TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 738-755

SOLVING INVERSE PARTIAL DIFFERENTIAL EQUATIONS PROBLEMS BY USING TEACHING LEARNING BASED OPTIMIZATION ALGORITHM

A. A. BOROUJENI[1], R. POURGHOLI[1*], S. H. TABASI[1], §

Abstract. In this paper, a numerical approach is combined with teaching learning based optimization algorithm (TLBO) to solve problems of inverse partial differential equations. The most important point in the way we have presented for these problems, for example an inverse heat condition problem (IHCP), is that we can get an answer to them without any conjecture about the unknown. Numerical experiments in implementation of this method, even without guessing the type of unknown function of the problem, show that these problems can be solved with great accuracy. Accurate results also show that after solving the problems, an excellent estimation of some unknowns can be obtained.

Keywords: Inverse partial differential equations problems, TLBO Algorithm, Optimization, Fully implicit method.

AMS Subject Classification: 83-02, 99A00.

1. Introduction

In many problems of partial equations we encounter a problem in which the problem conditions are known and the main equation is solved. In addition to these problems, there are another set of problems where in addition to the unknown principal factor, there are other unknowns in the equation or in terms of it. Such problems are called inverse problems [37]. Inverse Problems in modeling many issues arise in the physical and engineering fields, such as heat conduction problems [2, 3], infiltration issues such as soil contamination, laser cutting, mechanical problems, medical imaging and etc [37, 16, 19]. Inverse problems are a subset of indirect measurement problems. In fact, indirect measurements describe the nature of a problem. Inverse problems are difficult because they are very sensitive to the error of the measured value. For a inverse problem a mathematical model can be presented [17, 9, 10, 11]. But the process of solving an inverse problems is very difficult, which

[1] School of Mathematics and Computer Sciences, Damghan University, P.O. Box 36715-364, Damghan, Iran
e-mail: ahmadaliyari@outlook.com; ORCID: https://orcid.org/0000-0002-9318-3920.
e-mail: pourgholi@du.ac.ir; ORCID: https://orcid.org/0000-0002-0110-2856.
[*] Corresponding author.
e-mail: tabasi@du.ac.ir; ORCID: https://orcid.org/0000-0003-4111-5130.
§ Manuscript received: June 12, 2022; accepted: September 27, 2022.

usually does not give the exact answer. Therefore, for solving such problems, approximate methods such as: iterative methods, regularization technique (Tikhonov regularization) [5, 6], random methods and system identification, methods that search for approximate answer in subset of solutions, integrated techniques or direct numerical methods are used [21, 34, 36, 38].
Methods are also provided for one type of these problems such as inverse heat conduction problem (IHCP) and among the most versatile methods can be mentioned: Mollification [4], iterative regularization [35], BFM (Base Function Method) [6], tikhonov regularization [5, 6], HBM (Haar basis method) [7, 8], and the FSM (Function Specification Method) [34]. As mentioned, these methods are approximate and usually do not get the exact answer [12]. For example, regularization technique is used in [6], which is difficult to find the α value of the regularizer or in Haar basis method [7, 8], we face a limitation when the dimension of the operational matrix is high but one of the advantages of this method is the ability to solve non-linear problems. In this paper, a numerical approach is combining with fully-implicit method [13] and teaching learning based algorithm [28, 29] (TLBO) to solve problems of inverse partial differential equations.

2. Inverse Partial Differential Equations Problems (IPDEP)

Problems in engineering, which are defined as equations with partial derivatives or integral equations, are defined by the shape and size of the problem domain, boundary and initial conditions, the physical characteristics of the fluid in question, internal sources, external conditions, and input [14, 37].
If all this information is specified, the problems is straightforward, and it can generally be considered solvable. consider the following inverse parabolic problem.

$$U_t(x,t) = U_{xx}(x,t) + F(x,t), \qquad 0 < x < 1, \quad t > 0 \tag{1a}$$

$$U(x,0) = f(x), \qquad 0 \leqslant x \leqslant 1 \tag{1b}$$

$$U(0,t) = p(t), \qquad 0 \leqslant t \leqslant t_{fin} \tag{1c}$$

$$U(1,t) = q(t), \qquad 0 \leqslant t \leqslant t_{fin} \tag{1d}$$

and the over-specified condition:

$$u(x_1,t) = g(t), \qquad 0 \leq x_1, \leq 1, \quad 0 \leq t \leq t_{fin} \tag{1e}$$

where *f(x)* is a continuous known function, *p(t)* and *q(t)* are infinitely differentiable known functions and t_{fin} represents the final existence time for the time evolution of the problem.
In (IPDE problems), finding unknown conditions is considered. Because by finding an unknown condition, the problem becomes a direct problem that can be solved with desirable accuracy using the methods described. In this paper we approximate the unknown condition of the problem using TLBO algorithm. To do this, we define the cost function as follows that Eq. (2) must be minimum:

$$f(S) = \sum_{j=1}^{m} (U_j - s_j)^2, \tag{2}$$

where U_j ,$j = 1, 2, \cdots, m$ are values calculated at a point $(x,t) = (\alpha, s_j)$ by numerical methods approximated bye the TLBO algorithm and $s_j = s(t_j)$,$j = 1, 2, \cdots, m$ are measured values.

Remark 2.1. Here we examined the problems using the implementation of the fully implicit method [13]. In this method, we first discretize the problem and then solve the algebraic system of equations:

$$
\begin{aligned}
&-ru_{i-1,j+1}+(1+2r)u_{i,j+1}-ru_{i+1,j+1}=u_{i,j}+kF_{i,l}, \qquad j\geq 0,\\
&\qquad i=1,2,\cdots,N-1, && \text{(3a)}\\
&u_{i,0}=f(ih), \qquad i=1,2,\cdots,N-1, \qquad j=0, && \text{(3b)}\\
&u_{0,j}=p(jk), \qquad i=0 \qquad j=0,1,2,\cdots,N-1, && \text{(3c)}\\
&u_{N,j}=q(jk), \qquad j=0,1,2,\cdots,N-1, \qquad Nh=1, && \text{(3d)}
\end{aligned}
$$

where $r=\dfrac{k}{h}$, $x=ih$ and $t=jk$. By using Eq. (3) we obtain the following linear algebraic system of equations:

$$
\begin{pmatrix}
1+2r & -r & 0 & 0 & 0 & 0 & 0\\
-r & 1+2r & -r & 0 & 0 & 0 & 0\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
. & . & . & . & . & . & .\\
0 & 0 & 0 & 0 & -r & 1+2r & -r\\
0 & 0 & 0 & 0 & 0 & -r & 1+2r
\end{pmatrix}
\begin{pmatrix}
u_{1,j+1}\\ u_{2,j+1}\\ .\\ .\\ .\\ u_{N-2,j+1}\\ u_{N-1,j+1}
\end{pmatrix}
\begin{pmatrix}
-rp_{j+1}\\ 0\\ .\\ .\\ .\\ 0\\ -rq_{j+1}
\end{pmatrix}
$$

$$
=\begin{pmatrix}
u_{1,j}\\ u_{2,j}\\ .\\ .\\ .\\ u_{N-2,j}\\ u_{N-1,j}
\end{pmatrix}
\begin{pmatrix}
kF_{1,j}\\ kF_{2,j}\\ .\\ .\\ .\\ kF_{N-2,j}\\ kF_{N-1,j}
\end{pmatrix}. \tag{4}
$$

3. Teaching-Learning Based Optimization (TLBO) Algorithm for solving IPDEP

The teaching learning-based Optimization algorithm is an efficient optimization method that was first introduced by Mr. Rao et al. This method is similar to other existing optimization techniques of an algorithm derived from different natural phenomena. TLBO works based on the influence of a teacher on classroom learning. The algorithm was introduced in 2011 and is one of the new algorithms that can be improved in a variety of ways, as has been done in recent years and the desired results have been obtained. See in [15, 23, 26, 30, 31, 32]

The most common method of evolutionary optimization is genetic algorithm (GA) [20]. However, GA offers an almost optimal near-solution solution to a complex problem with a large number of variables and constraints. This is mainly due to the difficulty in determining control parameters such as population size, crossover rate and mutation rate. Changing the algorithm parameters changes the effect of the algorithm. The same is true of PSO [25], which uses inertial, social, and cognitive weight components. Likewise, Artificial Bee Colony algorithm (ABC) [22] needs control parameters in optimization, such as the number of employees, scouts and onlookers bees. Harmony Search (HS) [24] needs to consider memory coordination, step adjustment, and so on. This can also be seen in the [1, 18], therefore, efforts are being made to develop new optimization techniques that are independent of the algorithm parameters. The TLBO algorithm is free of algorithm parameters, i.e. no algorithmic parameters are required for the algorithm to work. This advantage helps us to solve the problem without guessing the answer and without the need for parameters.

In TLBO the teacher is generally regarded as a highly educated person who shares his knowledge with the learners. Teacher quality affects learners' outcomes. Obviously, a good teacher trains learners so that they can have better results. This algorithm uses an initial population of answers to achieve the overall answer. The population is considered a group of learners or students of a class. A teacher strives to increase the level of classroom knowledge by educating learners, and the student achieves a good grade or rank according to his or her ability. In fact, a good teacher is one who brings his students to the level of his knowledge. The teacher is a person with high knowledge in society who shares his knowledge with his students, so that in the same repetition the best answer (the best member of the population) acts as a teacher. But it is necessary to point out that students acquire knowledge according to the situation of the students present in the class and the quality of education provided by the teacher. Additionally, students learn from the interaction between themselves (group discussions, presentations, etc.) that helps their situation [28, 29].

If we consider two different teachers, T_1 and T_2 in two different classes, the figure 1 shows the distribution of the scores obtained by the students of the two different classes evaluated by teachers. Curves 1 and 2, which obtain from teaching students with the same level of learning and content, represent the scores obtained by students who are taught by the teacher T_1 and T_2. A normal distribution is considered for the symptoms obtained, but in practice it can be skewed. The normal distribution is defined as follows:

$$f(X) = \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-(x-\mu)^2}{2\sigma^2} \tag{5}$$

where σ^2 is the variance, x is any value of which normal distribution function is required and μ is the means.

3.1. **Teaching Phase.** At this stage, a member of the community who is better than the others is selected as a teacher. As you can see in the Fig. 2, The grade point average increases from M_1 to M_2 according to the teacher performance [28, 29].

According to the teacher's knowledge, learners update their knowledge using the following expression,

$$DifferenceMean_i = r_i(M_{new} - T_f M_i) \tag{6}$$

$$X_{new,i} = X_{old,i} + DifferenceMean_i \tag{7}$$

Where T_F is a teaching factor and r_i is a random value in the range [0, 1].

3.2. **Learning Phase.** At this stage, students interact randomly with others to enhance their knowledge, so that the learner interacts randomly with other learners through group discussions, presentations, formal communication, etc., and learners interact with others. They increased their knowledge among themselves. Learner updates their knowledge according to the difference with another learner by using the following expression [28, 29]:

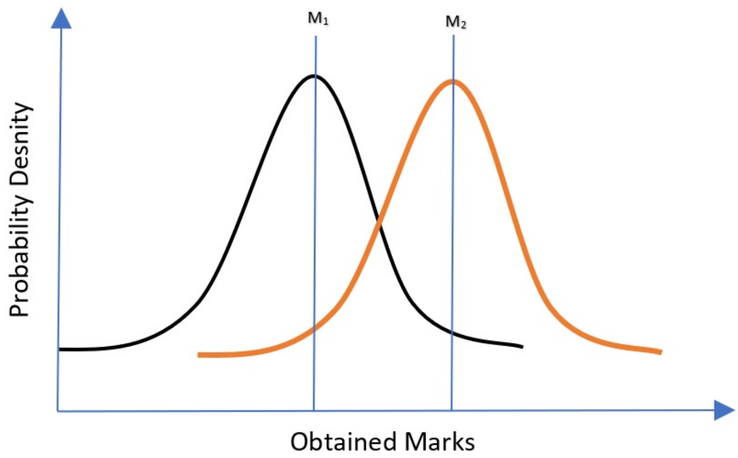


FIGURE 1. Distribution of marks obtained by learners taught by two different teachers.

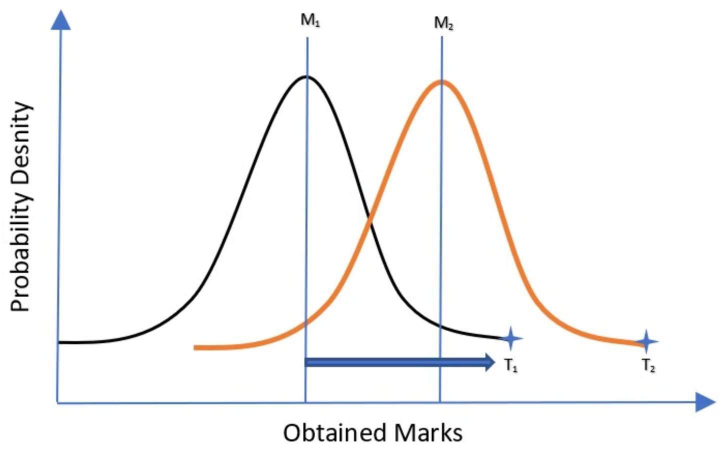


FIGURE 2. Model for obtained marks distribution for a group of learners.

$$x_i^{new} = x_i + r(x_j - x_i) \qquad iff(x_i) \leq f(x_j) \tag{8a}$$
$$x_i^{new} = x_i + r(x_i - x_j) \qquad iff(x_i) > f(x_j) \tag{8b}$$

The overall flowchart of teaching-learning based optimization algorithm is shown in Fig. 3.

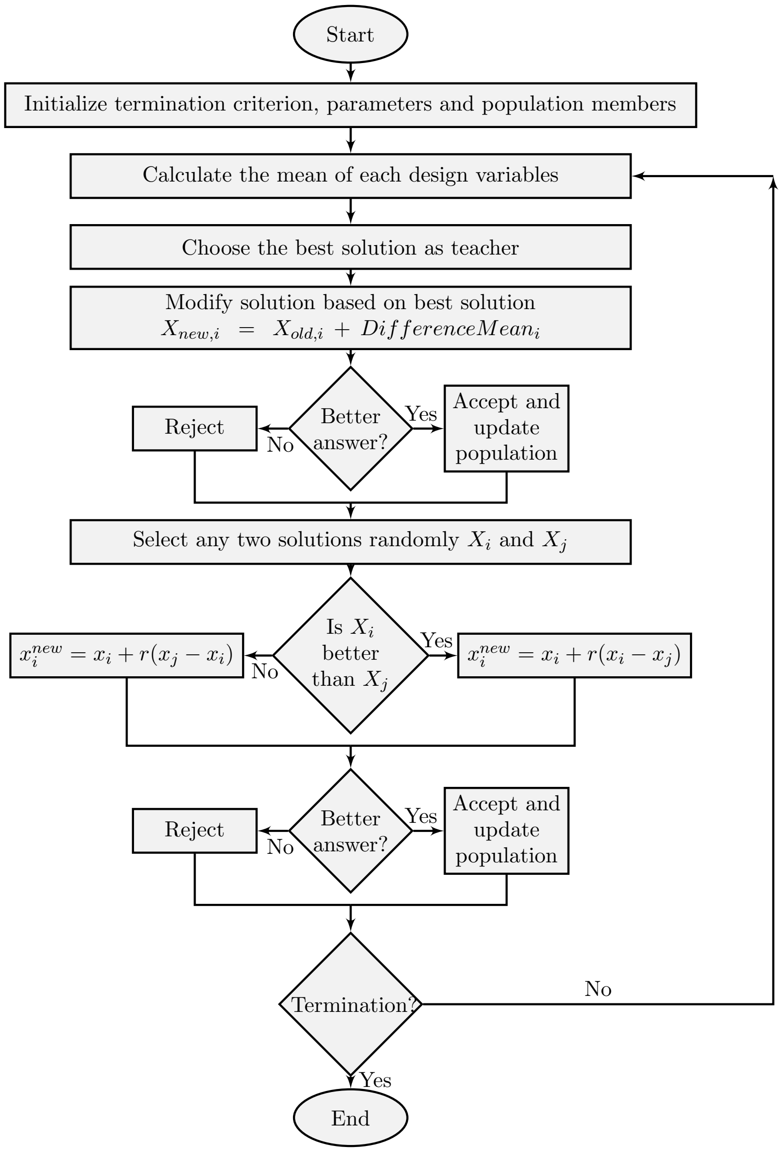


FIGURE 3. Flowchart showing the working of TLBO algorithm.

In this study for determining $q(t)$, TLBO is considered for solving IPDEP. Consider students:

$S_i = \{s_{i,1}, s_{i,2}, ..., s_{i,m}\}, \qquad i = 1, 2, 3, ..., n.$

In the methods that have been proposed to solve IPDEP to obtain $q(t)$, they guess the type of function $q(t)$ [27, 12] and continue to checking it until the answer is obtained with the desired accuracy. But then we show that even without guessing the unknown answer, we get exact values for it.

4. Solving IPDEP without guessing q(t)

As mentioned, in the methods proposed to solve problem IPDEP so far, for determining the unknown of the problem, for example $q(t)$ here, $q(t)$ was considered as a polynomial or exponential function [27, 12] and then, they estimates coefficients of function $q(t)$, but in this study, we determine $q(t)$ without considering the type of function $q(t)$.

We have used the TLBO algorithm with real values. In this algorithm, we consider each student as a string of real numbers in which each element of the string represents the value of the unknown condition or conditions at the corresponding point. In this study, we use TLBO algorithm and estimate each member of the class at times m to determine $q(t)$. Each student estimates value of $q(t)$ at t_j, $j = 1, 2, 3, ..., m$. Solving then direct partial differential equations problem by fully-implicit method and selecting the best student by considering Eq. 2. We run the algorithm until we get the desired answer and then for determining $q(t)$ we find the interpolation of m-points of the best student. The steps of the TLBO algorithm for determining $q(t)$ can be divided into the following steps:

Step 1. Generate the initial population of the class at random.
Step 2. Evaluate fitness of each class members.
Step 3. Perform the teaching phase by selecting the best member with better fitness as a teacher.
Step 4. Perform the learning phase. Learners increase their knowledge by interacting with each other during the learning phase. We select two variables at random and quantify the learner according to the explanations given in the learning phase.
Step 5. Evaluate fitness value of new members and update main class population based on the fitness of all members.
Step 6. Repeat steps 2 to 5 to find an acceptable member. The end condition of the TLBO algorithm can be determined by the problem or conditions such as the execution time of the algorithm or the failure to change the best answer for a certain number of production steps.

Remark 4.1. We set the problem optimization parameters, i.e. population size, number of generations, number of variables and the desired interval for $q(t)$. The range for $q(t)$ can be considered the range of high values or the range of low values, which if we consider the range of values is low, our unknown values may not be in that range, and if we consider the range of values is high, we may not get the exact answer.

5. Numerical results and discussion

IPDE problems, such as an Inverse Heat Conduction Problem (IHCP), is examined. In the inverse heat conduction problems, finding the condition or unknown conditions is considered. Because by finding an unknown condition , the problem becomes a direct problem that can be solved with desirable accuracy using the numerical methods described in Section 2. In this study to solve the inverse thermal conductivity problem, we approximate the unknown condition or condition of the problem using the TLBO algorithm.

Now we are going to demonstrate numerically, some of results for the unknown boundary condition and the answer to the problem, i.e. $U(x,t)$ in the 1 (IPDEP). Considering that the IPDE problems is ill-posed, in this section we test the stability of the method presented in section 2.

In this example we consider the following IHCP (one-dimensional) which is discussed in [27], where it is solved in [27] by considering the type of $q(t)$ function and guessing the coefficients of the function, but in the paper we solve the problems without any guessing the type of $q(t)$ function, using the said method.

$$T_t(x,t) = T_{xx}(x,t), \qquad x \in (0,1), \quad t \in (0,1) \tag{9a}$$
$$T(x,0) = \sin(\pi x) + \cos(\pi x), \qquad x \in [0,1] \tag{9b}$$
$$T(0,t) = e^{-\pi^2 t}, \qquad t \in [0,1] \tag{9c}$$
$$T(1,t) = q(t), \qquad t \in [0,1] \tag{9d}$$

and the over-specified condition:

$$T(0.5,t_i) = s(t_i), \quad t_i = i \times 0.1, \quad i = 1,2,...,10, \tag{9e}$$

Where unknown function is the continuous function $q(t)$ as,

$$q(t) = -e^{-\pi^2 t}, \qquad t \in [0,1].$$

And the exact solution of this problem is

$$T(x,t) = e^{-\pi^2 t}(\sin(\pi x) + \cos(\pi x))$$

Remark 5.1. In an IHCP there are two sources of error in the estimation. The first source is the deterministic error (or unavoidable bias deviation). The second source of error is the variance due to the amplification of measurement errors (stochastic error). The global effect of deterministic and stochastic errors are considered is considered as follows, [33].

$$S = \left[\frac{1}{n-1}\sum_{i=1}^{N}(\widehat{q_i} - q_i)^2\right]^{\frac{1}{2}}, \tag{10}$$

Where n is the total number of estimated values, $\widehat{q_i}$ are calculated values from interpolated equation and q_i are exact values of $q(t)$.

In this example to determining the unknown condition, $q(t)$ and get the answer to the problem ($T(x,t)$), using the TLBO algorithm, real values are used. In this example, a population of 30 and 60 class students is generated where the student estimate 10 points of $q(t)$ at $t_i = i \times 0.1, i = 1,2,...,10$, and the described steps, are applied to the population of the class to find unknown conditions and answer the problem. The number of iterations to run the algorithm in both populations of the class to determining the exact answer was 40. In this method, the best student is entered with the least fitness to approximate the unknown condition.

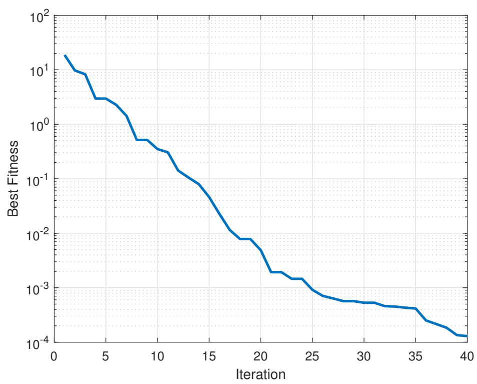
It should be noted that the interval we chose randomly in this issue to create the initial class population was [-20 20], which is a relatively large interval. We know that the interval that boundary condition ($q(t)$) in this problem, is [0 1], but we chose a much larger interval to show the execution power of the proposed method and to show, without any conjecture and dependence on the unknown boundary condition, we determine the unknown of the problem with the desired accuracy. Obviously, if we chose a smaller initial population interval, the rate of convergence to the exact answer would also increase.

Remark 5.2. In all the problems that we examine in this section, we change the number of variables for the class population according to the length of the step and the conditions of the problem. We also consider the teaching factor (T_F) randomly to increase the accuracy of problem solving.

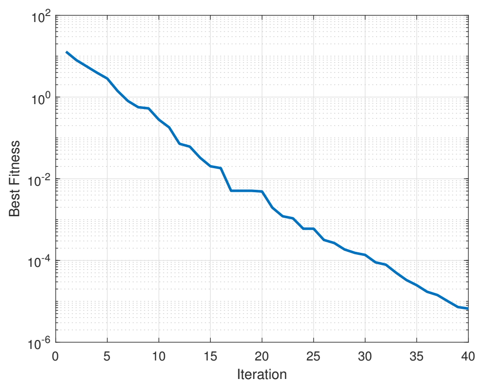
Fig. 4, illustrates the best fitness at each iteration for a population of 30 and 60 students in 40 iterations of the algorithm. The implementation of this method shows that it always tries to improve the answer and with a slight increase in the number of people in the population, the results are obtained with very good accuracy.

Fig. 5(a), illustrates exact solution of this problem $(T(x,t))$ and solution using $q(t)$ estimation using TLBO algorithm with the initial population of 30 and 60 students. In this algorithm, a population of 30 and 60 students is used as the initial population to obtain for numerical results of algorithm for 40 iterations. In this method each student are used to determine $q(t)$ and with the best value of $q(t)$ specified, we solve the problem to determine $(T(x,t))$. Table 1 also presents the numerical values of the answer at 10 time points.

Fig. 5(b) ,illustrates the exact values of $q(t)$ and best estimated $q(t)$ by implementing the TLBO algorithm to interpolate the best student for determining $q(t)$. In this method, a population of 30 and 60 students is used as the initial population to obtain the numerical results; and our algorithm is run for 40 iterations. Table 2 presents the numerical values of the best $q(t)$ at 10 time points. Table 3 also presents the results. We should note that because of high calculations it consumes a long execution time to compute finesses of students.

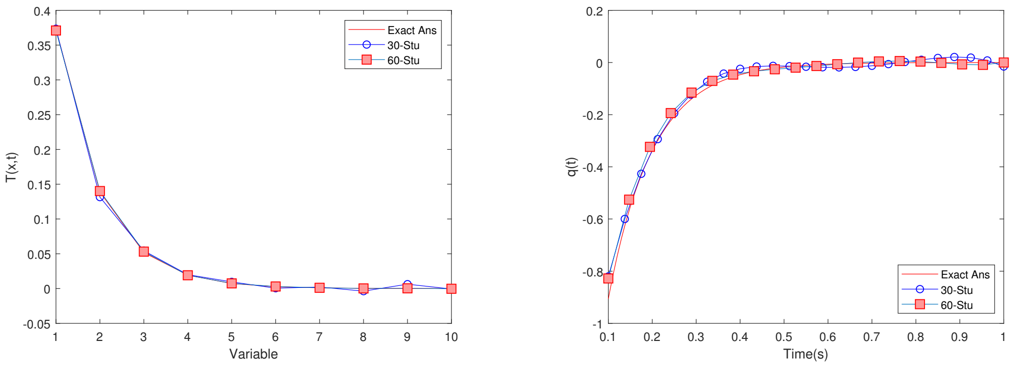


(a) The best amount of fitness in each iteration with a population of 30 students for 40 iteration.



(b) The best amount of fitness in each iteration with a population of 60 students for 40 iteration.

FIGURE 4. The best amount of fitness in each iteration.



(a) Determining $T(x,t)$ by estimating $q(t)$, with a population of 30 and 60 students for 40 iteration

(b) Determining $q(t)$, with a population of 30 and 60 students for 40 iteration.

FIGURE 5. Results of $T(x,t)$ and $q(t)$.

TABLE 1. The answer of the IPDEP (T(x,t)), by using the obtained $q(t)$.

Population	$T(x,t_i) = s(t_i)$				
30	$s(t_1)$ 0.373	$s(t_2)$ 0.131	$s(t_3)$ 0.054	$s(t_4)$ 0.0199	$s(t_5)$ 0.009
	$s(t_6)$ 5.2398E-04	$s(t_7)$ 0.002	$s(t_8)$ -0.003	$s(t_9)$ 0.006	$s(t_{10})$ -6.5535E-04
60	$s(t_1)$ 0.371	$s(t_2)$ 0.140	$s(t_3)$ 0.053	$s(t_4)$ 0.019	$s(t_5)$ 0.007
	$s(t_6)$ 0.002	$s(t_7)$ 0.001	$s(t_8)$ 1.8972E-04	$s(t_9)$ 3.4978E-04	$s(t_{10})$ -2.8148E-04
Exact	s_1 0.374	s_2 0.154	s_3 0.058	s_4 0.0197	s_5 0.005
	s_6 0.001	s_7 1.6605E-04	s_8 -8.6864E-05	s_9 -8.8096E-05	s_{10} -5.1723E-05

TABLE 2. The best student at 40 iteration for population of 30 and 60 students.

Population	s_i				
30	s_1 -0.819	s_2 -0.341	s_3 -0.084	s_4 -0.046	s_5 -0.006
	s_6 -0.021	s_7 0.008	s_8 -0.022	s_9 0.037	s_{10} -0.018
60	s_1 -0.830	s_2 -0.296	s_3 -0.111	s_4 -0.046	s_5 -0.014
	s_6 -0.005	s_7 -0.023	s_8 -0.0019	s_9 0.0010	s_{10} -0.0021
Exact	s_1 -0.610	s_2 -0.215	s_3 -0.075	s_4 -0.026	s_5 -0.009
	s_6 -0.003	s_7 -0.001	s_8 -4.1550E-04	s_9 -1.4659E-04	s_{10} -5.1723E-05

TABLE 3. The results of 40 iteration for determining $q(t)$, for a population of 30 and 60 students.

Population	Iteration	Best fitness	Time(s)
30	40	1.29371E-04	293.444
60	40	6.65814E-06	590.425

Example 2.

In this example let us consider the following one-dimensional inverse parabolic problem [6, 7],

$$T_t(x,t) = T_{xx}(x,t), \qquad 0 < x < 1, \quad 0 < t < 1 \tag{11a}$$

$$T(x,0) = 2\sin(2x) + 2\cos(2x) + \frac{1}{4}x^4, \qquad 0 \leqslant x \leqslant 1 \tag{11b}$$

$$T(0,t) = p(t), \qquad 0 \leqslant t \leqslant 1 \tag{11c}$$

$$T(1,t) = 2e^{-4t}(\cos(2) + \sin(2)) + 3(t^2 + t + \frac{1}{12}), \qquad 0 \leqslant t \leqslant 1 \tag{11d}$$

and the over-specified condition:

$$T(0.2, t_i) = s(t_i), \quad t_i = i \times 0.1, \quad i = 1, 2, ..., 10, \tag{11e}$$

Where unknown function is the continuous function $p(t)$ as,

$$p(t) = 2e^{-4t} + 3t^2, \qquad 0 \leqslant t \leqslant 1.$$

And the exact solution of this problem is

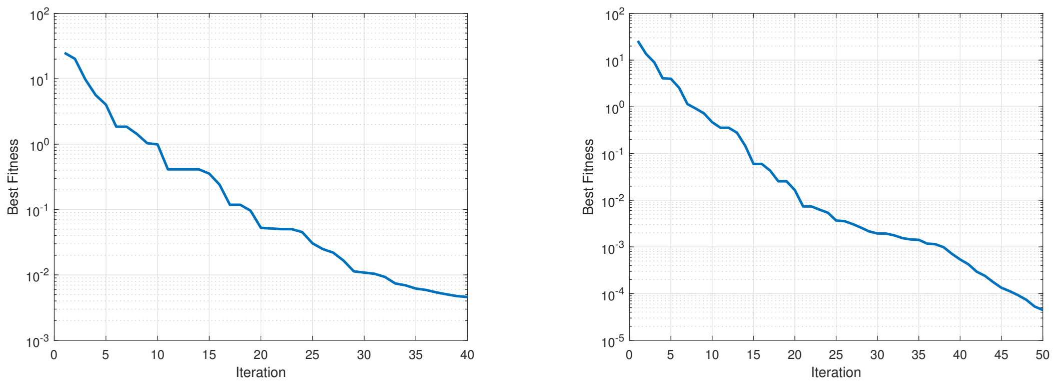
$$T(x,t) = 2e^{-4t}(\sin(2x) + \cos(2x)) + 3(t^2 + tx^2 + \frac{1}{12}x^4)$$

It should be noted that the interval we chose randomly in this issue to create the initial class population was [-10 10], which is a relatively large interval. We know that the interval that boundary condition ($p(t)$) in this problem, is [0 1], but we chose a much larger interval to show the execution power of the proposed method and to show without any conjecture and dependence on the unknown boundary condition, we determine the unknown of the problem with the desired accuracy. Obviously, if we choose a smaller initial population interval, the rate of convergence to the exact answer would also increase.

Fig. 6, illustrates the best fitness at each iteration for a population of 30 and 50 students in 50 iterations of the algorithm. The implementation of this method shows that it always tries to improve the answer and with a slight increase in the number of people in the population, the results are obtained with very good accuracy.

Fig. 7(a), illustrates exact solution of this problem ($T(x,t)$) and solution using $p(t)$ estimation using TLBO algorithm with the initial population of 30 and 60 students. In this algorithm, a population of 30 and 50 students is used as the initial population to obtain for numerical results of algorithm for 40 and 50 iterations. In this method each student are used to determine $p(t)$ and with the best value of $p(t)$ specified, we determine ($T(x,t)$). Table 4 also presents the numerical values of the answer at 10 time points.

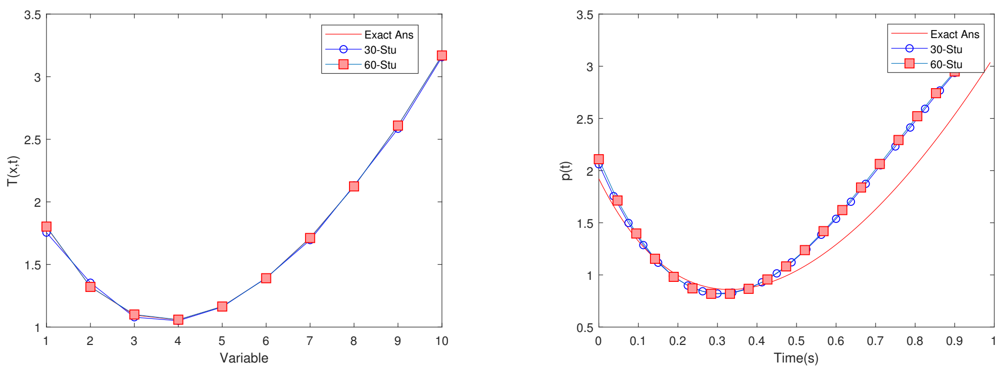
Fig. 7(b), illustrates the exact values of $p(t)$ and best estimated $p(t)$ by implementing the TLBO algorithm to interpolate the best student for determining $p(t)$. In this method, a population of 30 and 50 students is used as the initial population to obtain the numerical results; and our algorithm is run for 40 and 50 iterations. Table 5 presents the numerical values of the best $p(t)$ at 10 time points. Table 6 also presents the results. We should note that because of high calculations it consumes a long execution time to compute finesses of students.



(a) The best amount of fitness in each iteration with a population of 30 students for 40 iteration.

(b) The best amount of fitness in each iteration with a population of 50 students for 50 iteration.

FIGURE 6. The best amount of fitness in each iteration.



(a) Determining $T(x,t)$ by estimating $p(t)$, with a population of 30 and 50 students for 40 and 50 iteration.

(b) Determining $p(t)$, with a population of 30 and 50 students for 40 and 50 iteration.

FIGURE 7. Results of $T(x,t)$ and $p(t)$.

Table 4. The answer of the IPDEP (T(x,t)), by using the obtained $p(t)$.

Population	$T(x,t_i)=s(t_i)$				
30	$s(t_1)$	$s(t_2)$	$s(t_3)$	$s(t_4)$	$s(t_5)$
	1.754	1.354	1.078	1.050	1.160
	$s(t_6)$	$s(t_7)$	$s(t_8)$	$s(t_9)$	$s(t_{10})$
	1.393	1.695	2.130	2.584	3.156
50	$s(t_1)$	$s(t_2)$	$s(t_3)$	$s(t_4)$	$s(t_5)$
	1.802	1.320	1.100	1.059	1.164
	$s(t_6)$	$s(t_7)$	$s(t_8)$	$s(t_9)$	$s(t_{10})$
	1.390	1.711	2.123	2.609	3.169
30	$s(t_1)$	$s(t_2)$	$s(t_3)$	$s(t_4)$	$s(t_5)$
	1.799	1.322	1.095	1.057	1.165
	$s(t_6)$	$s(t_7)$	$s(t_8)$	$s(t_9)$	$s(t_{10})$
	1.390	1.713	-2.123	2.610	3.168

Table 5. The best student at 40 and 50 iteration for population of 30 and 50 students.

Population	s_i				
30	s_1	s_2	s_3	s_4	s_5
	2.099	1.331	0.856	0.846	-0.966
	s_6	s_7	s_8	s_9	s_{10}
	1.218	1.508	1.982	2.393	2.989
50	s_1	s_2	s_3	s_4	s_5
	2.189	1.242	0.912	0.851	0.968
	s_6	s_7	s_8	s_9	s_{10}
	1.211	1.539	1.960	2.443	3.001
Exact q(t)	s_1	s_2	s_3	s_4	s_5
	2	1.319	0.970	,0.860	0.930
	s_6	s_7	s_8	s_9	s_{10}
	1.142	1.472	1.903	2.427	3.036

Table 6. The results of 40 and 50 iteration for determining $p(t)$, for a population of 30 and 50 students.

Population	Iteration	Best fitness	Time(s)
30	40	0.0045981	309.333
50	50	4.4672e-05	645.560

Example 3.

In the next example, we examine and solve one of the non-homogeneous inverse problems, which are considered as inverse problems that are time-consuming and difficult to solve, but can be solved with the desired method with good accuracy:

$$U_t(x,t) = U_{xx}(x,t) + e^{-t^2}(\pi^2\cos(\pi x) + \pi^2\sin(\pi x)) - 2te^{-t^2}(\cos(\pi x) + \sin(\pi x))$$
$$0 < x < 1, \quad 0 < t < 1 \tag{12a}$$
$$U(x,0) = \sin(\pi x) + \cos(\pi x), \qquad 0 \leqslant x \leqslant 1 \tag{12b}$$
$$U(0,t) = e^{-t^2}, \qquad 0 \leqslant t \leqslant 1 \tag{12c}$$
$$U(1,t) = q(t), \qquad 0 \leqslant t \leqslant 1 \tag{12d}$$

and the overspecified condition:

$$T(0.5, t_i) = s(t_i), \quad t_i = i \times 0.1, \quad i = 1, 2, 3, ..., 10, \tag{12e}$$

Where unknown function is the continuous function $q(t)$ as,

$$q(t) = -e^{-t^2}, \qquad 0 \leqslant t \leqslant 1.$$

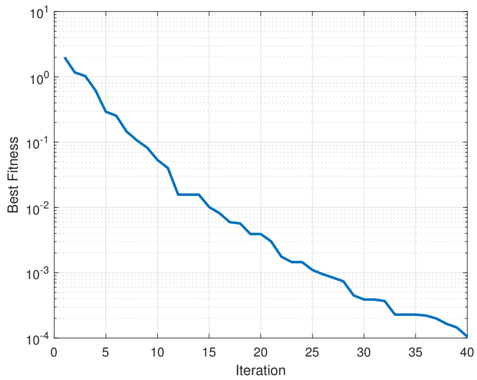
And the exact solution of this problem is

$$U(x,t) = e^{-t^2}(sin(\pi x) + cos(\pi x)).$$

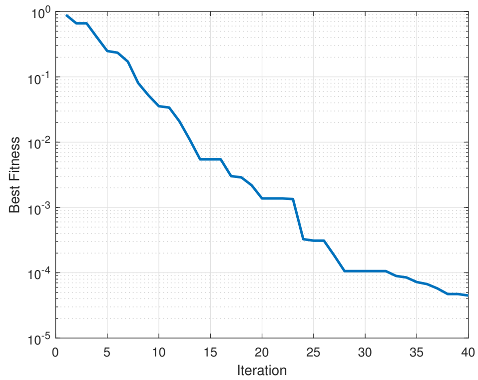
Fig. 8, illustrates the best fitness at each iteration for a population of 30 and 60 students in 40 iterations of the algorithm. The implementation of this method shows that it always tries to improve the answer and with a slight increase in the number of people in the population, the results are obtained with very good accuracy.

Fig. 9(a), illustrates exact solution of this problem ($T(x,t)$) and solution using $q(t)$ estimation using TLBO algorithm with the initial population of 30 and 60 students. In this algorithm, a population of 30 and 60 students is used as the initial population to obtain for numerical results of algorithm for 40 iterations. In this method each student are used to determine $q(t)$ and with the best value of $q(t)$ specified, we solve the problem to determine ($T(x,t)$). Table 7 also presents the numerical values of the answer at 10 time points.

Fig. 9(b) ,illustrates the exact values of $q(t)$ and best estimated $q(t)$ by implementing the TLBO algorithm to interpolate the best student for determining $q(t)$. In this method, a population of 30 and 60 students is used as the initial population to obtain the numerical results; and our algorithm is run for 40 iterations. Table 8 presents the numerical values of the best $q(t)$ at 10 time points. Table 9 also presents the results. We should note that because of high calculations it consumes a long execution time to compute finesses of students.

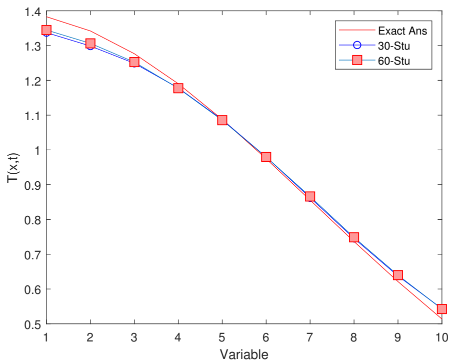


(a) The best amount of fitness in each iteration with a population of 30 students for 40 iteration.

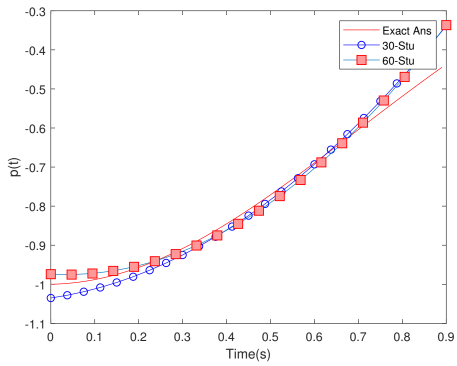


(b) The best amount of fitness in each iteration with a population of 60 students for 40 iteration.

FIGURE 8. The best amount of fitness in each iteration.



(a) Determining $T(x,t)$ by estimating $q(t)$, with a population of 30 and 60 students for 40 iteration.



(b) Determining $q(t)$, with a population of 30 and 60 students for 40 iteration.

FIGURE 9. Results of $T(x,t)$ and $p(t)$.

TABLE 8. The best student at 40 iteration for population of 30 and 60 students.

Population	s_i				
30	s_1 -1.123	s_2 -0.964	s_3 -0.915	s_4 -0.873	s_5 -0.837
	s_6 -0.802	s_7 -0.763	s_8 -0.652	s_9 -0.479	s_{10} -0.269
60	s_1 -0.997	s_2 -0.957	s_3 -0.945	s_4 -0.910	s_5 -0.837
	s_6 -0.787	s_7 -0.707	s_8 -0.661	s_9 -0.467	s_{10} -0.310
Exact	s_1 -1	s_2 -0.987	s_3 -0.951	s_4 -0.894	s_5 -0.820
	s_6 -0.734	s_7 -0.641	s_8 -0.546	s_9 -0.453	s_{10} -0.367

TABLE 7. The answer of the IPDEP (T(x,t)), by using the obtained $q(t)$.

Population	$T(x,t_i)=s(t_i)$				
30	$s(t_1)$ 1.337	$s(t_2)$ 1.298	$s(t_3)$ 1.248	$s(t_4)$ 1.178	$s(t_5)$ 1.086
	$s(t_6)$ 0.979	$s(t_7)$ 0.862	$s(t_8)$ 0.745	$s(t_9)$ 0.637	$s(t_{10})$ 0.543
60	$s(t_1)$ 1.344	$s(t_2)$ 1.306	$s(t_3)$ 1.252	$s(t_4)$ 1.177	$s(t_5)$ 1.085
	$s(t_6)$ 0.979	$s(t_7)$ 0.866	$s(t_8)$ 0.748	$s(t_9)$ 0.639	$s(t_{10})$ 0.542
Exact	s_1 1.382	s_2 1.342	s_3 1.276	s_4 1.190	s_5 1.087
	s_6 -0.974	s_7 0.855	s_8 0.736	s_9 0.621	s_{10} 0.513

TABLE 9. The results of 40 iteration for determining $q(t)$, for a population of 30 and 60 students.

Population	Iteration	Best fitness	Time(s)
30	40	4.7708E-04	442.123
60	40	4.4822e-05	887.848

6. Conclusions

A numerical method to determine unknown boundary condition is proposed for IPDEP and the following results are obtained:

(1) The present study successfully applies the numerical method to IPDEPs.
(2) The most important point in this method that we have presented for IPDEPs, for example an inverse heat condition problem (IHCP), is that we can get an answer to them without any conjecture about the unknown. Numerical experiments in running this algorithm, even without an unknown guess, show that these problems can be solved with great accuracy.
(3) The TLBO algorithm is free of algorithm parameters. This advantage helps us to solve the problem without guessing the unknown so that guessing the values of an unknown function instead of its coefficient using TLBO algorithm, does not increase the execution time of the algorithm.
(4) Information on the numerical interval of the boundary or initial condition of the IPDEPs is usually available, but our numerical results show that even without an exact interval, the unknown of the problem can be determined and interpolated.
(5) Both homogeneous and non-homogeneous IPDEPs can be solved using this method. Also IPDEPs can be solved with this method, whether its boundary condition or its initial condition is unknown.
(6) By implementation this method at pentium (R) CPU 3.70 GHz, numerical results show that an excellent estimation can be obtained within a couple of minutes.

References

[1] Ahrari A., Atai AA., (2010), Grenade explosion method-a novel tool for optimization of multimodal functions, Applied Soft Computing., 10(4), pp. 1132-40.

[2] Jianhua Zhou., Yuwen Zhang., J.K. Chen., Z.C. Feng., (2010), Inverse heat conduction in a composite slab with pyrolysis effect and temperature-dependent thermophysical properties, J. Heat Transfer., 132(3), 034502(3 pages).

[3] Cheng-Hung Huanga., Chun-Ying Yeha., Helcio R.B Orlande., (2003), A nonlinear inverse problem in simultaneously estimating the heat and mass production rates for a chemically reacting fluid, Chemical Engineering Science., 58(16), pp. 3741-3752.

[4] D.A. Murio., (1993), The Mollification Method and the Numerical Solution of Ill-Posed Problems, Wiley-Interscience New York.

[5] Tikhonov, A.N., Arsenin, V.A., (1977), Solution of Ill-posed Problems, Winston and Sons Washington., ISBN 0470991240.

[6] Pourgholi, R., Rostamian, R., (2012), A numerical technique for solving IHCPs using Tikhonov regularization method, J. Math. Chem., 50(8), pp. 2317-2337.

[7] Pourgholi, R., Tavallaie, N., Foadian, S., (2012) Applications of Haar basis method for solving some ill-posed inverse problems, J. Math. Chem., 50(8), pp. 2317-2337.

[8] Pourgholi, R., Foadian, S., Esfahani, A., (2013), Haar basis method to solve some inverse problems for two-dimensional parabolic and hyperbolic equations, TWMS J. App. Eng. Math., 3(1), pp. 10-32.

[9] Azimi, R., Pourgholi, R., Tahmasbi, A., (2021), An efficient method for a class of integro-differential equations with a weakly singular kernel, TWMS J. App. and Eng. Math., 11(4), pp. 1036-1049.

[10] Pourgholi, R., Molai, A. A., Houlari, T., (2013), Resolution of an inverse parabolic problem using sinc-galerkin method, TWMS J. App. Eng. Math., 3(2), pp. 160-181.

[11] Pourgholi, R., Abtahi, M., Tabasi, S. H., (2012), A numerical solution of an inverse parabolic problem, TWMS J. App. Eng. Math., 2(2), pp. 195-209

[12] Dana, H., Pourgholi, R., Houlari, T., (2017), Combining genetic algorithm and sinc-galerkin method for solving an inverse diffusion problem, TWMS J. App. Eng. Math., 7(1), pp. 33-50

[13] Smith, G. D., (1965), Numerical Solution of Partail Differential Equations. Oxford University Press New York.

[14] Tan, S. M., Fox, C., (2009), Physics 707 Inverse Problems, The University of Auckland.

[15] Amiri, B., (2012), Application of teaching-learning-based optimization algorithm on cluster analysis, Journal of Basic and Applied Scientific Research., 2(11), pp. 11795-11802.

[16] Anderssen, R. S., (2005), Inverse problems: A pragmatist's approach to the recovery of information from indirect measurements, Australian and New Zealand industrial and applied mathematics journal., 46, pp. C588-C622, ISSN 1445-8735.

[17] Bakushinsky, A. B., Kokurin, M. Yu., (2004), Iterative methods for approximate solution of inverse problems, Springer, ISBN 1-4020-3121-1.

[18] Blum C., (2005), Ant colony optimization: introduction and recent trends, Physics of Life Reviews., 2, pp. 353-73.

[19] Cannon, J.R., Duchateau, p., (1980). An Inverse Problem for a Nonlinear Diffusion Equation. SIAM J.Appl. Math., 39(2), pp. 272-289.

[20] Goldberg D., (1989), Genetic algorithms in search, optimization, and machine learning, New York Addison-Wesley.

[21] Endure Suli., David F. Mayers., (2003). An Introduction to Numerical Analysis, University of Oxford, Cambridge University Press.

[22] Karaboga D., (2005), An idea based on honey bee swarm for numerical optimization, Technical report-TR06, Erciyes University, Engineering Faculty Computer Engineering Department.

[23] Kunjie Yu., XinWang·, Zhenlei Wang., (2014), An improved teaching-learning-based optimization algorithm for numerical and engineering optimization problems, Springer Science, Business Media New York.

[24] Lee KS., Geem ZW., (2004), A new meta-heuristic algorithm for continuous engineering optimization: harmony search theory and practice, Computer Methods in Applied Mechanics and Engineering., 194, pp. 3902-33.

[25] Kennedy, J., Eberhart, R., (1995), Particle Swarm Optimization, Proceedings of IEEE International Conference on Neural Networks., 4, pp. 1942–1948.

[26] Niknam, T., Azizipanah-Abarghooee, R., Narimani, M. R. (2012), A new multi objective optimization approach based on TLBO for location of automatic voltage regulators in distribution systems, Engineering Applications of Artificial Intelligence., 25(8), pp. 1577-1588.

[27] Pourgholi R., Dana H., Tabasi H., (2014), Solving an inverse heat conduction problem using genetic algorithm: Sequential and multi-core parallelization approach, Applied Mathematical Modelling., 38, pp. 1948-1958.

[28] Rao R.V., Savsani V.J., Vakharia D.P., (2011), Teaching-learning-based optimization: a novel method for constrained mechanical design optimization problems, Comput-Aided Design., 43, pp. 303-15.

[29] Rao R.V., Savsani VJ., Vakharia D.P., (2012), Teaching-Learning-Based Optimization: An optimization method for continuous non-linear large scale problems, Information Sciences., 183, pp. 1-15.

[30] Rao, R. V., Kalyankar, V.D., (2012), Parameter optimization of modern machining processes using teaching-learning-based optimization algorithm, Engineering Applications of Artificial., 26(1), pp. 524-531.

[31] Rao R.V., Patel V., (2013). An improved teaching-learning-based optimization algorithm for solving unconstrained optimization problems, Scientia Iranica., 20(3), pp. 710-720.

[32] Rao R.V., Patel V., (2013), Multi-objective optimization of heat exchangers using a modified teaching-learning-based optimization algorithm, Applied Mathematical Modelling., 37(3), pp. 1147-1162.

[33] J.M. G Cabeza., J.A. M Garcia., A.C. Rodriguez., (2005), A sequential algorithm of inverse heat conduction problems using singular value decomposition, Int. J. Thermal Sci., 44, pp. 235–244.

[34] J.V. Beck., D.C. Murio., (1986), Combined function specification-regularization procedure for solution of inverse heat condition problem, AIAA J., 24, pp. 180-185.

[35] O.M. Alifanov., (1994), Inverse Heat Transfer Problems, Springer New York.

[36] J.R. Shenefelt., R. Luck., R.P. Taylor., J.T. Berry., (2002), Solution to inverse heat conduction problems employing singular value decomposition and modelreduction, Int. J. Heat Mass Transfer., 45, pp. 67-74.

[37] Isakov V., (2017), Inverse Problems for Partial Differential Equations, Applied Mathematical Sciences., 127, DOI 10.1007/978-3-319-51658-5 1.

[38] Reinhardt, H.J., Hao, D. N., Frohne, J., Suttmeier, F.T. (2007), Numerical solution of inverse heat conduction problems in two spatial dimensions, Journal of Inverse and Ill-posed Problems., 15(5), pp. 181-198, ISSN: 0928-0219.

Ahmad Aliyari Boroujeni got his B.Sc. degree in electronic engineer (2015) and his M.Sc. degree in computer science (2020) from the Damghan University, Iran. He is now a Ph.D. student at Damghan University. His area of research is solving IPDE problems with meta-heuristic and machine learning algorithms.

Reza Pourgholi for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.13, N.2.

Seyed Hashem Tabasi got his Ph.D. degree from Kursk State Technical University. He is an associate professor at Damghan University. His area of research is numerical solution of linear and nonlinear inverse heat conduction problems.