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GLOBAL EXISTENCE OF SOLUTIONS FOR A WEAKLY COUPLED SYSTEM OF THREE DAMPED σ-EVOLUTION EQUATIONS

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ABSTRACT. In this paper our purpose is the study of the Cauchy problem for weakly coupled system of three semi-linear damped σ-evolution equations. Using $(L^m \cap L^2) - L^2$ linear estimates combined with fractional Gagliardo-Nirenberg inequality. We find the so-called $(p_1 - p_2 - p_3)$ planes for the global (in time) existence. Moreover, from the interaction between the parameters $m_1, m_2, m_3 \in [1, 2)$ in one hand and $\sigma_1, \sigma_2, \sigma_3 \geqslant 1$ in the other hand. We proved lower bounds for powers nonlinearities similarly to the modified Fujita exponent, which are in the form of planes $(p_1 - p_2)$, $(p_1 - p_3)$ and $(p_2 - p_3)$.

Weakly coupled system; σ-evolution equation; frictional damping, visco-elastic damping, Additional regularity; Global existence.

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1. INTRODUCTION

Let us consider the Cauchy problem for weakly coupled system of three semi-linear damped σ-evolution equations :

$$\begin{aligned} u_{tt} + (-\Delta)^{\sigma_1} u + u_t + (-\Delta)^{\sigma_1} u_t &= |w|^{p_1}, \\ v_{tt} + (-\Delta)^{\sigma_2} v + v_t + (-\Delta)^{\sigma_2} v_t &= |u|^{p_2}, \\ w_{tt} + (-\Delta)^{\sigma_3} w + w_t + (-\Delta)^{\sigma_3} w_t &= |v|^{p_3}, \end{aligned} \tag{1}$$

equipped with the initial data

$$u(0, x) = u_0(x),\ u_t(0, x) = u_1(x),$$
$$v(0, x) = v_0(x),\ v_t(0, x) = v_1(x),$$
$$w(0, x) = w_0(x),\ w_t(0, x) = w_1(x),$$

where

$$t \geqslant 0,\ x \in \mathbb{R}^n,\ \sigma_1, \sigma_2, \sigma_3 \geqslant 1,\quad p_1, p_2, p_3 > 1.$$

Here, we used the usual expressions for the time derivatives

$$u := u(t, x),\quad u_t := \frac{\partial u}{\partial t}(t, x),\quad u_{tt} := \frac{\partial^2 u}{\partial t^2}(t, x),\quad (t, x) \in [0, \infty) \times \mathbb{R}^n.$$

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The Laplacian operator $(-\Delta)^\sigma$ is defined as usual through the direct and inverse Fourier transform $\mathcal{F}$, $\mathcal{F}^{-1}$ as :

$$((-\Delta)^\sigma f)(x) = \mathcal{F}^{-1}\Big(|\xi|^{2\sigma}\mathcal{F}(f)(\xi)\Big)(x), \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \ |\xi| = \sqrt{\sum_{k=0}^{k=n} \xi_k^2}.$$

$|D|^a$ with $a \geqslant 0$ stand for the pseudo-differential operators with symbol $|\xi|$.
It is clear that the problem (1) is a generalization to the single semi-linear σ-evolution equation with frictional u_t and visco-elastic $(-\Delta)^\sigma u_t$ damping

$$\begin{aligned} &u_{tt} + (-\Delta)^\sigma u + u_t + (-\Delta)^\sigma u_t = |u|^p\,, \\ &u(0,x) = u_0(x), u_t(0,x) = u_1(x). \end{aligned} \tag{2}$$

Before state the main results of global (in time) existence to (1), let us recall some previous results for (2). The important goal in the study of single semi-linear equations or weakly coupled systems of two equations is to derive the so-called *critical exponent* or *critical $(p-q)$ curve* respectively. Indeed, the critical exponent pcrit or critical p_{crit} or critical $(p-q)$ curve is exactly the threshold between two important results, the first one is the global (in time) existence of small data solutions for any $p > p_{crit}$ or $h(p,q) < n$, while the second one is the blow-up of solutions (no global solution) for $p \leqslant p_{crit}$ or $n \leqslant h(p,q)$, where h is an appropriate function of p and q. Here, the relation $h(p,q) < n$ is called $(p-q)$ plane.
For (2) several papers [4], [6] and [7] have derived the modified Fujita exponent where $\sigma = 1$. In [10] the general case $\sigma \geqslant 1$, the authors derived the following form of $p_{crit}(n,m,\sigma)$:

$$p_{crit}(n,m,\sigma) = 1 + \frac{2m\sigma}{n}, \quad m \in [1,2), \tag{3}$$

where m is the parameter of additional L^m regularity of the data (u_0,u_1)

$$(u_0,u_1) \in (H^\sigma \cap L^m) \times (L^2 \cap L^m).$$

We remark that the frictional damping has the dominant influence, then $p_{crit}(n,m,\sigma)$ in (3) is exactly the critical exponent to

$$\begin{aligned} &u_{tt} + (-\Delta)^\sigma u + u_t = |u|^p\,, \\ &u(0,x) = u_0(x), \ u_t(0,x) = u_1(x). \end{aligned} \tag{4}$$

For more details the reader can see [3] if $m = 1$. Let us now consider the weakly coupled system :

$$\begin{aligned} &u_{tt} + (-\Delta)^{\sigma_1} u + u_t = |v|^p\,, \\ &v_{tt} + (-\Delta)^{\sigma_2} v + v_t = |u|^q\,, \\ &u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \\ &v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{aligned} \tag{5}$$

where $\sigma_1, \sigma_2 \geqslant 1, \ p, q > 1$. In [11] the authors studied the global (in time) existence of small data solutions by using $(L^1 \cap L^2) - L^2$ linear estimates to the corresponding linear equation, they shown also the influence of σ_1, σ_2 in the *critical $(p-q)$ curve.* Finally, they prove also the optimality of $(p-q)$ curve by using the test function method. For more detail about this method one can see [2, 5, 9, 10, 11] and reference therein. In this paper we take different additional regularities of the data, this method is inspired from

[12] where the author studied the following weakly coupled system

$$\begin{array}{l} u_{tt} - \Delta u + b(t)u_t = |v|^p\,, \\ v_{tt} - \Delta v + b(t)v_t = |u|^q\,, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x), \end{array} . \tag{6}$$

where $b(t)$ satisfies some suitable conditions. The author used different L^{m_1}, L^{m_2} regularities for the data in the treatment of global existence (in time) of small data solution. The authors proved the existence of a lower bounds less than the modified Fujita exponent which stated in (3). So, motivated by these results our goal is study the Cauchy problem of weakly coupled systems of three equations (1) under different additional regularities to get the lower planes than the $(p_1 - p_2)$, $(p_1 - p_3)$ or $(p_2 - p_3)$ planes. That is, we use different $(L^{m_k} \cap L^2) - L^2$ linear estimates for solutions to the corresponding linear equations appeared in (1)

$$\begin{array}{l} y_{tt} + (-\Delta)^\sigma y + y_t + (-\Delta)^\sigma y_t = 0, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \end{array} \tag{7}$$

where $m_k \in [1,2)$ and show the interaction between $\sigma_1, \sigma_2, \sigma_3$ and m_1, m_2, m_3 which leads to the global (in time) of small data solution to (1).

2. Notations

Through this paper, we use the following notations:

- $f \lesssim g$ when there exists a constant $C > 0$ such that $f \leqslant Cg$. This means that these constants does not play any role in our studies.
- Sobolev spaces (see [2] for more detail)

$$H^\sigma(\mathbb{R}^n) := \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{H^\sigma(\mathbb{R}^n)} = \|(1 + |\cdot|^2)^{\frac{\sigma}{2}} \mathcal{F}(f)\|_{L^2(\mathbb{R}^n)} < \infty \right\}.$$

- For the sake of simplicity, we omit the notation $\mathbb{R}^n$ in all spaces and write L^1, L^m, L^2, H^σ instead of $L^1(\mathbb{R}^n)$, $L^m(\mathbb{R}^n)$, $L^2(\mathbb{R}^n)$ and $H^\sigma(\mathbb{R}^n)$. In particular, we write the admissible data spaces as follows:

$$\mathcal{A}^{m_k}_{\sigma_k} := \left(H^{\sigma_k} \cap L^1\right) \times \left(L^2 \cap L^{m_k}\right), \; k = 1, 2, 3,$$

  such that if $(f,g) \in \mathcal{A}^{m_k}_{\sigma_k}$ we have the following norm:

$$\|(f,g)\|_{\mathcal{A}^m_{\sigma_k}} = \|f\|_{H^{\sigma_k}} + \|f\|_{L^1} + \|g\|_{L^2} + \|g\|_{L^{m_k}}.$$

In Section 3 we state the main results of the global (in time) existence to (1) with some examples. In the last section we prove our main theorems using linear estimates $(L^m \cap L^2) - L^2$ explained as well in the same section before the main proof.

3. Main results

It is naturally that the system behave like one single equation if the power nonlinearities satisfy the Fujita condition. So, for this reason we consider that two power nonlinearities not satisfied this condition. This assumptions generate two loss of decay estimates for (u,v) or (u,w) or (v,w) in order to obtain the $(p_1 - p_2 - p_3)$ curve.

Theorem 3.1. *Let $m_1, m_2, m_3 \in [1,2)$ and $\sigma_1, \sigma_2, \sigma_3 \geqslant 1$ such that*

$$\sigma_3 \leqslant \sigma_1 \leqslant \sigma_2$$

and

$$n_0 = \min\left\{\frac{4\sigma_1}{2-m_2},\ \frac{4\sigma_2}{2-m_3},\ \frac{4\sigma_3}{2-m_1}\right\}.$$

With respect to the dimension n we distinguish the following cases:

- *for $2\sigma_2 < n \leqslant n_0$, then we assume*

$$\frac{2}{m_1} \leqslant p_1 \leqslant \frac{n}{n-2\sigma_3}, \quad \frac{2}{m_2} \leqslant p_2 \leqslant \frac{n}{n-2\sigma_1}, \quad \frac{2}{m_3} \leqslant p_3 \leqslant \frac{n}{n-2\sigma_2}, \tag{8}$$

- *for $2\sigma_1 < n \leqslant 2\sigma_2$, then we assume*

$$\frac{2}{m_1} \leqslant p_1 \leqslant \frac{n}{n-2\sigma_3}, \quad \frac{2}{m_2} \leqslant p_2 \leqslant \frac{n}{n-2\sigma_1}, \quad \frac{2}{m_3} \leqslant p_3 < \infty, \tag{9}$$

- *for $2\sigma_3 < n \leqslant 2\sigma_1$, then we assume*

$$\frac{2}{m_1} \leqslant p_1 \leqslant \frac{n}{n-2\sigma_3}, \quad \frac{2}{m_2} \leqslant p_2 < \infty, \quad \frac{2}{m_3} \leqslant p_3 < \infty, \tag{10}$$

- *for $n \leqslant 2\sigma_3$, then we assume*

$$\frac{2}{m_1} \leqslant p_1 < \infty, \quad \frac{2}{m_2} \leqslant p_2 < \infty, \quad \frac{2}{m_3} \leqslant p_3 < \infty. \tag{11}$$

Moreover, we suppose

$$\begin{cases} \dfrac{2}{m_1} \leqslant p_1 \leqslant \dfrac{m_3}{m_1} + \dfrac{2m_3\sigma_3}{n}, \\ \dfrac{n}{2m_3\sigma_3} \leqslant \dfrac{1+p_2}{p_1p_2 - 1 + p_2\left(\frac{\sigma_3}{\sigma_1} - 1\right)\frac{m_3}{m_1} + \left(1 - \frac{m_3\sigma_3}{m_2\sigma_1}\right)}, \\ \frac{1+p_3+p_2p_3}{(p_1p_2p_3-1) - p_2p_3\left(1-\frac{\sigma_3}{\sigma_1}\right)\frac{m_3}{m_1} - p_3\left(\frac{\sigma_2}{\sigma_1}-1\right)\frac{m_3\sigma_3}{m_2\sigma_2} + \left(1-\frac{\sigma_3}{\sigma_2}\right)\frac{m_3}{2}} < \frac{n}{2m_3\sigma_3}. \end{cases} \tag{12}$$

There exists a constant $\varepsilon > 0$ such that for any data

$$\Big((u_0,u_1),(v_0,v_1),(w_0,w_1)\Big) \in \mathcal{B} =: \mathcal{A}_{\sigma_1}^{m_1} \times \mathcal{A}_{\sigma_2}^{m_2} \times \mathcal{A}_{\sigma_3}^{m_3}$$

with

$$I_0 = \left\|\Big((u_0,u_1),(v_0,v_1),(w_0,w_1)\Big)\right\|_{\mathcal{B}} < \varepsilon,$$

then there exists a uniquely globally determined (in time) solution

$$(u,v,w) \in \prod_{k=1}^{3}\Big(\mathcal{C}\left([0,\infty), H^{\sigma_k}\right) \cap \mathcal{C}^1\left([0,\infty), L^2\right)\Big)$$

to (1). Furtheremore the solution satisfies the estimates

$$
\begin{aligned}
\|u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\gamma(p_1)} I_0,\\
\|u_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1+\gamma(p_1)} I_0,\\
\||D|^{\sigma_1}u(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}+\gamma(p_1)} I_0,\\
\|v(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)+\delta(p_1,p_2)} I_0,\\
\|v_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1+\delta(p_1,p_2)} I_0\\
\||D|^{\sigma_2}v(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}+\delta(p_1,p_2)} I_0,\\
\|w(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)} I_0,\\
\|w_t(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)-1} I_0,\\
\||D|^{\sigma_3}w(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)-\frac{1}{2}} I_0.
\end{aligned}
$$

Here, $\gamma(p_1)$ represents the loss of decay in comparison with the corresponding decay estimates for the solutions u of the corresponding linear Cauchy problems with vanishing right-hand sides which defined

- *If $p_1 < \frac{m_3}{m_1} + \frac{2m_3\sigma_3}{n}$, then* $\gamma(p_1) = 1 - \frac{np_1}{2m_3\sigma_3} + \frac{n}{2m_1\sigma_3} > 0,$
- *If $p_1 = \frac{m_3}{m_1} + \frac{2m_3\sigma_3}{n}$, then* $\gamma(p_1) = \varepsilon_0.$

For $\delta(p_1,p_2)$ represents the loss of decay in comparison with the corresponding decay estimates for the solutions u of the corresponding linear Cauchy problems with vanishing right-hand sides which defined

- *If $\frac{n}{2m_3\sigma_3} < \frac{1+p_2}{p_1p_2-1+p_2\left(\frac{\sigma_3}{\sigma_1}-1\right)\frac{m_3}{m_1}+\left(1-\frac{m_3\sigma_3}{m_2\sigma_1}\right)}$, then*
$$\delta(p_1,p_2) = 1 - \frac{np_2}{2m_1\sigma_1} + \frac{n}{2m_2\sigma_1} + p_2\gamma(p_1) > 0,$$
- *If $\frac{n}{2m_3\sigma_3} = \frac{1+p_2}{p_1p_2-1+p_2\left(\frac{\sigma_3}{\sigma_1}-1\right)\frac{m_3}{m_1}+\left(1-\frac{m_3\sigma_3}{m_2\sigma_1}\right)}$, then*
$$\delta(p_1,p_2) = \varepsilon_1,$$

where ε_0 and ε_1 are a sufficiently small positive numbers.

Remark 3.1. *The upper bounds $n/(n-2\sigma_k)$ with $k = 1,2,3$ appear due to the application of the fractional Gagliardo-Nirenberg inequality from Lemma 4.2, while we assume the last condition (12) to get the same decay estimates for solutions w as those to the corresponding linear model (7).*

Remark 3.2. *If we change in Theorem 3.1 the order of σ_1, σ_2 and σ_3 for example $\sigma_2 \leqslant \sigma_3 \leqslant \sigma_1$, then we get a similar (by summitry) result to Theorem 3.1, where we feel the modification in condition (12) and the loss of decay for u changed to $\delta(p_1,p_3)$, whereas for w changed to $\gamma(p_3)$. In this case there is no loss of decay for v. The same remark if we take $\sigma_1 \leqslant \sigma_2 \leqslant \sigma_3$.*

Remark 3.3. *If we also change in weakly coupled system (1) the order of power nonlinearties, then we get a similar result to Theorem 3.1, where we also feel the modification in conditions (8)-(12).*

Example 3.1. *Let us illustrate the obtained result with an example of a weakly coupled system of a very well-known doubly damped semi-linear wave equation, that is*

$$\begin{aligned} u_{tt} - \Delta u + u_t - \Delta u_t &= |w|^{p_1}, \\ v_{tt} - \Delta v + v_t - \Delta v_t &= |u|^{p_2}, \\ w_{tt} - \Delta w + w_t - \Delta w_t &= |v|^{p_3}. \end{aligned}$$

On the one hand, we can see that the upper bound of p_1 can be chosen smaller or larger than the modified Fujita exponent (3) if $m_3 < m_1$ or $m_3 > m_1$, respectively.
On the other hand, one can shows after a straightforward calculations that the $(p_1 - p_2)$ curve which guarantees global (in time) existence of small data solutions to the following weakly coupled system

$$\begin{aligned} u_{tt} - \Delta u + u_t - \Delta u_t &= |v|^{p_1}, \\ v_{tt} - \Delta v + v_t - \Delta v_t &= |u|^{p_2}, \end{aligned}$$

is exactly similar to that in [12] *(or also in* [11] *when $m_2 = 1$). So, in comparison with second condition in (12) we can feel the nice influence of m_2 and m_3 on the (p_1-p_2) curve which is smaller or larger than that in* [11], [12] *if $m_3 < m_2$ or $m_3 > m_2$.*

To prove these results, we need to show some new linear estimates which are the main tools for the following sections.

4. Linear estimates

Let us show the derived $(L^m \cap L^2) - L^2$ and $L^2 - L^2$ linear estimates proved in [10] for solution to (7), where the authors chose the data spaces

$$(y_0, y_1) \in (H^\sigma \cap L^m) \times \left(L^2 \cap L^m\right), \quad m \in [1, 2).$$

Proposition 4.1. *Let $\sigma \geqslant 1$ in (7). For all $m \in [1, 2)$, the solutions y to (7) satisfy the following $(L^m \cap L^2) - L^2$ estimates*

$$\|\partial_t^j |D|^a y(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{a}{2\sigma}-j} \|(y_0, y_1)\|_{(L^m \cap H^a) \times (L^m \cap H^{[a+2(j-1)\sigma]^+})},$$

and the $L^2 - L^2$ estimates

$$\|\partial_t^j |D|^a y(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{a}{2\sigma}-j} \|(y_0, y_1)\|_{H^a \times H^{[a+2(j-1)\sigma]^+}},$$

for any $a \geqslant 0$, $j = 0, 1$ and for all space dimensions $n \geqslant 1$, where $[\cdot]^+ = \max\{0, \cdot\}$.

For the proof see [10].
In some parts of the proof of our main theorem we need to assume L^1 regularity for the initial data y_0, then for this reason we prove the following lemma which is important if the data

$$(y_0, y_1) \in \left(H^\sigma \cap L^1\right) \times \left(L^2 \cap L^m\right), \quad m \in [1, 2).$$

Lemma 4.1. *Let $\sigma \geqslant 1$ in (7) and $m \in [1, 2)$. The solutions y to (7) satisfy the following estimates*

$$\|y(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)} \left(\|y_0\|_{L^1 \cap L^2} + \|y_1\|_{L^m \cap L^2}\right), \tag{13}$$

$$\|y_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-1} \left(\|y_0\|_{L^1 \cap H^\sigma} + \|y_1\|_{L^m \cap L^2}\right), \tag{14}$$

$$\||D|^\sigma y(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma}\left(\frac{1}{m}-\frac{1}{2}\right)-\frac{1}{2}} \left(\|y_0\|_{L^1 \cap H^\sigma} + \|y_1\|_{L^m \cap L^2}\right). \tag{15}$$

Proof. We apply the Fourier transform to the linear equation (7), then we get for any fixed $\xi \in \mathbb{R}^n$ the following differential equation of second order:

$$\hat{y}_{tt} + (1 + |\xi|^{2\sigma})\hat{y}_t + |\xi|^{2\sigma}\hat{y} = 0, \;\; \hat{y}(0,\xi) = \hat{y}_0(\xi), \;\; \hat{y}_t(0,\xi) = \hat{y}_1(\xi),$$

where $\hat{y}(t,\xi) = \mathcal{F}(y)(t,x)$.
Straightforward calculations implies that the solution to the above equation can be written as follows:

$$\begin{aligned}|\xi|^a \partial_t^j \hat{y}(t,\xi) &= \frac{(-1)^j|\xi|^{2\sigma j+a}e^{-|\xi|^{2\sigma}t} + (-1)^{j+1}|\xi|^{2\sigma+a}e^{-t}}{1-|\xi|^{2\sigma}}\hat{y_0}(\xi) \\ &+ \frac{(-1)^j|\xi|^{2\sigma j+a}e^{-|\xi|^{2\sigma}t} + (-1)^{j+1}|\xi|^{a}e^{-t}}{1-|\xi|^{2\sigma}}\hat{y_1}(\xi) \\ &= \hat{K}(t,\xi)\hat{y_0}(\xi) + \hat{G}(t,\xi)\hat{y_1}(\xi),\end{aligned}$$

with $j = 0,1$ and $a \geqslant 0$.
In order to estimate the L^2 norm of the solution and its derivatives in small frequencies $|\xi| < 1$, we need only to estimate the L^2 norm of $\hat{K}(t,\xi)$ and L^{m_0} norm of $\hat{G}(t,\xi)$ where $m_0 = 2m/(2-m)$. Then, we write:

$$\begin{aligned}\|\partial_t^j|D|^a y(t,\cdot)\|_{L^2} &\lesssim \left\|\frac{(-1)^j|\xi|^{2\sigma j+a}e^{-|\xi|^{2\sigma}t} + (-1)^{j+1}|\xi|^{2\sigma+a}e^{-t}}{1-|\xi|^{2\sigma}}\right\|_{L^2}\|y_0\|_{L^1} \\ &+ \left\|\frac{(-1)^j|\xi|^{2\sigma j+a}e^{-|\xi|^{2\sigma}t} + (-1)^{j+1}|\xi|^{a}e^{-t}}{1-|\xi|^{2\sigma}}\right\|_{L^{m_0}}\|y_1\|_{L^m}.\end{aligned}$$

Here, we can estimate the above two norms as follows:

$$\begin{aligned}\left\|\frac{(-1)^j|\xi|^{2\sigma j+a}e^{-|\xi|^{2\sigma}t} + (-1)^{j+1}|\xi|^{2\sigma+a}e^{-t}}{1-|\xi|^{2\sigma}}\right\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{a}{2\sigma}-j} + e^{-t}, \\ \left\|\frac{(-1)^j|\xi|^{2\sigma j+a}e^{-|\xi|^{2\sigma}t} + (-1)^{j+1}|\xi|^{a}e^{-t}}{1-|\xi|^{2\sigma}}\right\|_{L^{m_0}} &\lesssim (1+t)^{-\frac{n}{2\sigma}(\frac{1}{m}-\frac{1}{2})-\frac{a}{2\sigma}-j} + e^{-t},\end{aligned}$$

where we used the following inequality:

$$\||\xi|^a e^{-(1+t)|\xi|^b}\|_{L^r} \lesssim (1+t)^{-\frac{n}{rb}-\frac{a}{b}}, \quad a \geqslant 0, \quad b > 0, \ n \geqslant 1, \ r \geqslant 1.$$

Summarizing, we obtain

$$\begin{aligned}\|\partial_t^j|D|^a y(t,\cdot)\|_{L^2} &\lesssim \left((1+t)^{-\frac{n}{4\sigma}-\frac{a}{2\sigma}-j} + e^{-t}\right)\|y_0\|_{L^1} \\ &+ \left((1+t)^{-\frac{n}{2\sigma}(\frac{1}{m}-\frac{1}{2})-\frac{a}{2\sigma}-j} + e^{-t}\right)\|y_1\|_{L^m},\end{aligned}$$

this implies the following desired estimate for low frequencies:

$$\begin{aligned}\|\partial_t^j|D|^a y(t,\cdot)\|_{L^2} &\lesssim (1+t)^{-\frac{n}{4\sigma}-\frac{a}{2\sigma}-j}\|y_0\|_{L^1} + (1+t)^{-\frac{n}{2\sigma}(\frac{1}{m}-\frac{1}{2})-\frac{a}{2\sigma}-j}\|y_1\|_{L^m} \\ &\lesssim (1+t)^{-\frac{n}{2\sigma}(\frac{1}{m}-\frac{1}{2})-\frac{a}{2\sigma}-j}\left(\|y_0\|_{L^1} + \|y_1\|_{L^m}\right),\end{aligned}$$

for $m \in [1,2)$.
For large frequencies $|\xi| > 1$, we use the same procedure as in [10] and the proof of Lemma 4.1 is completed. □

In the following we present two important tools used in the next section. The first one is the fractional Gagliardo-Nirenberg inequality in general form.

Lemma 4.2. *Let $1 < q_0, q_1, q_2 < \infty$, $\sigma > 0$ and $s \in [0,\sigma)$. Then, the following fractional Gagliardo-Nirenberg inequality holds for all $y \in L^{q_0} \cap \dot{H}^{\sigma,q_1}$*

$$\||D|^s y\|_{L^{q_2}} \lesssim \||D|^\sigma y\|_{L^{q_1}}^{\theta_{s,\sigma}(q_0,q_1,q_2)}\|y\|_{L^{q_0}}^{1-\theta_{s,\sigma}(q_0,q_1,q_2)},$$

where

$$\theta_{s,\sigma}(q_0,q_1,q_2)=\frac{\frac{1}{q_0}-\frac{1}{q_2}+\frac{s}{n}}{\frac{1}{q_0}-\frac{1}{q_1}+\frac{\sigma}{n}}\in\left[\frac{s}{\sigma},1\right].$$

The proof can be found in [1]

Lemma 4.3. *Let $a,b\in\mathbb{R}$. Then, it holds*

$$\int_0^t(1+t-s)^{-a}(1+s)^{-b}ds\leqslant\begin{cases}C(1+t)^{-\min\{a,b\}} & if\quad \max\{a,b\}>1,\\ C(1+t)^{-\min\{a,b\}}\log(2+t) & if\quad \max\{a,b\}=1,\\ C(1+t)^{1-a-b} & if\quad \max\{a,b\}<1.\end{cases}$$

For the proof one can see [2]

5. Proof of Theorems 3.1

Proof. To prove Theorems 3.1 we will use Banach's fixed point theorem. We define a family of Banach spaces $\{B(T)\}_{T>0}$ and the operator

$$\mathcal{S}:B(T)\to B(T).$$

For any $T>0$, we introduce the Banach spaces $B(T)$ as follows:

$$B(T):=\prod_{k=1}^{3}\Big(\mathcal{C}\left([0,T],H^{\sigma_k}\right)\cap\mathcal{C}^1\left([0,T],L^2\right)\Big),$$

equipped with the usual norm

$$\begin{aligned}\|(u,v,w)\|_{B(T)}&=\sup_{0\leqslant t\leqslant T}\Big\{(1+t)^{-\gamma(p_1)}M_1(t,u)+(1+t)^{-\delta(p_1,p_2)}M_2(t,v)\\&\qquad+M_3(t,w)\Big\}.\end{aligned}$$

The functions $M_1(t,u)$, $M_2(t,v)$ and $M_3(t,w)$ are defined with respect to the linear estimates with some loss of decay

$$\begin{aligned}M_1(t,u)&=(1+t)^{\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)}\|u(t,\cdot)\|_{L^2}+(1+t)^{\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\frac{1}{2}}\||D|^{\sigma_1}u(t,\cdot)\|_{L^2}\\&\quad+(1+t)^{\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+1}\|u_t(t,\cdot)\|_{L^2},\end{aligned}\tag{16}$$

$$\begin{aligned}M_2(t,v)&=(1+t)^{\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)}\|v(t,\cdot)\|_{L^2}+(1+t)^{\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)+\frac{1}{2}}\||D|^{\sigma_2}v(t,\cdot)\|_{L^2}\\&\quad+(1+t)^{\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)+1}\|\|v_t(t,\cdot)\|_{L^2},\end{aligned}\tag{17}$$

$$\begin{aligned}M_3(t,w)&=(1+t)^{\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)}\|w(t,\cdot)\|_{L^2}+(1+t)^{\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)+\frac{1}{2}}\||D|^{\sigma_3}w(t,\cdot)\|_{L^2}\\&\quad+(1+t)^{\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)+1}\|\|w_t(t,\cdot)\|_{L^2}\end{aligned}\tag{18}$$

We know that homogeneous Cauchy problem corresponding to (1) has exactly the following representation of solution:

$$\begin{aligned}u^{ln}(t,x)&=K_{\sigma_1}(t,x)*_{(x)}u_0(x)+G_{\sigma_1}(t,x)*_{(x)}u_1(x),\\v^{ln}(t,x)&=K_{\sigma_2}(t,x)*_{(x)}v_0(x)+G_{\sigma_2}(t,x)*_{(x)}v_1(x),\\w^{ln}(t,x)&=K_{\sigma_3}(t,x)*_{(x)}w_0(x)+G_{\sigma_3}(t,x)*_{(x)}w_1(x),\end{aligned}$$

where $*_{(x)}$ denotes the convolution product with respect to space variable x, and the kernels $K_{\sigma_k}(t,x)$, $G_{\sigma_k}(t,x)$ are defined in the proof of Lemma 4.1. We apply Duhamel's principle to obtain the integrals representation of solution to (1)

$$\begin{aligned} u(t,x) &= u^{ln}(t,x) + \int_0^t G_{\sigma_1}(t-s,x) *_{(x)} |w(s,x)|^{p_1} ds \\ &= u^{ln}(t,x) + u^{nl}(t,x), \\ v(t,x) &= v^{ln}(t,x) + \int_0^t G_{\sigma_2}(t-s,x) *_{(x)} |u(s,x)|^{p_2} ds \\ &= v^{ln}(t,x) + v^{nl}(t,x), \\ w(t,x) &= w^{ln}(t,x) + \int_0^t G_{\sigma_3}(t-s,x) *_{(x)} |v(s,x)|^{p_3} ds \\ &= w^{ln}(t,x) + w^{nl}(t,x). \end{aligned}$$

Now, we can define our operator $\mathcal{S} : B(T) \longrightarrow B(T)$ by the same formula

$$(u,v,w) \longmapsto \mathcal{S}(u,v,w) = \begin{pmatrix} u^{ln} + u^{nl} \\ v^{ln} + v^{nl} \\ w^{ln} + w^{nl} \end{pmatrix}.$$

The main goal now is to prove that the operator $\mathcal{S}$ satisfies for any $(u,v,w) \in B(T)$ and $(\bar{u},\bar{v},\bar{w}) \in B(T)$ the following inequalities:

$$\begin{aligned} \|\mathcal{S}(u,v,w)\|_{B(T)} \lesssim\; & \|(u_0,u_1)\|_{\mathcal{A}^m_{\sigma_1}} + \|(v_0,v_1)\|_{\mathcal{A}^m_{\sigma_2}} + \|(w_0,w_1)\|_{\mathcal{A}^m_{\sigma_3}} \\ & + \|(u,v,w)\|^{p_1}_{B(T)} + \|(u,v,w)\|^{p_2}_{B(T)} + \|(u,v,w)\|^{p_3}_{B(T)}, \end{aligned} \tag{19}$$

$$\begin{aligned} & \|\mathcal{S}(u,v,w) - \mathcal{S}(\bar{u},\bar{v},\bar{w})\|_{B(T)} \lesssim \|(u,v,w) - (\bar{u},\bar{v},\bar{w})\|_{B(T)} \times \\ & \Big(\|(u,v,w)\|^{p_1-1}_{B(T)} + \|(u,v,w)\|^{p_2-1}_{B(T)} + \|(u,v,w)\|^{p_3-1}_{B(T)} + \|(\bar{u},\bar{v},\bar{w})\|^{p_1-1}_{B(T)} \\ & \quad + \|(\bar{u},\bar{v},\bar{w})\|^{p_2-1}_{B(T)} + \|(\bar{u},\bar{v},\bar{w})\|^{p_3-1}_{B(T)} \Big). \end{aligned} \tag{20}$$

Using linear estimates and the fact that $\gamma(p_1), \delta(p_1,p_2) > 0$, we can reduce the proof of (19) to

$$\|(u^{nl},v^{nl},w^{nl})\|_{B(T)} \lesssim \|(u,v,w)\|^{p_1}_{B(T)} + \|(u,v,w)\|^{p_2}_{B(T)} + \|(u,v,w)\|^{p_3}_{B(T)}. \tag{21}$$

To prove (21) we divide the interval $[0,t]$ into two sub-intervals $[0,t/2]$ and $(t/2,t]$, where we use from Lemma 4.1 the $(L^{m_k} \cap L^2) - L^2$ linear estimates if $s \in [0,t/2]$ and $L^2 - L^2$

estimates if $s \in [t/2, t]$, then we have

$$
\begin{aligned}
\|u^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \||w(s,\cdot)|^{p_1}\|_{L^{m_1}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} \||w(s,\cdot)|^{p_1}\|_{L^2}\, ds, \qquad (22)\\
\|u_t^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1} \||w(s,\cdot)|^{p_1}\|_{L^{m_1}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} (1+t-s)^{-1} \||w(s,\cdot)|^{p_1}\|_{L^2}\, ds, \qquad (23)\\
\||D|^{\sigma_1} u^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}} \||w(s,\cdot)|^{p_1}\|_{L^{m_1}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} (1+t-s)^{-\frac{1}{2}} \||w(s,\cdot)|^{p_1}\|_{L^2}\, ds, \qquad (24)\\
\|v^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)} \||u(s,\cdot)|^{p_2}\|_{L^{m_2}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} \||u(s,\cdot)|^{p_2}\|_{L^2}\, ds, \qquad (25)\\
\|v_t^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-1} \||u(s,\cdot)|^{p_2}\|_{L^{m_2}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} (1+t-s)^{-1} \||u(s,\cdot)|^{p_2}\|_{L^2}\, ds, \qquad (26)\\
\||D|^{\sigma_2} v^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_2}\left(\frac{1}{m_2}-\frac{1}{2}\right)-\frac{1}{2}} \||u(s,\cdot)|^{p_2}\|_{L^{m_2}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} (1+t-s)^{-\frac{1}{2}} \||u(s,\cdot)|^{p_2}\|_{L^2}\, ds, \qquad (27)\\
\|w^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)} \||v(s,\cdot)|^{p_3}\|_{L^{m_3}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} \||v(s,\cdot)|^{p_3}\|_{L^2}\, ds, \qquad (28)\\
\|w_t^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)-1} \||v(s,\cdot)|^{p_3}\|_{L^{m_3}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} (1+t-s)^{-1} \||v(s,\cdot)|^{p_3}\|_{L^2}\, ds, \qquad (29)\\
\||D|^{\sigma_3} w^{nl}(t,\cdot)\|_{L^2} &\lesssim \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)-\frac{1}{2}} \||v(s,\cdot)|^{p_3}\|_{L^{m_3}\cap L^2}\, ds \\
&\qquad + \int_{t/2}^{t} (1+t-s)^{-\frac{1}{2}} \||v(s,\cdot)|^{p_3}\|_{L^2}\, ds. \qquad (30)
\end{aligned}
$$

To control integrals in (22) to (24) we need to estimate the following norms:

$$
\|w(s,\cdot)\|^{p_1}_{L^{2p_1}}, \quad \||w(s,\cdot)|^{p_1}\|_{L^{m_1}\cap L^2} = \|w(s,\cdot)\|^{p_1}_{L^{m_1 p_1}} + \|w(s,\cdot)\|^{p_1}_{L^{2p_1}}.
$$

From the definition of the norm we have

$$\||D|^{\sigma_3} w(s,\cdot)\|_{L^2} \lesssim (1+s)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)-\frac{1}{2}} \|(u,v,w)\|_{B(T)}, \tag{31}$$

$$\|w(s,\cdot)\|_{L^2} \lesssim (1+s)^{-\frac{n}{2\sigma_3}\left(\frac{1}{m_3}-\frac{1}{2}\right)} \|(u,v,w)\|_{B(T)}. \tag{32}$$

Applying the fractional Gagliardo-Nirenberg inequality from Lemma 4.2 we can estimate the above two norms as follows:

$$\|w(s,\cdot)\|^{p_1}_{L^{m_1p_1}} \lesssim (1+s)^{-p_1\frac{n}{2m_3\sigma_3}+\frac{n}{2m_1\sigma_3}} \|(u,v,w)\|^{p_1}_{B(T)},$$

$$\|w(s,\cdot)\|^{p_1}_{L^{2p_1}} \lesssim (1+s)^{-p_1\frac{n}{2m_3\sigma_3}+\frac{n}{4\sigma_3}} \|(u,v,w)\|^{p_1}_{B(T)}. \tag{33}$$

Hence, we conclude

$$\||w(s,\cdot)|^{p_1}\|_{L^{m_1}\cap L^2} \lesssim (1+s)^{-p_1\frac{n}{2m_3\sigma_3}+\frac{n}{2m_1\sigma_3}} \|(u,v,w)\|^{p_1}_{B(T)}, \tag{34}$$

provided that

$$\frac{2}{m_1} \leqslant p_1 \leqslant \frac{n}{n-2\sigma_3} \quad forall \quad 2\sigma_3 < n \leqslant \frac{4\sigma_3}{2-m_1}.$$

We have

$$(1+t-s) \approx (1+t) \ for \ s \in [0,t/2], \quad (1+s) \approx (1+t) \ for \ s \in [t/2,t]. \tag{35}$$

Using (35), then we obtain from (34) the estimate for $[0,t/2]$

$$\int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \||w(s,\cdot)|^{p_1}\|_{L^{m_1}\cap L^2}\, ds \lesssim$$

$$\begin{cases} (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+1-\frac{np_1}{2m_3\sigma_3}+\frac{n}{2m_1\sigma_3}} \|(u,v,w)\|^{p_1}_{B(T)} & \text{if} \quad p_1 < \frac{m_3}{m_1}+\frac{2m_3\sigma_3}{n}, \\ (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\varepsilon_0} \|(u,v,w)\|^{p_1}_{B(T)} & \text{if} \quad p_1 = \frac{m_3}{m_1}+\frac{2m_3\sigma_3}{n}, \end{cases}$$

with sufficiently small positive number ε_0.
For the second integral over $[t/2,t]$ we can conclude

$$(1+t)^{-\frac{np_1}{2m_3\sigma_3}+\frac{n}{4\sigma_3}} \|(u,v,w)\|^{p_1}_{B(T)} \int_{t/2}^{t} ds \lesssim (1+t)^{1-\frac{np_1}{2m_3\sigma_3}+\frac{n}{4\sigma_3}} \|(u,v,w)\|^{p_1}_{B(T)}.$$

Using the same way one can obtain

$$\|u^{nl}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)+\gamma(p_1)} \|(u,v,w)\|^{p_1}_{B(T)},$$

$$\|u^{nl}_t(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1+\gamma(p_1)} \|(u,v,w)\|^{p_1}_{B(T)},$$

$$\||D|^{\sigma_1} u^{nl}(t,\cdot)\|_{L^2} \lesssim (1+t)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}+\gamma(p_1)} \|(u,v,w)\|^{p_1}_{B(T)},$$

where $\sigma_3 \leqslant \sigma_1$.
Similarly to the function u we can after straightforward computations get the desired estimates for v and w under conditions (8) to (12). Finaly inequality (19) proved. To prove (20) we assume that (u,v,w) and $(\bar{u},\bar{v},\bar{w})$ belong to $B(T)$, then we write

$$\mathcal{S}(u,v,w) - S(\bar{u},\bar{v},\bar{w}) = \begin{pmatrix} u^{nl} - \bar{u}^{nl} \\ v^{nl} - \bar{v}^{nl} \\ w^{nl} - \bar{w}^{nl} \end{pmatrix},$$

where

$$u^{nl}(t,x) - \bar{u}^{nl}(t,x) = \int_0^t G_{\sigma_1}(t-s,x) *_{(x)} (|w(s,x)|^{p_1} - |\bar{w}(s,x)|^{p_1})ds.$$

Hölder's inequality leads to

$$\begin{aligned} \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^k} \quad &\lesssim \quad \|w(s,\cdot) - \bar{w}(s,\cdot)\|_{L^{kp_1}} \\ &\times \left(\|w(s,\cdot)\|^{p_1-1}_{L^{kp_1}} + \|\bar{w}(s,\cdot)\|^{p_1-1}_{L^{kp_1}} \right), \end{aligned} \tag{36}$$

where $k = m_1, 2$ to control all norms of $u^{nl}(t,x) - \bar{u}^{nl}(t,x)$ appearing in (16). In fact, we will use the same approach that we proved from (22) to (24), that is

$$\begin{aligned} &\|(u^{nl} - \bar{u}^{nl})(t,\cdot)\|_{L^2} \lesssim \\ &\quad \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)} \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^{m_1}\cap L^2}\, ds \\ &\qquad + \int_{t/2}^t \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^2}\, ds, \\ &\|(u^{nl} - \bar{u}^{nl})_t(t,\cdot)\|_{L^2} \lesssim \\ &\quad \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-1} \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^{m_1}\cap L^2}\, ds \\ &\qquad + \int_{t/2}^t (1+t-s)^{-1} \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^2}\, ds, \\ &\| |D|^{\sigma_1}(u^{nl} - \bar{u}^{nl})(t,\cdot)\|_{L^2} \lesssim \\ &\quad \int_0^{t/2} (1+t-s)^{-\frac{n}{2\sigma_1}\left(\frac{1}{m_1}-\frac{1}{2}\right)-\frac{1}{2}} \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^{m_1}\cap L^2}\, ds \\ &\qquad + \int_{t/2}^t (1+t-s)^{-\frac{1}{2}} \| |w(s,\cdot)|^{p_1} - |\bar{w}(s,\cdot)|^{p_1} \|_{L^2}\, ds. \end{aligned}$$

Using again the norm of the solution space $B(T)$ and fractional Gagliardo-Nirenberg inequality we can estimate the norms in (37) as follows

$$\begin{aligned} \|w(s,\cdot) - \bar{w}(s,\cdot)\|_{L^{kp_1}} &\lesssim (1+s)^{-\frac{n}{2m_3\sigma_3}+\frac{n}{2kp_1\sigma_3}} \|(u,v,w) - (\bar{u},\bar{v},\bar{w})\|_{B(T)}, \\ \|w(s,\cdot)\|^{p_1-1}_{L^{kp_1}} &\lesssim (1+s)^{\left(-\frac{n}{2m_3\sigma_3}+\frac{n}{2kp_1\sigma_3}\right)(p_1-1)} \|(u,v,w)\|^{p_1-1}_{B(T)}, \\ \|\bar{w}(s,\cdot)\|^{p_1-1}_{L^{kp_1}} &\lesssim (1+s)^{\left(-\frac{n}{2m_3\sigma_3}+\frac{n}{2kp_1\sigma_3}\right)(p_1-1)} \|(\bar{u},\bar{v},\bar{w})\|^{p_1-1}_{B(T)}. \end{aligned}$$

Now, we use the same conditions again for p_1 and n as in (8) to (12) to obtain the loss of decay for $u^{nl} - \bar{u}^{nl}$, and in the same way for $v^{nl} - \bar{v}^{nl}$ and $w^{nl} - \bar{w}^{nl}$. Summarizing, the proof of Theorem 3.1 is completed. □

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