TWMS J. App. and Eng. Math. V.14, N.2, 2024, pp. 756-774

GENERALIZED q-DIFFERENCE EQUATION FOR THE GENERALIZED q-OPERATOR $_r\Phi_s(D_q)$ AND ITS APPLICATIONS IN q-INTEGRALS

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Abstract. In 2014, Fang [12] discovered a general q-exponential operator identity by solving a q-difference equation. Fang [12] developed some generalizations of q-integrals using this q-difference equation. Reshem and Saad [20] presented the solution to a generalized q-difference equation in q-operator form, which is a generalization of Fang's work [12]. Using the q-difference equation technique, Reshem and Saad [20] discussed some properties of q-polynomials. In this paper, the generalized q-difference equation technique is used to generalize some well-known integrals such as fractional q-integrals, the q-Barnes contour integral, and Ramanujan q-integrals.

Keywords: q-difference equation, q-operator, q-integral, fractional q-integrals, q-Barnes contour integral, Ramanujan q-integrals

AMS Subject Classification: 05A30, 33D45

1. Introduction

In this paper, the notations that was used in [13] is followed, and we assume that $|q| < 1$. We mention to some notations that we depend on during this paper.

The q-shifted factorial is defined by [13]:

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \qquad and \qquad (a,q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

Also the multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \ldots (a_m;q)_n.$$

The basic hypergeometric series $_t\phi_s$ is given by [13]:

$$_t\phi_s\left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, b_2, \ldots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_0, a_1 \ldots, a_{t-1};q)_n}{(q, b_1, b_2 \ldots, b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} x^n,$$

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§ Manuscript received: May 06, 2022; accepted: October 10, 2022.

where $q \neq 0$ when $r > s+1$. Note that

$$ {}_{s+1}\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_{s+1} \\ b_1, b_2, \ldots, b_s \end{matrix}; q, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{s+1}; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} x^n, \quad |x| < 1. $$

The q-binomial coefficients is given by [13]:

$$ \begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad 0 \leqslant k \leqslant n. $$

We will use the following identities in this paper [13]:

$$ (a;q)_n = (q^{1-n}/a;q)_n (-a)^n q^{\binom{n}{2}}. \tag{1} $$

$$ (q^{-n};q)_k = \frac{(q;q)_n}{(q;q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk}. \tag{2} $$

The q-Chu-Vandermonde sums are:

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, q\right) = \frac{(c/a;q)_n}{(c;q)_n} a^n. \tag{3} $$

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, a \\ c \end{matrix}; q, cq^n/a\right) = \frac{(c/a;q)_n}{(c;q)_n}. \tag{4} $$

The q-Gauss sum is:

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, c/ab\right) = \frac{(c/a, c/b;q)_\infty}{(c, c/ab;q)_\infty}. \tag{5} $$

Heine's transformation of ${}_3\phi_2$ series [13, Appendix III, equations (III.1),(III.12)] are:

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right) = \frac{(b, az;q)_\infty}{(c, z;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, z \\ az \end{matrix}; q, b\right). \tag{6} $$

$$ {}_3\phi_2\left(\begin{matrix} q^{-n}, b, c \\ d, e \end{matrix}; q, q\right) = \frac{(e/c;q)_n}{(e;q)_n} c^n {}_3\phi_2\left(\begin{matrix} q^{-n}, c, d/b \\ d, cq^{1-n}/e \end{matrix}; q, \frac{bq}{e}\right). \tag{7} $$

The Thomae-Jackson q-integral [13, 14, 23] is

$$ \int_a^b f(x) d_q x = (1-q) \sum_{n=0}^{\infty} [bf(q^n b) - af(q^n a)] q^n. $$

In 1910, Watson [24] introduced the q-Barnes contour integral

$$ \int_{-i\infty}^{i\infty} \frac{(q^{1+s}, cq^s;q)_\infty}{(aq^s, bq^s;q)_\infty} \frac{\pi(-z)^s}{\sin \pi s} ds = -2i\pi \frac{(q, c;q)_\infty}{(a, b;q)_\infty} {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, z\right). \tag{8} $$

The generalized Riemann-Liouville fractional q-integral is given by [1, 18, 19]

$$ I_{q,a}^{\alpha} f(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x;q)_{\alpha-1} f(t) d_q t, \quad \alpha \in R^+. $$

The discrete fractional differences are studied deeply and extensively by many scientists, see [15, 16, 17].

Two integrals of Ramanujan are [2, 4, 9]

$$ \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx};q)_\infty} dx = \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm};q)_\infty}{(abq;q)_\infty}. \tag{9} $$

$$ \int_{-\infty}^{\infty} e^{-x^2+2mx} (-aqe^{2kx}, -bqe^{-2kx};q)_\infty dx = \sqrt{\pi} e^{m^2} \frac{(abq;q)_\infty}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km};q)_\infty}. \tag{10} $$

The q-differential operator is defined by [11]

$$D_q\{f(b)\} = D_{q,b}\{f(b)\} = \frac{f(b)-f(bq)}{b}. \tag{11}$$

In 2003, Chen et al. [10] defined the homogeneous q-difference operator D_{xy} as follows:

$$D_{xy}\{f(x,y)\} = \frac{f(x,q^{-1}y)-f(qx,y)}{x-q^{-1}y}.$$

In 2014, Cao [3] established the homogeneous q-operator as follows:

$$\mathbb{T}(a, zD_{xy}) = \sum_{k=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} \, (zD_{xy})^n.$$

Cao [3] introduced the following q-difference equation:

Theorem 1.1. [3]. *Let $f(x,y,z)$ be a three variables analytic function in a neighbourhood of $(x,y,z) = (0,0,0) \in \mathbb{C}^3$. If $f(x,y,z)$ satisfies the equation*

$$\begin{aligned}(x-&q^{-1}y)[f(a,x,y,z)-f(a,x,y,qz)]\\ &= z[f(a,x,q^{-1}y,z)-f(a,qx,y,z)] - az[f(a,x,q^{-1}y,qz)-f(a,qx,y,qz)],\end{aligned} \tag{12}$$

then we have

$$f(a,x,y,z) = \mathbb{T}(a, zD_{xy})\left\{f(a,x,y,0)\right\}.$$

In 2014, Cao [4] defined the generalized q-exponential operator

$$\mathbb{T}\left[\begin{matrix} w,r \\ v \end{matrix} ;q;cD_{q,b}\right] = \sum_{n=0}^{\infty} \frac{(w,r;q)_n}{(q,v;q)_n}\,(cD_{q,b})^n$$

Cao [4] constructed the following q-difference equation:

Theorem 1.2. [4]. *Let $f(w,r,v,b,c)$ be a five-variable analytic function in a neighborhood of $(w,r,v,b,c) = (0,0,0,0,0) \in \mathbb{C}^5$. If $f(w,r,v,b,c)$ satisfies the q-difference equation*

$$\begin{aligned} b[f(w,&r,v,b,c) - (1+q^{-1}v)f(w,r,v,b,cq) + q^{-1}vf(w,r,v,b,cq^2)]\\ =&c\big\{[f(w,r,v,b,c)-f(w,r,v,qb,c)] - (w+r)[f(w,r,v,b,qc)-f(w,r,v,qb,qc)]\\ &+ wr[f(w,r,v,b,q^2c)-f(w,r,v,qb,q^2c)]\big\},\end{aligned} \tag{13}$$

then we have

$$f(w,r,v,b,c) = \mathbb{T}\left[\begin{matrix} w,r \\ v \end{matrix} ;q;cD_{q,b}\right]\{f(w,r,v,b,0)\}.$$

Using equation (13), Cao [4] verified the following Ramanujan integral:

Corollary 1.1. [4]. *For $m \in R, N \in \mathbb{N}$, $r = q^{-N}$ and $|abq| < 1$, we have*

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx};q)_\infty}\, {}_3\phi_1\left(\begin{matrix} r,w,-aqe^{2ikm} \\ qwr/v \end{matrix} ;q, \frac{-q^{1/2}e^{2ik(x-m)}}{v}\right)dx\\ &= \sqrt{\pi}e^{m^2}\frac{(-aqe^{2ikm}, -bqe^{-2ikm}, qw/v, qr/v;q)_\infty}{(abq, qwr/v, q/v;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} r,w \\ v \end{matrix} ;q, \frac{-qb}{e^{2ikm}}\right).\end{aligned} \tag{14}$$

Also, Cao [4] defined the generalized Al-Salam-Carlitz polynomials

$$\Phi_n^{(a)}(b,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (a;q)_k b^k c^{n-k}.$$

$$\Psi_n^{(a,b,c)}(x,y|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(a,b;q)_k}{(c;q)_k}(-1)^k q^{\binom{k+1}{2}-nk}x^k y^{n-k}.$$

In 2014, Fang [12] used the q-difference equation method to generalize Andrews-Askey and Askey-Wilson integrals.

In 2017, Cao and Niu [7] introduced the following q-difference equation:

Lemma 1.1. [7]. *Let $f(a,b,c)$ be a three-variable analytic function at $(0,0,0)\in\mathbb{C}^3$. Then The function f can be expanded in terms of $\Phi_n^{(a)}(b,c|q)$ if and only if f satisfies the following functional equation:*

$$abf(a,bq,cq)-bf(a,b,cq)+(c-ab)f(a,bq,c)=(c-b)f(a,b,c). \tag{15}$$

In 2018, Cao and et al. [8] derived the following fraction q-integral:

Lemma 1.2. [8]. *For $\alpha\in R_0$ and $0<a<x<1$, it is asserted that*

$$I_{q,a}^{\alpha}\left\{\frac{1}{(xt;q)_\infty}\right\}=\frac{(1-q)^\alpha}{(at;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}t^k}{(q;q)_{\alpha+k}},\quad \max\{|at|,|xt|\}<1 \tag{16}$$

$$I_{q,a}^{\alpha}\{(xt;q)_\infty\}=(1-q)^\alpha(at;q)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}}}{(at;q)_k}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}t^k}{(q;q)_{\alpha+k}},\quad |xt|<1. \tag{17}$$

Cao and et al. [8] used the q-difference equation (15) to get the following:

Lemma 1.3. [8]. *For $\alpha\in R_0$, $0<a<x<1$ and max$\{|as|,|at|,|xt|\}<1$ it is asserted that*

$$I_{q,a}^{\alpha}\left\{\frac{(rsx;q)_\infty}{(xt,xs;q)_\infty}\right\}=\frac{(1-q)^\alpha(ars;q)_\infty}{(as,at;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}t^k}{(q;q)_{\alpha+k}}\,{}_3\phi_1\left(\begin{matrix}q^{-k},r,at\\ars\end{matrix};q,\frac{q^k s}{t}\right). \tag{18}$$

In 2019, Cao [5] used equation (12) to prove the following:

Lemma 1.4. [5]. *For $\alpha\in R^+$* and $0<a<x<1$, if *max$\{|as|,|az|\}<1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{(xbz,xt;q)_\infty}{(xs,xz;q)_\infty}\right\}=\frac{(1-q)^\alpha(abz,at;q)_\infty}{(as,az;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_3\phi_2\left(\begin{matrix}q^{-k},as,az\\abz,at\end{matrix};q,q\right). \tag{19}$$

In 2019, Cao [6] obtained the following q-difference equation:

Theorem 1.3. [6]. *Let $f(a,b,c,x,y)$ be a five-variables analytic function in a neighborhood of $(a,b,c,x,y)=(0,0,0,0,0)\in\mathbb{C}^5$. Then $f(a,b,c,x,y)$ can be expanded in terms of $\Psi_n^{(a,b,c)}(x,y|q)$ if and only if*

$$\begin{aligned}&q^{-1}y\big[f(a,b,c,x,y)-(1+q^{-1}c)f(a,b,c,qx,y)+q^{-1}cf(a,b,c,q^2x,y)\big]\\&\quad=x\big\{[f(a,b,c,x,y)-f(a,b,c,x,q^{-1}y)]-(a+b)[f(a,b,c,qx,y)-f(a,b,c,qx,q^{-1}y)]\\&\qquad+ab[f(a,b,c,q^2x,y)-f(a,b,c,q^2x,q^{-1}y)]\big\}.\end{aligned} \tag{20}$$

By using equation (20), Cao [6] proved the q-integral:

Theorem 1.4. [6] *Suppose that* $\max\{|aq^{1/2}e^{2mk}|,|bq^{1/2}e^{-2mk}|,|cq^{1/2}e^{2mk}|\}<1$. *For $m\in R$ and $0<q=e^{-2k^2}<1$, we have*

$$\int_{-\infty}^{\infty}e^{-x^2+2mx}(-aqe^{2kx},-bqe^{-2kx};q)_\infty\,{}_2\phi_2\left(\begin{matrix}r,s\\t,-aqe^{2kx}\end{matrix};q,-qce^{2kx}\right)dx$$

$$= \sqrt{\pi} e^{m^2} \frac{(abq;q)_\infty}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km};q)_\infty} {}_3\phi_2 \left(\begin{matrix} r, s, bq^{1/2}e^{-2km} \\ t, abq \end{matrix} ; q, cq^{1/2}e^{2km} \right). \quad (21)$$

In 2020, Cao and et al. [9] defined the q-operator $\mathbb{T}(a, b, c, d, e, yD_x)$ as:

$$\mathbb{T}(a, b, c, d, e, yD_x) = \sum_{n=0}^{\infty} \frac{(a, b, c; q)_n}{(q, d, e; q)_n} (yD_x)^n,$$

where D_x is the operator D_q acts on x.

Cao and et al. [9] set up the following q-difference equation:

Theorem 1.5. [9]. *Let $f(a, b, c, d, e, x, y)$ be a seven-variable analytic function in a neighborhood of $(a, b, c, d, e, x, y) = (0, 0, 0, 0, 0, 0, 0) \in \mathbb{C}^7$. If $f(a, b, c, d, e, x, y)$ satisfies the difference equation*

$$\begin{aligned} & x\big\{f(a,b,c,d,e,x,y) - f(a,b,c,d,e,x,yq) \\ & \quad - (d+e)q^{-1}[f(a,b,c,d,e,x,yq) - f(a,b,c,d,e,x,yq^2)] \\ & \quad + deq^{-2}[f(a,b,c,d,e,x,yq^2) - f(a,b,c,d,e,x,yq^3)]\big\} \\ = & y\big\{f(a,b,c,d,e,x,y) - f(a,b,c,d,e,xq,y) \\ & \quad - (a+b+c)[f(a,b,c,d,e,x,yq) - f(a,b,c,d,e,xq,yq)] \\ & \quad + (ab+ac+bc)[f(a,b,c,d,e,x,yq^2) - f(a,b,c,d,e,xq,yq^2)] \\ & \quad - abc[f(a,b,c,d,e,x,yq^3) - f(a,b,c,d,e,xq,yq^3)]\big\}, \end{aligned} \quad (22)$$

then

$$f(a, b, c, d, e, x, y) = \mathbb{T}(a, b, c, d, e, yD_x)\{f(a, b, c, d, e, x, 0)\}.$$

They used equation (22) to show the following result:

Theorem 1.6. [9] *For $0 < q = e^{-2k^2} < 1$ and $m \in R$. Suppose that $|abq| < 1$, we have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx};q)_\infty} {}_3\phi_2 \left(\begin{matrix} r, s, t \\ u, v \end{matrix} ; q, yq^{1/2}e^{2ikx} \right) dx$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm};q)_\infty}{(abq;q)_\infty} {}_4\phi_3 \left(\begin{matrix} r, s, t, e^{2ikm}/b \\ u, v, -aqe^{2ikm} \end{matrix} ; q, ybq \right).$$

In 2021, Saad and Hassan [21, 22] introduced the generalized q-operator as follows:

$$\begin{aligned} & F(a_0, a_1 \ldots, a_{t-1}, b_1 \ldots, b_s, cD_{q,b}) \\ & = \sum_{n=0}^{\infty} \frac{(a_0, a_1 \ldots, a_{t-1};q)_n}{(q, b_1, \ldots, b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} (cD_{q,b})^n. \end{aligned}$$

In 2022, Reshem and Saad [20] obtained the following general q-difference equation:

Theorem 1.7. [20] *Let $f(a_0, a_1 \ldots, a_{t-1}, b_1, \ldots, b_s, b, c)$ be an $(t+s+2)$-variable analytic function in a neighborhood of $(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, c) = (0, 0, \ldots, 0) \in \mathbb{C}^{t+s+2}$ satisfying the q-difference equation*

$$\begin{aligned} & (-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^{j+t-s-1}) \\ & \qquad = c \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \left\{ f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^j) \right\}, \end{aligned} \quad (23)$$

where

$$b_0 = q, \quad B_0 = A_0 = 1, \quad B_1 = \sum_{j=0}^{s} b_i, \quad B_2 = \sum_{0\le i<j\le s} b_i b_j$$

$$B_3 = \sum_{0\le i<j<k\le s} b_i b_j b_k, \ldots, \quad B_{s+1} = b_0 b_1 \ldots b_s, \quad A_1 = \sum_{i=0}^{t-1} a_i$$

$$A_2 = \sum_{0\le i<j\le t} a_i a_j, \quad A_3 = \sum_{0\le i<j<k\le t} a_i a_j a_k, \ldots, \quad A_t = a_0 a_1 \ldots a_{t-1}.$$

Then

$$f(a_0, \ldots, a_{t-1}, b_1, \ldots, b_s, b, c) = F(a_0, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b})\{f(a_0, \ldots, a_{t-1}, b_1, \ldots, b_s, b, 0)\}.$$

Lemma 1.5. [20]. *Let $D_{q,b}$ be defined as in (11), then*

$$D^n_{q,b}\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\} = \frac{1}{b^n}\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw \end{matrix}; q; q\right). \tag{24}$$

Theorem 1.8. [20]. *For $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, r, u, v, a_i, b_j \in \mathbb{C}, i = 0, \ldots, t-1, j = 1, \ldots, s,$ we have*

$$F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b})\left\{\frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\right\}$$
$$= \frac{(bw, br; q)_\infty}{(bu, bv; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{c}{b}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} {}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ br, bw \end{matrix}; q, q\right),$$

provided that $\max\{|bu|, |bv|\} < 1$.

Corollary 1.2. [20]. *For $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, u, v, a_i, b_j \in \mathbb{C}, i = 0, 1, \ldots, t-1, j = 1, 2, \ldots, s,$ we have*

$$F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b})\left\{\frac{(bw; q)_\infty}{(bu, bv; q)_\infty}\right\} = \frac{(bw; q)_\infty}{(bu, bv; q)_\infty}$$
$$\times\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{c}{b}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} {}_3\phi_2\left(\begin{matrix} q^{-n}, bu, bv \\ 0, bw \end{matrix}; q, q\right), \tag{25}$$

provided that $\max\{|bu|, |bv|\} < 1$.

$$F(a_0, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b})\left\{\frac{1}{(bu; q)_\infty}\right\}$$
$$= \frac{1}{(bu; q)_\infty}\, {}_t\phi_s\left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix}; q, cu\right), \quad |bu| < 1. \tag{26}$$

- Letting $v = 0$ in (25) and then applying q-Chu-Vandermonde sum (3), we obtain

  Corollary 1.3. *If $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, u, a_i, b_j \in \mathbb{C}, i = 0, \ldots, t-1, j = 1, \ldots, s,$ then*

$$F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b})\left\{\frac{(bw; q)_\infty}{(bu; q)_\infty}\right\}$$
$$= \frac{(bw; q)_\infty}{(bu; q)_\infty}\, {}_{t+1}\phi_{s+1}\left(\begin{matrix} a_0, a_1, \ldots, a_{t-1}, w/u \\ b_1, \ldots, b_s, bw \end{matrix}; q, cu\right), \quad |bu| < 1. \tag{27}$$

- Putting $u = 0$ in (27), we get

Corollary 1.4. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}, b, w, a_i, b_j \in \mathbb{C}, i = 0, 1, \ldots, t-1, j = 1, 2, \ldots, s,$ *we have*

$$F(a_0, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b})\{(bw;q)_\infty\} = (bw;q)_\infty\ {}_t\phi_{s+1}\left(\begin{matrix} a_0, \ldots, a_{t-1} \\ b_1, \ldots, b_s, bw \end{matrix};q, cw\right). \quad (28)$$

In this paper, we generalize some q-integrals by using the method q-difference equation. In section 2, we use the q-difference equation method to generalize fractional q-integrals. In section 3, we extend q-Barnes contour integral as general to this integral. In section 4, we construct a generalizations of Ramanujan integrals by using the q-difference equation technique to offer another types of this integral.

2. Generalization of the Fractional q-Integrals

This section concern with the generalization of fractional q-integrals given in [5, 8] by using q-difference equation method.

Theorem 2.1. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $\alpha \in R^+$ *and* $0 < a < x < 1$, *if* $\max\{|ay|, |az|\} < 1$, *we have*

$$I^{\alpha}_{q,a}\left\{\frac{(xbz, xr;q)_\infty}{(xy, xz;q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1};q)_n}{(q, b_1, \ldots, b_s;q)_n}\left(\frac{c}{x}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}\ {}_3\phi_2\left(\begin{matrix} q^{-n}, xy, xz \\ xbz, xr \end{matrix};q, q\right)\right\}$$
$$= \frac{(1-q)^\alpha(abz, ar;q)_\infty}{(ay, az;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\sum_{n=0}^{k}\frac{(q^{-k}, ay, az;q)_n}{(q, abz, ar;q)_n}q^n$$
$$\times\ {}_{t+1}\phi_{s+1}\left(\begin{matrix} a_0, a_1, \ldots, a_{t-1}, q^{-\alpha-k} \\ b_1, \ldots, b_s, xq^{1-\alpha-k}/a \end{matrix};q, \frac{cq}{a}\right). \quad (29)$$

Proof. Let $f_R = f_R(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, c) =$ RHS of equation (29).

$$(-q)^{1+s-t}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, b, cq^{j+t-s-1})$$
$$= (-q)^{1+s-t}\sum_{j=0}^{s+1}\frac{(-1)^j B_j}{q^j}\frac{(1-q)^\alpha(abz, ar;q)_\infty}{(ay, az;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\sum_{n=0}^{k}\frac{(q^{-k}, ay, az;q)_n}{(q, abz, ar;q)_n}q^n$$
$$\times\ {}_{t+1}\phi_{s+1}\left(\begin{matrix} a_0, a_1, \ldots, a_{t-1}, q^{-\alpha-k} \\ b_1, \ldots, b_s, xq^{1-\alpha-k}/a \end{matrix};q, \frac{cq^{j+t-s}}{a}\right)$$
$$= \frac{(1-q)^\alpha(abz, ar;q)_\infty}{(ay, az;q)_\infty}\sum_{m=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1};q)_m}{(q, b_1, \ldots, b_s;q)_m}\left[(-1)^{m+1}q^{\binom{m}{2}-(m-1)}\right]^{1+s-t}c^m\sum_{k=0}^{\infty}\frac{1}{a^k(q;q)_{\alpha+k}}$$
$$\times\sum_{n=0}^{k}\frac{(q^{-k}, ay, az;q)_n}{(q, abz, ar;q)_n}q^n x^{\alpha+k}(a/x;q)_{\alpha+k}\frac{(q^{-\alpha-k};q)_m}{(xq^{1-\alpha-k}/a;q)_m}\left(\frac{q}{a}\right)^m\sum_{j=0}^{s+1}(-1)^j B_j q^{j(m-1)}$$
$$= \frac{(1-q)^\alpha(abz, ar;q)_\infty}{(ay, az;q)_\infty}\sum_{m=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1};q)_m}{(q, b_1, \ldots, b_s;q)_m}\left[(-1)^{m+1}q^{\binom{m}{2}-(m-1)}\right]^{1+s-t}c^m\sum_{k=0}^{\infty}\frac{1}{a^k(q;q)_{\alpha+k}}$$
$$\times\sum_{n=0}^{k}\frac{(q^{-k}, ay, az;q)_n}{(q, abz, ar;q)_n}q^n\frac{(q^{-\alpha-k};q)_m}{(xq^{1-\alpha-k}/a;q)_m}\left(\frac{q}{a}\right)^m(xq^{1-\alpha-k}/a;q)_{\alpha+k}(-a)^{\alpha+k}q^{\binom{\alpha+k}{2}}$$
$$\times\sum_{j=0}^{s+1}(-1)^j B_j q^{j(m-1)} \quad \text{(by using (1))}$$
$$= \frac{(1-q)^\alpha(abz, ar;q)_\infty}{(ay, az;q)_\infty}\sum_{m=0}^{\infty}\frac{(a_0, a_1, \ldots, a_{t-1};q)_m}{(q, b_1, \ldots, b_s;q)_m}\left[(-1)^{m+1}q^{\binom{m}{2}-(m-1)}\right]^{1+s-t}c^m\sum_{k=0}^{\infty}\frac{1}{a^k(q;q)_{\alpha+k}}$$

$$\times \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n (-a)^{\alpha+k} q^{\binom{\alpha+k}{2}} \left(\frac{1}{x}\right)^m \frac{(xq^{1-\alpha-k}/a; q)_\infty}{(xq/a; q)_\infty} {}_2\phi_1 \left(\begin{matrix} q^{-m}, xq/a \\ xq^{1-\alpha-k}/a \end{matrix} ; q, q \right)$$

$$\times \sum_{j=0}^{s+1} (-1)^j B_j q^{j(m-1)} \quad \text{(by using (3))}$$

$$= \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{m=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_m}{(q, b_1, \ldots, b_s; q)_m} \left[(-1)^{m+1} q^{\binom{m}{2}-(m-1)}\right]^{1+s-t} c^m \sum_{k=0}^{\infty} \frac{1}{a^k (q;q)_{\alpha+k}}$$

$$\times \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n (-a)^{\alpha+k} q^{\binom{\alpha+k}{2}} D_q^m \left\{ \frac{(xq^{1-\alpha-k}/a; q)_\infty}{(xq/a; q)_\infty} \right\} \prod_{j=0}^{s} (1 - b_j q^{m-1})$$

(by using (24))

$$= \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{m=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_m}{(q, b_1, \ldots, b_s; q)_m} \left[(-1)^{m+1} q^{\binom{m}{2}-(m-1)}\right]^{1+s-t} c^m \sum_{k=0}^{\infty} \frac{1}{a^k (q;q)_{\alpha+k}}$$

$$\times \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n D_{q,x}^m \{x^{\alpha+k} (a/x; q)_{\alpha+k}\} \prod_{j=0}^{s} (1 - b_j q^{m-1}) \quad \text{(by using (1))}$$

$$= \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{m=1}^{\infty} \frac{(a_0, \ldots, a_{t-1}; q)_{m-1}}{(q, b_1, \ldots, b_s; q)_{m-1}} \left[(-1)^{m+1} q^{\binom{m}{2}-(m-1)}\right]^{1+s-t} c^m \sum_{k=0}^{\infty} \frac{1}{a^k (q;q)_{\alpha+k}}$$

$$\times \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n D_{q,x}^m \{x^{\alpha+k} (a/x; q)_{\alpha+k}\} \prod_{j=0}^{t-1} (1 - a_j q^{m-1})$$

$$= \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{m=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_m}{(q, b_1, \ldots, b_s; q)_m} \left[(-1)^{m+2} q^{\binom{m+1}{2}-m}\right]^{1+s-t} c^{m+1} \sum_{k=0}^{\infty} \frac{1}{a^k (q;q)_{\alpha+k}}$$

$$\times \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n D_{q,x} D_{q,x}^m \{x^{\alpha+k} (a/x; q)_{\alpha+k}\} \prod_{j=0}^{t-1} (1 - a_j q^m)$$

$$= c D_{q,x} \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{m=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_m}{(q, b_1, \ldots, b_s; q)_m} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t} c^m \sum_{k=0}^{\infty} \frac{1}{a^k (q;q)_{\alpha+k}}$$

$$\times \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n x^{\alpha+k} (a/x; q)_{\alpha+k} \frac{(q^{-\alpha-k}; q)_m}{(xq^{1-\alpha-k}/a; q)_m} \left(\frac{q}{a}\right)^m \sum_{j=0}^{t} (-1)^j A_j q^{jm}$$

$$= c \sum_{j=0}^{t} (-1)^j A_j D_{q,x} \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k} (a/x; q)_{\alpha+k}}{a^k (q;q)_{\alpha+k}} \sum_{n=0}^{k} \frac{(q^{-k}, ay, az; q)_n}{(q, abz, ar; q)_n} q^n$$

$$\times \sum_{m=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}, q^{-\alpha-k}; q)_m}{(q, b_1, \ldots, b_s, xq^{1-\alpha-k}/a; q)_m} \left[(-1)^m q^{\binom{m}{2}}\right]^{1+s-t} \left(\frac{cq^j q}{a}\right)^m$$

$$= c \sum_{j=0}^{t} (-1)^j A_j D_{q,x} \left\{ f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, cq^j) \right\}.$$

So $f_R = f_R(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, c)$ satisfies the q-difference equation (23) and from Theorem 1.7, we have

$$f_R = F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x}) \{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, 0)\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x}) \left\{ \frac{(1-q)^\alpha (abz, ar; q)_\infty}{(ay, az; q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k} (a/x; q)_{\alpha+k}}{a^k (q;q)_{\alpha+k}} \right.$$

$$\times\ {}_3\phi_2\left(\begin{matrix} q^{-k}, ay, az \\ abz, ar \end{matrix}; q, q\right)\Bigg\}$$
$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x})\left\{I_{q,a}^{\alpha}\left\{\frac{(xbz, xr; q)_\infty}{(xy, xz; q)_\infty}\right\}\right\}$$
$$= I_{q,a}^{\alpha}\left\{F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x})\left\{\frac{(xbz, xr; q)_\infty}{(xy, xz; q)_\infty}\right\}\right\}$$
$$= I_{q,a}^{\alpha}\left\{\frac{(xbz, xr; q)_\infty}{(xy, xz; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{c}{x}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}{}_3\phi_2\left(\begin{matrix} q^{-n}, xy, xz \\ xbz, xr \end{matrix}; q, q\right)\right\}.$$

□

- Setting $(c, r, z, b, y) = (0, 0, s, r, t)$ in equation (29), we recover equation (18) obtained by Cao and et al. [8].
- When $(c, y, r) = (0, s, t)$ in equation (29), we attain equation (19) obtained by Cao [5].

Theorem 2.2. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $\alpha \in R^+$, $0 < a < x < 1$, *and* $max\{|xz|, |ay|, |az|\} < 1$, *we have*

$$I_{q,a}^{\alpha}\left\{\frac{(ryx; q)_\infty}{(xz, xy; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{c}{x}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}{}_3\phi_2\left(\begin{matrix} q^{-n}, xz, xy \\ ryx, 0 \end{matrix}; q, q\right)\right\}$$
$$= \frac{(1-q)^\alpha (ary; q)_\infty}{(ay, az; q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k} z^k}{(q; q)_{\alpha+k}}\sum_{n=0}^{k}\frac{(q^{-k}, r, az; q)_n}{(q, ary; q)_n}\left(-\frac{q^k y}{z}\right)^n q^{-\binom{n}{2}}$$
$$\times\ {}_{t+1}\phi_{s+1}\left(\begin{matrix} a_0, a_1 \ldots, a_{t-1}, q^{-\alpha-k} \\ b_1, \ldots, b_s, xq^{1-\alpha-k}/a \end{matrix}; q, \frac{cq}{a}\right). \tag{30}$$

Proof. Let $f_R = f_R(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, c) =$ RHS of equation (30). By using the same technique used in Theorem 2.1, we can show that f_R satisfies the q-difference equation (23), so

$$f_R = F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x})\{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, 0)\}$$
$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x})\Bigg\{\frac{(1-q)^\alpha (ary; q)_\infty}{(ay, az; q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x; q)_{\alpha+k} z^k}{(q; q)_{\alpha+k}}$$
$$\times\sum_{n=0}^{k}\frac{(q^{-k}, r, az; q)_n}{(q, ary; q)_n}\left(-\frac{q^k y}{z}\right)^n q^{-\binom{n}{2}}\Bigg\}$$
$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x})\left\{I_{q,a}^{\alpha}\left\{\frac{(ryx; q)_\infty}{(xz, xy; q)_\infty}\right\}\right\}$$
$$= I_{q,a}^{\alpha}\left\{F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,x})\left\{\frac{(ryx; q)_\infty}{(xz, xy; q)_\infty}\right\}\right\}$$
$$= I_{q,a}^{\alpha}\left\{\frac{(ryx; q)_\infty}{(xz, xy; q)_\infty}\sum_{n=0}^{\infty}\frac{(a_0, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n}\left(\frac{c}{x}\right)^n\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t}{}_3\phi_2\left(\begin{matrix} q^{-n}, xz, xy \\ ryx, 0 \end{matrix}; q, q\right)\right\}.$$

□

- Setting $(c, y, z) = (0, s, t)$ in equation (30), we get equation (18) obtained by Cao and et al. [8].

Theorem 2.3. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $\alpha \in R_0$, $0 < a < x < 1$ and $max\{|az|, |xz|\} < 1$, *we have*

$$I_{q,a}^{\alpha}\left\{ \frac{1}{(xz;q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0,\ldots,a_{t-1};q)_n}{(q,b_1,\ldots,b_s;q)_n} (cz)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} \right\}$$
$$= \frac{(1-q)^\alpha}{(az;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} z^k}{(q;q)_{\alpha+k}} {}_{t+1}\phi_{s+1}\left(\begin{matrix} a_0, a_1 \ldots, a_{t-1}, q^{-\alpha-k} \\ b_1,\ldots,b_s, xq^{1-\alpha-k}/a \end{matrix} ; q, \frac{cq}{a} \right). \quad (31)$$

Proof. Let $f_R = f_R(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, c)$ =RHS of equation (31). By using the same technique used in Theorem 2.1, f_R satisfies the q-difference equation (23), we have

$$f_R = F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x})\{f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,x,0)\}$$
$$= F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x}) \left\{ \frac{(1-q)^\alpha}{(az;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} z^k}{(q;q)_{\alpha+k}} \right\}$$
$$= F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x}) \left\{ I_{q,a}^{\alpha} \left\{ \frac{1}{(xz;q)_\infty} \right\} \right\} \quad \text{(by using (16))}$$
$$= I_{q,a}^{\alpha} \left\{ F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x}) \left\{ \frac{1}{(xz;q)_\infty} \right\} \right\}$$
$$= I_{q,a}^{\alpha} \left\{ \frac{1}{(xz;q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0,\ldots,a_{t-1};q)_n}{(q,b_1,\ldots,b_s;q)_n} (cz)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} \right\}. \quad \text{(by using (26))}$$

□

- Letting $(c, z) = (0, t)$ in equation (31), we get equation (16) given by Cao and et al. [8].

Theorem 2.4. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $\alpha \in R_0$, $0 < a < x < 1$ *and* $|xz| < 1$ *we have*

$$I_{q,a}^{\alpha}\left\{ (xz;q)_\infty \sum_{n=0}^{\infty} \frac{(a_0,\ldots,a_{t-1};q)_n}{(q,b_1,\ldots,b_s,xz;q)_n} (cz)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{2+s-t} \right\} = (1-q)^\alpha (az;q)_\infty$$
$$\times \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(az;q)_k} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} z^k}{(q;q)_{\alpha+k}} {}_{t+1}\phi_{s+1}\left(\begin{matrix} a_0, \ldots, a_{t-1}, q^{-\alpha-k} \\ b_1,\ldots,b_s, xq^{1-\alpha-k}/a \end{matrix} ; q, \frac{cq}{a} \right). \quad (32)$$

Proof. Let $f_R = f_R(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, x, c)$ =RHS of equation (32). By using the same technique used in Theorem 2.1, f_R satisfies the q-difference equation (23), we have

$$f_R = F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x})\{f(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,x,0)\}$$
$$= F(a_0,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x}) \left\{ (1-q)^\alpha (az;q)_\infty \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(az;q)_k} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} z^k}{(q;q)_{\alpha+k}} \right\}$$
$$= F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x}) \left\{ I_{q,a}^{\alpha} \left\{ (xz;q)_\infty \right\} \right\} \quad \text{(by using (17))}$$
$$= I_{q,a}^{\alpha} \left\{ F(a_0,a_1,\ldots,a_{t-1},b_1,\ldots,b_s,cD_{q,x}) \left\{ (xz;q)_\infty \right\} \right\}$$
$$= I_{q,a}^{\alpha} \left\{ (xz;q)_\infty \sum_{n=0}^{\infty} \frac{(a_0,\ldots,a_{t-1};q)_n}{(q,b_1,\ldots,b_s,xz;q)_n} (cz)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{2+s-t} \right\}. \quad \text{(by using (28))}$$

□

- Letting $(c, z) = (0, t)$ in equation (32), we get equation (17) obtained by Cao and et al. [8].

3. Generalization of q-Barne's contour integral

In 1910, Watson [24] showed that Barnes contour integral has a q-analogue. We use the q-difference equation method to generalize q-Barnes contour integral and show how to obtain another generalization of q-Barnes contour integral.

Theorem 3.1. *For $a_0 = q^{-G}$, $G \in \mathbb{N}$, $|z| < 1$,$|\arg(-z)| < \pi$, we have*

$$\int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(aq^x, bq^x; q)_\infty} {}_{t+1}\phi_{s+1}\left(\begin{matrix} q^{-x}, a_0, a_1, \ldots, a_{t-1} \\ a, b_1, \ldots, b_s \end{matrix} ; q, eq^x \right) \frac{\pi(-z)^x}{\sin \pi x} dx$$
$$= -2i\pi \frac{(q, c; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, c; q)_n} z^n {}_{t+1}\phi_{s+1}\left(\begin{matrix} q^{-n}, a_0, a_1, \ldots, a_{t-1} \\ a, b_1, \ldots, b_s \end{matrix} ; q, eq^n \right). \quad (33)$$

Proof. Rewrite equation (33) as follows:

$$\int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty} \frac{(a; q)_\infty}{(aq^x; q)_\infty} {}_{t+1}\phi_{s+1}\left(\begin{matrix} q^{-x}, a_0, a_1, \ldots, a_{t-1} \\ a, b_1, \ldots, b_s \end{matrix} ; q, eq^x \right) \frac{\pi(-z)^x}{\sin \pi x} dx$$
$$= -2i\pi \frac{(q, c; q)_\infty}{(b; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, c; q)_n} z^n {}_{t+1}\phi_{s+1}\left(\begin{matrix} q^{-n}, a_0, a_1, \ldots, a_{t-1} \\ a, b_1, \ldots, b_s \end{matrix} ; q, eq^n \right). \quad (34)$$

Let $f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s; a; e) =$ LHS of equation (34), we have

$$(-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, eq^{j+t-s-1})$$
$$= (-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty} \frac{(a; q)_\infty}{(aq^x; q)_\infty} \sum_{n=0}^{x} \frac{(q^{-x}, a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, a, b_1, \ldots, b_s; q)_n}$$
$$\times \left(eq^s q^{j+t-s-1}\right)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} \frac{\pi(-z)^x}{\sin \pi x} dx$$
$$= \sum_{n=0}^{x} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-t} e^n \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty}$$
$$\times q^{nx} \frac{(a; q)_\infty}{(aq^x; q)_\infty} \frac{(q^{-x}; q)_n}{(a; q)_n} \frac{\pi(-z)^x}{\sin \pi x} dx \sum_{j=0}^{s+1} (-1)^j B_j q^{j(n-1)}$$
$$= \sum_{n=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-t} e^n \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty}$$
$$\times D_{q,a}^n \left\{ \frac{(a; q)_\infty}{(aq^x; q)_\infty} \right\} \frac{\pi(-z)^x}{\sin \pi x} dx \prod_{j=0}^{s} (1 - b_j q^{n-1}) \qquad \text{(by using (3) and (24))}$$
$$= \sum_{n=1}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_{n-1}}{(q, b_1, \ldots, b_s; q)_{n-1}} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)}\right]^{1+s-t} e^n \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty}$$
$$\times D_{q,a}^n \left\{ \frac{(a; q)_\infty}{(aq^x; q)_\infty} \right\} \frac{\pi(-z)^x}{\sin \pi x} dx \prod_{j=0}^{t-1} (1 - a_j q^{n-1})$$
$$= \sum_{n=0}^{\infty} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^{n+2} q^{\binom{n+1}{2}-n}\right]^{1+s-t} e^{n+1} \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty}$$

$$\times D_{q,a} D_{q,a}^n \left\{ \frac{(a;q)_\infty}{(aq^x;q)_\infty} \right\} \frac{\pi(-z)^x}{\sin \pi x} dx \prod_{j=0}^{t-1} (1 - a_j q^n)$$

$$= e \sum_{n=0}^{x} \frac{(a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, b_1, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-t} e^n \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty}$$

$$\times D_{q,a} \frac{q^{nx}(a;q)_\infty}{(aq^x;q)_\infty} \frac{(q^{-x};q)_n}{(a;q)_n} \frac{\pi(-z)^x}{\sin \pi x} dx \sum_{j=0}^{t} (-1)^j A_j q^{jn}$$

$$= e \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty} \frac{(a;q)_\infty}{(aq^x;q)_\infty} \sum_{n=0}^{x} \frac{(q^{-x}, a_0, a_1, \ldots, a_{t-1}; q)_n}{(q, a, b_1, \ldots, b_s; q)_n}$$

$$\times \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-t} \left(eq^x q^j \right)^n \frac{\pi(-z)^x}{\sin \pi x} dx$$

$$= e \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \left\{ f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, eq^j) \right\}$$

So, $f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, e)$ satisfies the q-difference equation (23) and from Theorem 1.7, we have

$$\begin{aligned} f_L &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, eD_{q,a}) f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, 0) \\ &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, eD_{q,a}) \left\{ \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x; q)_\infty} \frac{(a;q)_\infty}{(aq^x;q)_\infty} \frac{\pi(-z)^x}{\sin \pi x} dx \right\} \\ &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, eD_{q,a}) \left\{ -2i\pi \frac{(q, c; q)_\infty}{(b;q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) \right\} \\ &\qquad \text{(by using (8))} \\ &= -2i\pi \frac{(q, c; q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(b;q)_n z^n}{(q, c; q)_n} F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, eD_{q,a}) \left\{ \frac{(a;q)_\infty}{(aq^n;q)_\infty} \right\} \\ &= -2i\pi \frac{(q, c; q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, c; q)_n} z^n \; {}_{t+1}\phi_{s+1} \left(\begin{matrix} q^{-n}, a_0, a_1, \ldots, a_{t-1} \\ a, b_1, \ldots, b_s \end{matrix} ; q, eq^n \right). \end{aligned}$$

This patently completes the proof of Theorem 3.1. □

- Letting $(e, x) = (0, s)$ in equation (33), we get the q-Barnes contour integral obtained by Watson [24] (equation (8)).
- When $(t, s, e) = (1, 0, a/a_0)$ in (33) and by using (5), (4) and (6), we get

Corollary 3.1. *For* $|a/a_0| < 1, |\arg(-z)| < \pi$, *we have*

$$\int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(bq^x, aq^x/a_0; q)_\infty} \frac{\pi(-z)^x}{\sin \pi x} dx = -2i\pi \frac{(q, bz; q)_\infty}{(b, z; q)_\infty} \sum_{n=0}^{\infty} \frac{(ca/a_0, z; q)_n}{(q, bz; q)_n} \left(\frac{a}{a_0} \right)^n.$$

Theorem 3.2. *For* $a_0 = q^{-G}$, $G \in \mathbb{N}$, $|z| < 1, |\arg(-z)| < \pi$, *we have*

$$\int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(aq^x, bq^x; q)_\infty} \; {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, eq^x \right) \frac{\pi(-z)^x}{\sin \pi x} dx$$
$$= -2i\pi \frac{(q, c; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, c; q)_n} z^n \; {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, eq^n \right). \qquad (35)$$

Proof. Let $f_L = f_L(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s; a; e)$ =LHS of equation (35). By using the same technique used in Theorem 3.1, f_L satisfies the q-difference equation (23), we have

$$\begin{aligned} f_L &= F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, eD_{q,a})\{f(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, a, 0)\} \\ &= F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, eD_{q,a}) \left\{ \int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x; q)_\infty}{(aq^x, bq^x; q)_\infty} \frac{\pi(-z)^x}{\sin \pi x} dx \right\} \\ &= F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, eD_{q,a}) \left\{ -2i\pi \frac{(q, c; q)_\infty}{(a, b; q)_\infty} {}_2\phi_1 \left(\begin{matrix} a, b \\ c \end{matrix} ; q, z \right) \right\} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by using (8))} \\ &= -2i\pi \frac{(q, c; q)_\infty}{(b; q)_\infty} \sum_{n=0}^{\infty} \frac{(b; q)_n z^n}{(q, c; q)_n} F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, eD_{q,a}) \left\{ \frac{1}{(aq^n; q)_\infty} \right\} \\ &= -2i\pi \frac{(q, c; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, c; q)_n} z^n \, {}_t\phi_s \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s \end{matrix} ; q, eq^n \right). \end{aligned}$$

□

- Letting $(e, x) = (0, s)$ in equation (35), we get the q-Barnes contour integral obtained by Watson [24] (equation (8)).
- For $(t, s) = (2, 1)$, $(a_0, a_1, b_1) = (q^{-G}, 1/e, q^{x-G})$ in equation (35) and applying q-Gauss sum (5), we get

Corollary 3.2. *For* $|\arg(-z)| < \pi$ *and* $|z| < 1$*, we have*

$$\begin{aligned} &\int_{-i\infty}^{i\infty} \frac{(q^{1+x}, cq^x, q^x, eq^{x-G}; q)_\infty}{(aq^x, bq^x, q^{x-G}, eq^x; q)_\infty} \frac{\pi(-z)^x}{\sin \pi x} dx \\ &\qquad = -2i\pi \frac{(q, c; q)_\infty}{(a, b; q)_\infty} \sum_{n=0}^{\infty} \frac{(a, b; q)_n}{(q, c; q)_n} z^n \, {}_2\phi_1 \left(\begin{matrix} q^{-G}, 1/e \\ q^{x-G} \end{matrix} ; q, eq^n \right). \end{aligned}$$

4. Generalization of Ramanujan integrals

Using the q-difference equation method, we present several generalizations of Ramanujan integrals.

Theorem 4.1. *For* $m \in R$, $a_0 = q^{-G}$, $G \in \mathbb{N}$ *and* $|abq| < 1$*, we have*

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, \dots, a_{t-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left(\frac{c}{a}\right)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} \\ &\quad \times {}_3\phi_2 \left(\begin{matrix} q^{-n}, aq^{1/2}e^{2ikx}, -aqe^{2ikm} \\ 0, 0 \end{matrix} ; q, q \right) dx \\ &= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} \, {}_t\phi_s \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s \end{matrix} ; q, cbq \right). \end{aligned} \tag{36}$$

Proof. First rewrite equation (36) as follows:

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm}, bq^{1/2}e^{-2ikx}; q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, \dots, a_{t-1}; q)_n}{(q, b_1, \dots, b_s; q)_n} \left(\frac{c}{a}\right)^n \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-t} \\ &\times {}_3\phi_2 \left(\begin{matrix} q^{-n}, aq^{1/2}e^{2ikx}, -aqe^{2ikm} \\ 0, 0 \end{matrix} ; q, q \right) dx \end{aligned}$$

$$= \sqrt{\pi} e^{m^2} \frac{(-bqe^{-2ikm};q)_\infty}{(abq;q)_\infty} \, {}_t\phi_s \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s \end{matrix} ; q, cbq \right). \tag{37}$$

Let $f_L = f_L(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s; a; c) =$ LHS of (37),

$$(-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} f_L(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, b, cq^{j+t-s-1})$$

$$= (-q)^{1+s-t} \sum_{j=0}^{s+1} \frac{(-1)^j B_j}{q^j} \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm}, bq^{1/2}e^{-2ikx};q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_n}{(q, b_1, \dots, b_s;q)_n}$$
$$\times \left(\frac{cq^{j+t-s-1}}{a} \right)^n \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-t} {}_3\phi_2 \left(\begin{matrix} q^{-n}, aq^{1/2}e^{2ikx}, -aqe^{2ikm} \\ 0, 0 \end{matrix} ; q, q \right) dx$$

$$= \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_n}{(q, b_1, \dots, b_s;q)_n} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)} \right]^{1+s-t} c^n \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty}$$
$$\times \frac{1}{a^n} \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm};q)_\infty} \, {}_3\phi_2 \left(\begin{matrix} q^{-n}, aq^{1/2}e^{2ikx}, -aqe^{2ikm} \\ 0, 0 \end{matrix} ; q, q \right) dx \sum_{j=0}^{s+1} (-1)^j B_j q^{j(n-1)}$$

$$= \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_n}{(q, b_1, \dots, b_s;q)_n} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)} \right]^{1+s-t} c^n \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty}$$
$$\times D_{q,a}^n \left\{ \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm};q)_\infty} \right\} dx \prod_{j=0}^{s} (1 - b_j q^{n-1})$$

$$= \sum_{n=1}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_{n-1}}{(q, b_1, \dots, b_s;q)_{n-1}} \left[(-1)^{n+1} q^{\binom{n}{2}-(n-1)} \right]^{1+s-t} c^n \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty}$$
$$\times D_{q,a}^n \left\{ \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm};q)_\infty} \right\} dx \prod_{j=0}^{t-1} (1 - a_j q^{n-1})$$

$$= \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_n}{(q, b_1, \dots, b_s;q)_n} \left[(-1)^{n+2} q^{\binom{n+1}{2}-n} \right]^{1+s-t} c^{n+1} \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty}$$
$$\times D_{q,a} D_{q,a}^n \left\{ \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm};q)_\infty} \right\} dx \prod_{j=0}^{t-1} (1 - a_j q^n)$$

$$= c \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_n}{(q, b_1, \dots, b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-t} c^n \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(bq^{1/2}e^{-2ikx};q)_\infty}$$
$$\times D_{q,a} D_{q,a}^n \left\{ \frac{1}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm};q)_\infty} \right\} dx \sum_{j=0}^{t} (-1)^j A_j q^{jn}$$

$$= c \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm}, bq^{1/2}e^{-2ikx};q)_\infty} \sum_{n=0}^{\infty} \frac{(a_0, a_1, \dots, a_{t-1};q)_n}{(q, b_1, \dots, b_s;q)_n}$$
$$\times \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-t} \left(\frac{cq^j}{a} \right)^n {}_3\phi_2 \left(\begin{matrix} q^{-n}, aq^{1/2}e^{2ikx}, -aqe^{2ikm} \\ 0, 0 \end{matrix} ; q, q \right) dx$$

$$= c \sum_{j=0}^{t} (-1)^j A_j D_{q,b} \left\{ f_L(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, b, cq^j) \right\}.$$

So, $f_L = f_L(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, a, c)$ satisfies the q-difference equation (23) and from Theorem 1.7, we have

$$f_L = F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a})\{f(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, a, 0)\}$$

$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, -aqe^{2ikm}, bq^{1/2}e^{-2ikx}; q)_\infty} dx \right\}$$
$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \sqrt{\pi} e^{m^2} \frac{(-bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} \right\}$$
$$= \sqrt{\pi} e^{m^2} (-bqe^{-2ikm}; q)_\infty F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \frac{1}{(abq; q)_\infty} \right\}$$
$$= \sqrt{\pi} e^{m^2} \frac{(-bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cbq \right).$$

This completes the proof of Theorem 4.1. □

- Setting $(t, s) = (2, 1)$ and $(a_0, a_1, b_1, c) = (r, w, v, -e^{-2ikm})$ in equation (36), we recover the Ramanujan integral (14) proposed by Cao [4].

  Proof. When $(t, s) = (2, 1)$ and $(a_0, a_1, b_1, c) = (r, w, v, -e^{-2ikm})$, equation (36) reduce to the following equation:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} \sum_{n=0}^{\infty} \frac{(r, w; q)_n}{(q, v; q)_n} \left(\frac{-1}{ae^{2ikm}} \right)^n \times {}_3\phi_2 \left(\begin{matrix} q^{-n}, aq^{1/2}e^{2ikx}, -aqe^{2ikm} \\ 0, 0 \end{matrix} ; q, q \right) dx$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} {}_2\phi_1 \left(\begin{matrix} r, w \\ v \end{matrix} ; q, \frac{-bq}{e^{2ikm}} \right). \tag{38}$$

Setting $d = e = 0$ and $c = a$ in equation (7), we get

$${}_3\phi_2 \left(\begin{matrix} q^{-n}, a, b \\ 0, 0 \end{matrix} ; q, q \right) = a^n \, {}_2\phi_0 \left(\begin{matrix} q^{-n}, a \\ - \end{matrix} ; q, \frac{bq^n}{a} \right). \tag{39}$$

Substituting (39) into (38), we get

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} \sum_{n=0}^{\infty} \frac{(r, w; q)_n}{(q, v; q)_n} q^n \, {}_2\phi_0 \left(\begin{matrix} q^{-n}, -aqe^{2ikm} \\ 0 \end{matrix} ; q, \frac{-q^n e^{2ik(x-m)}}{q^{1/2}} \right) dx$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} {}_2\phi_1 \left(\begin{matrix} r, w \\ v \end{matrix} ; q, \frac{-bq}{e^{2ikm}} \right). \tag{40}$$

Interchange summations and then applying (2) and q-Chu-Vandermonde sum on left side (40), we get the required result. □

Theorem 4.2. *For $m \in R$, $a_0 = q^{-G}$, $G \in \mathbb{N}$ and $|abq| < 1$, we have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cq^{1/2}e^{2ikx} \right) dx$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1}, -e^{2ikm}/b \\ b_1, \ldots, b_s, -aqe^{2ikm} \end{matrix} ; q, cbq \right). \tag{41}$$

Proof. $f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s; a; c)$ =LHS of (41), use the same technique in Theorem 4.1 to check f_L satisfies q-difference equation (23), we have

$$f_L = F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a})\{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, 0)\}$$
$$= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} dx \right\}$$

$$= F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a}) \left\{ \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} \right\}$$
$$= \sqrt{\pi} e^{m^2} (-bqe^{-2ikm}; q)_\infty F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a}) \left\{ \frac{(-aqe^{2ikm}; q)_\infty}{(abq; q)_\infty} \right\}$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1}, -e^{2ikm}/b \\ b_1, \dots, b_s, -aqe^{2ikm} \end{matrix} ; q, cbq \right).$$

This completes the proof of Theorem 4.2. □

- If $(t, s) = (3, 2)$ and $(a_0, a_1, a_2, b_1, b_2, c) = (r, s, t, u, v, y)$ in Theorem 4.2, we retain Theorem 1.6 obtained by Cao and et al. [9].

Theorem 4.3. *For $m \in R$, $a_0 = q^{-G}$, $G \in \mathbb{N}$ and if $|abq| < 1$, we have*

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1}, bq^{1/2}e^{-2ikx} \\ b_1, \dots, b_s, abq \end{matrix} ; q, cq^{1/2}e^{2ikx} \right) dx$$
$$= \sqrt{\pi} e^{m^2} \frac{(-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty}{(abq; q)_\infty} {}_{t}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s, -aqe^{2ikm} \end{matrix} ; q, -cqe^{2ikm} \right). \quad (42)$$

Proof. Rewrite equation (42) as follows:

$$\int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}(abq; q)_\infty}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1}, bq^{1/2}e^{-2ikx} \\ b_1, \dots, b_s, abq \end{matrix} ; q, cq^{1/2}e^{2ikx} \right) dx$$
$$= \sqrt{\pi} e^{m^2} (-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty {}_{t}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s, -aqe^{2ikm} \end{matrix} ; q, -cqe^{2ikm} \right). \quad (43)$$

Let $f_L = f_L(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s; a; c)$ =L.H.S of (43) which satisfies (23), so we have

$$f_L = F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a})\{f(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, a, 0)\}$$
$$= F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a}) \left\{ \int_{-\infty}^{\infty} \frac{e^{-x^2+2mx}(abq; q)_\infty}{(aq^{1/2}e^{2ikx}, bq^{1/2}e^{-2ikx}; q)_\infty} dx \right\}$$
$$= F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a}) \left\{ \sqrt{\pi} e^{m^2} (-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty \right\}$$
$$= \sqrt{\pi} e^{m^2} (-bqe^{-2ikm}; q)_\infty F(a_0, a_1, \dots, a_{t-1}, b_1, \dots, b_s, cD_{q,a}) \left\{ (-aqe^{2ikm}; q)_\infty \right\}$$
$$= \sqrt{\pi} e^{m^2} (-aqe^{2ikm}, -bqe^{-2ikm}; q)_\infty {}_{t}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s, -aqe^{2ikm} \end{matrix} ; q, -cqe^{2ikm} \right).$$

□

- Setting $c = 0$ in the equation (42), we get equation (9).

Theorem 4.4. *For $m \in \mathbb{R}$, $a_0 = q^{-G}$, $G \in \mathbb{N}$ and $max\{|aq^{1/2}e^{2km}|, |bq^{1/2}e^{-2km}|\} < 1$, we have*

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(-aqe^{2kx}, -bqe^{-2kx}; q)_\infty {}_{t}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1} \\ b_1, \dots, b_s, -aqe^{2kx} \end{matrix} ; q, -cqe^{2kx} \right) dx$$
$$= \sqrt{\pi} e^{m^2} \frac{(abq; q)_\infty}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty}$$
$$\times {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \dots, a_{t-1}, bq^{1/2}e^{-2mk} \\ b_1, \dots, b_s, abq \end{matrix} ; q, cq^{1/2}e^{2km} \right). \quad (44)$$

Proof. Let $f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s; a; c)$ =LHS of equation (44) which satisfies (23), we have

$$\begin{aligned}
f_L &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a})\{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, 0)\} \\
&= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \int_{-\infty}^{\infty} e^{-x^2+2mx}(-aqe^{2kx}, -bqe^{-2kx}; q)_\infty dx \right\} \\
&= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \sqrt{\pi} e^{m^2} \frac{(abq;q)_\infty}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty} \right\} \\
&= \sqrt{\pi} e^{m^2} \frac{1}{(bq^{1/2}e^{-2km}; q)_\infty} F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{(abq;q)_\infty}{(aq^{1/2}e^{2km}; q)_\infty} \right\} \\
&= \sqrt{\pi} e^{m^2} \frac{(abq;q)_\infty}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty} \ {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1}, bq^{1/2}e^{-2mk} \\ b_1, \ldots, b_s, abq \end{matrix} ; q, cq^{1/2}e^{2km} \right).
\end{aligned}$$

□

- Setting $c = 0$ in the equation (44), we get equation (10).
- Letting $(t, s) = (2, 1)$ and $(a_0, a_1, b_1) = (r, s, t)$ in the equation (44), we get equation (21) obtained by Cao [6].

Theorem 4.5. *For $m \in \mathbb{R}$, $a_0 = q^{-G}$, $G \in \mathbb{N}$ and $max\{|aq^{1/2}e^{2km}|, |bq^{1/2}e^{-2km}|\} < 1$, we have*

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-x^2+2mx}(-aqe^{2kx}, -bqe^{-2kx}; q)_\infty \ {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1 \ldots, a_{t-1}, -e^{2kx}/b \\ b_1 \ldots, b_s, -aqe^{2kx} \end{matrix} ; q, cbq \right) dx \\
&\quad = \sqrt{\pi} e^{m^2} \frac{(abq;q)_\infty}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty} \ {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cq^{1/2}e^{2km} \right). \qquad (45)
\end{aligned}$$

Proof. Rewrite (45) as follows:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-x^2+2mx}(-bqe^{-2kx}; q)_\infty \frac{(-aqe^{2kx}; q)_\infty}{(abq;q)_\infty} \ {}_{t+1}\phi_{s+1} \left(\begin{matrix} a_0, a_1 \ldots, a_{t-1}, -e^{2kx}/b \\ b_1 \ldots, b_s, -aqe^{2kx} \end{matrix} ; q, cbq \right) dx \\
&\quad = \sqrt{\pi} e^{m^2} \frac{1}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty} \ {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cq^{1/2}e^{2km} \right). \qquad (46)
\end{aligned}$$

Let $f_L = f_L(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s; a; c)$ =L.H.S of (46) which satisfies (23), we have

$$\begin{aligned}
f_L &= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a})\{f(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, a, 0)\} \\
&= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \int_{-\infty}^{\infty} e^{-x^2+2mx} \frac{(-aqe^{2kx}, -bqe^{-2kx}; q)_\infty}{(abq;q)_\infty} dx \right\} \\
&= F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,a}) \left\{ \sqrt{\pi} e^{m^2} \frac{1}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty} \right\} \\
&\qquad \text{(by using (10))} \\
&= \sqrt{\pi} e^{m^2} \frac{1}{(bq^{1/2}e^{-2km}; q)_\infty} F(a_0, a_1, \ldots, a_{t-1}, b_1, \ldots, b_s, cD_{q,b}) \left\{ \frac{1}{(aq^{1/2}e^{2km}; q)_\infty} \right\} \\
&= \sqrt{\pi} e^{m^2} \frac{1}{(aq^{1/2}e^{2km}, bq^{1/2}e^{-2km}; q)_\infty} \ {}_t\phi_s \left(\begin{matrix} a_0, a_1, \ldots, a_{t-1} \\ b_1, \ldots, b_s \end{matrix} ; q, cq^{1/2}e^{2km} \right). \\
&\qquad \text{(by using (26))}
\end{aligned}$$

□

5. Conclusions

(1) With the aid of the q-difference equation approach, we generalized the fractional q-integrals presented by Cao [5] and Cao and et al. [8].
(2) We provide two generalizations of the q-Barnes contour integral presented by Watson in 1910 [24] using the q-difference equation technique.
(3) We give numerous generalizations of Ramanujan integrals using the q-difference equation method.
(4) Certain parameter values can be substituted into the generalized integrals to obtain previously obtained or new findings.

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Husam Luti Saad for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.13, N.2.