: Let M denote the strong geodetic set of l-ST_n^k. Choose the vertices which were extreme vertices in ST_1^k, ST_2^k before adding new edges between each vertex in the last level of ST_1^k and the corresponding vertex of ST_2^k which lies in the last level such that one vertex lies in one tree and its corresponding pair in the diagonal position in the other tree.

Output: Strong geodetic number of l-ST_n^k.

Proof of correctness:

Vertices in l-$ST_n^k \setminus r_k^{'}$ where $r_k^{'} = \{ r_1^{'}, r_2^{'}, r_3^{'},..., r_k^{'} \}$ are connected by atmost 2 isometric paths. Only the vertices in $r_k^{'}$ are connected by more than 2 geodesics. When the vertices are chosen in the above pattern, alternate vertices lie in the same level. The vertices in the same level are connected by an unique geodesic, and there are 2 geodesics between the vertices which doesn't lie in the same level. When the vertices which lie on one half of the graph are covered by the isometric path joining vertices in the same level, the vertices lying on the other half are covered by the isometric path joining vertices which doesn't lie in the same level. Thus the vertices in the set M form a strong geodetic basis of l-ST_n^k. As a consequence of the above algorithm, we arrive at the following theorem.

Theorem 4.4. *For $n \geq 2$, $sg(l\text{-}ST_n^k) = k(2^n)$.*

Theorem 4.5. *For $n \geq 2$, $sg_e(l\text{-}ST_n^k) = k(2^n + 2^{n-1})$.*

Proof. Let M denote the strong edge geodetic set of l-ST_n^k. By definition, l-ST_n^k contains ST_1^k, ST_2^k where ST_1^k, ST_2^k refers to two copies of ST_n^k. Choose the two extreme vertices such that one vertex lies in ST_1^k and the other vertex lies in ST_2^k, before adding new edges between each vertex in the last level of ST_1^k and the corresponding vertex in the last level of ST_2^k. The set M forms a strong edge geodetic cover of l-ST_n^k. Hence $sg_e(l\text{-}ST_n^k) \leq k(2^n + 2^{n-1})$. Next our claim is that, M is a strong edge geodetic basis. For if $sg_e(l\text{-}ST_n^k) < k(2^n + 2^{n-1})$, then there exists a strong edge geodetic cover M with $|M| = k(2^n + 2^{n-1}) - 1$. If one vertex is omitted from the set M, then atleast one edge of l-ST_n^k will be left uncovered. Hence M is a strong edge geodetic basis. $\square$

5. Slim tree

Definition 5.1. [5] *The nth slim tree denoted by $SL(n)$, $n \geq 2$ has a recursive definition as follows:*

1. $SL(2)$ is the complete graph on 3 vertices.

2. The nth slim tree $SL(n)$, with $n \geq 3$ consists of a vertex $x(root)$ and two disjoint copies of $(n-1)th$ slim trees. The vertex set of $SL(n) = (V, E, x, e, f)$ is the union of the vertex sets of the two $(n-1)th$ slim trees and x. The edge set consists of the union of the edge sets of the two $(n-1)th$ slim trees together with the edges $\{(x, x_1), (x, x_2), (f_1, e_2)\}$, where x, e and f stand for the root, left and right vertex, respectively.

Observation 5.1. *$SL(n)$ is a bigeodetic graph.*

5.1. **Geodetic parameters of slim tree.**

5.1.1. *Algorithm to find the geodetic number and the strong geodetic number of SL_n.*

Input: Slim tree SL_n, $n \geq 4$.

Algorithm: Let the geodetic set of SL_n be denoted by M. The neighbors of the left vertex and the neighbors of the right vertex induce a complete subgraph and hence the left and the right vertices are members of any geodetic set and strong geodetic set. To cover the remaining vertices,

i) vertices in the level $(n-1)$ of SL_n are chosen such that, they are the left and right child of the parents in the $(n-2)th$ level (which are from the same parent in the $(n-3)rd$ level) and

ii) vertices in the level n are chosen such that they are the vertices l and r of $SL(n-2)s$.
Output: Geodetic number and strong geodetic number of $SL(n)$.
Proof of correctness:
Each $SL(n)$ contains 2 $SL(n-1)s$, 4 $SL(n-2)s$. There is a unique geodesic between the left and right vertices and hence the vertices (boundary vertices) which lie on this geodesic are covered. The vertices in the level $(n-1)$ are covered only when they are chosen in the geodetic set or when one of their corresponding children are chosen in the geodetic set. Hence the set constructed in the above manner is a geodetic cover and strong geodetic cover of $SL(n)$ and it is a geodetic basis and strong geodetic basis of $SL(n)$.
As a consequence of the above algorithm we arrive at the following theorem.

Theorem 5.1. *For $n \geq 4$, $g(SL_n) = sg(SL_n) = 2^{n-2}$.*

5.1.2. *Algorithm to find the edge geodetic number and the strong edge geodetic number of SL_n.*
Input: Slim tree SL_n, $n \geq 3$.
Algorithm: Choose the root, the extreme vertices l, r and all the vertices in the level n in the set M.
Output: Edge geodetic number and strong edge geodetic number of SL_n.
Proof of correctness:
Let M denote the geodetic set which covers the edges of $SL(n)$. As the vertices l and r are extreme vertices, they belong to M. There is a unique shortest path from the root to all the other vertices. Hence M acts as a strong geodetic set also. The geodesic from root to the chosen vertices covers all the edges connecting vertices in one level to vertices in the next level. Hence only when all the vertices in level n are chosen, the edges connecting level $(n-1)$ and level n and edges in the level n will be covered. Hence the set constructed in the above manner is an edge geodetic cover and a strong edge geodetic cover of $SL(n)$ and is an edge geodetic basis and a strong edge geodetic basis of $SL(n)$, respectively.
As a consequence of the above algorithm we arrive at the following theorem.

Theorem 5.2. *For $n \geq 3$, $g_e(SL(n)) = sg_e(SL(n)) = 2^{n-1} + 1$.*

6. l-COMPLETE BINARY TREE

Definition 6.1. [5] *Let T_k be a complete binary tree, $k \geq 1$. A graph which is obtained from two copies of complete binary tree T_k, say T_1, T_2 by joining each vertex in the last level (i.e., (k-1)th level) of T_1 and the corresponding vertex of T_2 is called the l-complete binary tree and its denoted by l-T_k.*

6.1. Geodetic parameters of l-complete binary tree.

Theorem 6.1. *For $k \geq 2$, $g(l\text{-}T_k) = g_e(l\text{-}T_k) = 2$.*

Proof. The root and the vertex in the last level of the l-complete binary tree form a geodetic basis and an edge geodetic basis of l-T_k. Hence $g(l\text{-}T_k) = g_e(l\text{-}T_k) = 2$. □

Remark 6.1. *l-T_k is a bigeodetic graph.*

Theorem 6.2. *For $k \geq 2$, $sg(l\text{-}T_k) = sg_e(l\text{-}T_k) = 2^k$.*

Proof. Let M denote the strong version of the geodetic set and the strong version of the edge geodetic set of l-T_k. There are 2^k vertices in the kth level of l-T_k and they are peripheral vertices. Choose them in the set M. Since l-T_k is a bigeodetic graph, there are atmost two geodesics joining any two vertices in the graph, in particular between the vertices chosen. All the remaining vertices of l-T_k, are internal vertices in atleast one of the isometric paths between the chosen vertices. By fixing one path between any two vertices in the set chosen, all the vertices and edges of l-T_k are covered and therefore M is a strong geodetic cover and strong edge geodetic cover and hence $|M| \leq 2^k$. We claim that $|M| \nless 2^k$. For if $|M| < 2^k$, i.e., if any one vertex from the kth level is not considered in the set M, then that vertex and the corresponding edges incident with it would be left uncovered. Therefore $sg(l\text{-}T_k) = sg_e(l\text{-}T_k) = 2^k$. □

7. Conclusions

The geodetic parameters of certain tree derived architectures are obtained. We have observed that in all the architectures at least two parameters are found to be equal.

References

[1] Harary, F., Loukakis, E. and Tsouros, C., (1993), The geodetic number of a graph, Mathematical and Computer Modelling, 17(11), pp. 89-95.

[2] Santhakumaran, A. P. and John, J., (2007), Edge geodetic number of a graph, Journal of Discrete Mathematical Sciences and Cryptography, 10(3), pp. 415-432.

[3] Chartrand, G. and Zhang, P., (2000), The geodetic number of an oriented graph, European Journal of Combinatorics, 21(2), pp. 181-189.

[4] Atici, M., (2002), Computational complexity of geodetic set, International journal of computer mathematics, 79(5), pp. 587-591.

[5] Paul, D., Rajasingh, I. and Rajan, R. S., (2015), Tree derived architectures with decycling number equal to cycle packing number, Procedia Computer Science, 57, pp. 716-726.

[6] Manuel, P., Rajan, B., Rajasingh, I. and Chris, M. M., (2011), Landmarks in binary tree derived architectures, Ars Combinatoria, 99, pp. 473-486.

[7] Buckley, F. and Harary, F., (1990), Distance in graphs, Addison-Wesley, Redwood City.

[8] Xu, J., (2013), Topological structure and analysis of interconnection networks, Springer Science and Business Media.

[9] Manuel, P., Klavzar, S., Xavier, A., Arokiaraj, A. and Thomas, E., (2018), Strong geodetic problem in networks, Discussiones Mathematicae Graph Theory, 40(1), pp. 307-321.

[10] Manuel, P., Klavzar, S., Xavier, A., Arokiaraj, A. and Thomas, E., (2017), Strong edge geodetic problem in networks, Open Mathematics, 15(1), pp.1225-1235.

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