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SOME RESULTS ON A SUBCLASS OF HARMONIC MAPPINGS OF ORDER ALPHA

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Abstract. Let S_H be the class of harmonic mappings defined by

$$S_H = \left\{ f = h(z) + \overline{g(z)} \mid h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, b_1 < 1 \right\}$$

where $h(z)$ and $g(z)$ are analytic. Additionally

$$f(z) \in S_H(\alpha) \Leftrightarrow \left| \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - \frac{1 - \overline{b_1}}{1 + \overline{b_1}} \right| < \left| \frac{1 - \overline{b_1}}{1 + \overline{b_1}} \right| - \alpha, \;\; z \in \mathcal{U}, \;\; 0 \leqslant \alpha < \frac{1 - \overline{b_1}}{1 + \overline{b_1}}$$

In the present work, by considering the analyticity of the functions defined by R. M. Robinson [7], we discuss the applications to the harmonic mappings.

Keywords: Harmonic Mappings, Subordination principle, Distortion theorem.

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1. Introduction

Let $\mathcal{U} = \{z| \; |z| < 1\}$ be the open unit disc in the complex plane $\mathbb{C}$. A complex-valued harmonic function $f : \mathcal{U} \to \mathbb{C}$ has the representation

$$f = h(z) + \overline{g(z)} \qquad (1)$$

where $h(z)$ and $g(z)$ are analytic and have the following power series expansion,

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathcal{U}$$

where a_n, $b_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$, choose i.e, $b_0 = 0$ so the representation (1) is unique in $\mathcal{U}$ and is called the canonical representation of f.

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For the univalent and sense-preserving harmonic mapping f in $\mathcal{U}$, it is convenient to make further normalization (without loss of generality), $h(0) = 0$ (i.e. , $a_0 = 0$) and $h'(0) = 1$ (i.e. , $a_1 = 1$). The family of such functions f is denoted by S_H [1] . The family of all functions $f \in S_H$ with the additional property that $g'(0) = 0$ (i.e. , $b_1 = 0$) is denoted by S_H^0 [1] . Observe that the classical family of univalent functions S consists of all functions $f \in S_H^0$ such that $g(z) \equiv 0$. Thus it is clear that $S \subset S_H^0 \subset S_H$ [1] .

Let Ω be the family of functions $\phi(z)$ regular in $\mathcal{U}$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathcal{U}$.

Next, for arbitrary fixed real numbers A, B, $-1 < A \leq 1$, $-1 \leq B < A$ denoted by $P(A, B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in $\mathcal{U}$ and such that $p(z)$ is in $P(A, B)$

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \tag{2}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathcal{U}$. This class was introduced by Janowski W. [5].

Next, let $s_1(z) = z + c_2 z^2 + c_3 z^3 + \dots$ and $s_2(z) = z + d_2 z^2 + d_3 z^3 + \dots$ be regular in $\mathcal{U}$. If there exists $\phi(z) \in \omega$ such that $s_1(z) = s_2(\phi(z))$ for all $z \in \mathcal{U}$, then we say that $s_1(z)$ is subordinated to $s_2(z)$ and denoted by $s_1(z) \prec s_2(z)$ and $S_1(\mathcal{U}) \subset S_2(\mathcal{U})$.

Finally, let $f = h(z) + \overline{g(z)}$ be an element of S_H. If f satisfies the condition

$$\frac{\partial}{\partial \theta}(Argf(re^{i\theta})) = Re\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} > 0$$

then f is called harmonic starlike function. The class of such functions is denoted by S_{HS}^*. Also let $f = h(z) + \overline{g(z)}$ be an element of S_H. If f satisfies the condition

$$\frac{\partial}{\partial \theta}(\frac{\partial}{\partial \theta}(Argf(re^{i\theta}))) = Re\left(\frac{z(zh'(z))' - \overline{z(zg'(z))'}}{zh'(z) + \overline{zg'(z)}}\right) > 0$$

then f is called convex harmonic function.The class of convex harmonic function is denoted by S_{HC}.

In this paper we will investigate the following subclass of harmonic mappings

$$S_H(\alpha) = \Big\{ f = h(z) + \overline{g(z)} \mid \left| \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - \frac{1 - \overline{b_1}}{1 + \overline{b_1}} \right| < \left| \frac{1 - \overline{b_1}}{1 + \overline{b_1}} \right| - \alpha,$$
$$z \in \mathcal{U}, 0 \leq \alpha < \left| \frac{1 - \overline{b_1}}{1 + \overline{b_1}} \right| \Big\} \tag{3}$$

For this investigation we will use the following lemma.

Lemma 1.1. *([5]) Let $\phi(z)$ be regular in the open unit disc $\mathcal{U}$ with $\phi(0) = 0$. Then if $\phi(z)$ attains its maximum value on the circle $|z| = r$ at z_0, then we can write $z_0.\phi'(z_0) = k\phi(z_0)$ where k is real and $k \geq 1$.*

2. Main Results

Theorem 2.1. *Let* $f = h(z) + \overline{g(z)} \in S_H(\alpha)$ *with*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = \sum_{n=2}^{\infty} b_n z^n, \quad (b_1 = 0)$$

. If $f(z)$ *satisfies*

$$\sum_{n=2}^{\infty} ((n-\alpha)|a_n| + (n+2-\alpha)|b_n|) \leq 1-\alpha \tag{4}$$

for some α $(0 \leq \alpha < 1)$*, then* $f(z) \in S_H(\alpha)$.

Proof. Since $f(z) \in S_H(\alpha)$ is equivalent to

$$\left| \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} - 1 \right| < 1-\alpha, \quad (z \in \mathcal{U})$$

we have to show that the inequality (4) implies that

$$|zh'(z) - h(z) - \overline{(zg'(z) + g(z))}| < (1-\alpha)|h(z) + \overline{g(z)}|.$$

It follows that

$$(1-\alpha)|h(z) + \overline{g(z)}| - |zh'(z) - h(z) - \overline{(zg'(z) + g(z))}| = (1-\alpha)\left| z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} \overline{b_n}\overline{z}^n \right|$$
$$- \left| \sum_{n=2}^{\infty} (n-1) a_n z^n - \sum_{n=2}^{\infty} (n+1) \overline{b_n}\overline{z}^n \right|$$
$$= (1-\alpha)|z| \left| 1 + \sum_{n=2}^{\infty} a_n z^{n-1} + \sum_{n=2}^{\infty} \overline{b_n}\overline{z}^{n-1}\frac{\overline{z}}{z} \right| - |z| \left| \sum_{n=2}^{\infty} (n-1) a_n z^{n-1} - \sum_{n=2}^{\infty} (n+1) \overline{b_n}\overline{z}^{n-1}\frac{\overline{z}}{z} \right|$$
$$= |z| \left\{ (1-\alpha) \left| 1 + \sum_{n=2}^{\infty} a_n z^{n-1} + \sum_{n=2}^{\infty} \overline{b_n}\overline{z}^{n-1}\frac{\overline{z}}{z} \right| - \left| \sum_{n=2}^{\infty} (n-1) a_n z^{n-1} - \sum_{n=2}^{\infty} (n+1) \overline{b_n}\overline{z}^{n-1}\frac{\overline{z}}{z} \right| \right\}$$
$$\geqslant |z| \left\{ (1-\alpha) \left(1 - \sum_{n=2}^{\infty} |a_n||z|^{n-1} - \sum_{n=2}^{\infty} |b_n||z|^{n-1} \right) \right.$$
$$\left. - \left(\sum_{n=2}^{\infty} (n-1)|a_n||z|^{n-1} + \sum_{n=2}^{\infty} (n+1)|b_n||z|^{n-1} \right) \right\}$$
$$= |z| \left\{ (1-\alpha) - \sum_{n=2}^{\infty} (n-\alpha)|a_n||z|^{n-1} - \sum_{n=2}^{\infty} (n+2-\alpha)|b_n||z|^{n-1} \right\}$$
$$> |z| \left\{ (1-\alpha) - \sum_{n=2}^{\infty} ((n-\alpha)|a_n| + (n+2-\alpha)|b_n|) \right\}.$$

Therefore, if the inequality (4) holds true, then we have that

$$|zh'(z) - h(z) - \overline{(zg'(z) + g(z))}| < (1-\alpha)|h(z) + \overline{g(z)}|$$

which implies that $f(z) \in S_H(\alpha)$

□

Corollary 2.1. *Let $f(z)$ satisfies*

$$\sum_{n=2}^{\infty}(n|a_n| + (n+2)|b_n|) \leq 1,$$

then $f(z) \in S_H(0)$.

Proof. If we take $\alpha = 0$ in Theorem 2.1, then we have the result. □

Definition 2.1. $f(z) \in C_H(\alpha) \Leftrightarrow |h'(z) - \overline{g'(z)} - (1 - \overline{b_1})| < |1 - b_1| - \alpha$, $z \in \mathcal{U}$ *for some* α, $(0 \leq \alpha < |1 - \overline{b_1}|)$.

Theorem 2.2. *If $f = h(z) + \overline{g(z)}$ satisfies*

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq |1 - \overline{b_1}| - \alpha \tag{5}$$

for some α, $(0 \leq \alpha < |1 - \overline{b_1}|)$, then $f(z) \in C_H(\alpha)$.

Proof. We note that

$$\begin{aligned}
\left|h'(z) - \overline{g'(z)} - (1 - \overline{b_1})\right| &= \left|1 + \sum_{n=2}^{\infty} n a_n z^{n-1} - \overline{b_1} - \sum_{n=2}^{\infty} n\overline{b_n}\overline{z}^{n-1} - 1 + \overline{b_1}\right| \\
&= \left|\sum_{n=2}^{\infty} n a_n z^{n-1} - \sum_{n=2}^{\infty} n\overline{b_n}\overline{z}^{n-1}\right| \\
&= \left|\sum_{n=2}^{\infty} n z^{n-1}\left(a_n - \overline{b_n}\left(\frac{\overline{z}}{z}\right)^{n-1}\right)\right| \\
&< |z|\sum_{n=2}^{\infty} n\left|a_n - \overline{b_n}\left(\frac{\overline{z}}{z}\right)^{n-1}\right| \\
&\leq |z|\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \\
&< \sum_{n=2}^{\infty} n(|a_n| + |b_n|).
\end{aligned}$$

□

Corollary 2.2. *If $f = h(z) + \overline{g(z)}$ satisfies*

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \leq |1 - \overline{b_1}|,$$

then $f(z) \in C_H(0)$.

Proof. If we take $\alpha = 0$ in Theorem 2.2, we have the result. □

Theorem 2.3. *$f = h(z) + \overline{g(z)} \in C_H(0)$ with $arg a_n = arg b_n = -n\pi$ for $n = 2, 3, 4, \ldots$, then*

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq 1 - Reb_1 - \alpha \tag{6}$$

Proof. In Definition 2.3, we know that $f(z) \in C_H(\alpha)$ satisfies

$$Re(h'(z) - g'(z)) > \alpha, \quad z \in \mathcal{U}$$

with $0 \leq \alpha < |1 - \overline{b_1}|$.
Since $arga_n = argb_n = -n\pi$, we have that

$$\begin{aligned} Re(h'(z) - g'(z)) &= Re(1 + \sum_{n=2}^{\infty} na_n z^{n-1} - b_1 - \sum_{n=2}^{\infty} nb_n z^{n-1}) \\ &= 1 - Reb_1 + Re(\sum_{n=2}^{\infty} n(a_n - b_n) z^{n-1}) \\ &= 1 - Reb_1 + Re(\sum_{n=2}^{\infty} n|a_n - b_n| e^{-in\pi} z^{n-1}) \\ &> \alpha \end{aligned}$$

for all $z \in \mathcal{U}$.
Let us consider a point $z = |z|e^{i\pi}$. Then we have that

$$\begin{aligned} Re(h'(z) - g'(z)) &= 1 - Reb_1 + Re(\sum_{n=2}^{\infty} n|a_n - b_n||z|^{n-1} e^{-i\pi}) \\ &= 1 - Reb_1 - \sum_{n=2}^{\infty} n|a_n - b_n||z|^{n-1} \\ &> \alpha \end{aligned}$$

for $|z| > 1$. Therefore, letting $|z| \to 1$, we see that

$$1 - Reb_1 - \sum_{n=2}^{\infty} n|a_n - b_n| \geqslant \alpha$$

for $f(z) \in C_H(\alpha)$. This completes the proof of the theorem. □

Corollary 2.3. *If $f = h(z) + \overline{g(z)} \in C_H(0)$ with $arga_n = argb_n = -n\pi$ for $n = 2, 3, 4, \dots$, then*

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq 1 - Reb_1.$$

Proof. If we take $\alpha = 0$ in Theorem 2.3, we have the result. □

Corollary 2.4. *If $f = h(z) + \overline{g(z)} \in C_H(\alpha)$ with $arga_n = argb_n = -n\pi$ for $n = 2, 3, 4, \dots,$ then*

$$|a_n - b_n| \leqslant \frac{1}{n}(1 - Reb_1 - \alpha), \quad n = 2, 3, 4, \dots$$

Proof. It is a simple consequence of Theorem 2.3. □

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