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HARMONIC RESONANCE PHENOMENA ON NONLINEAR SH WAVES

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ABSTRACT. The interaction of shear horizontal (SH) waves in a two layered elastic medium and its mth harmonic component is studied. The dispersion relation is analysed to obtain the wave number-phase velocity pairs where the third and fifth harmonic resonance phenomena emerge. By employing an asymptotic perturbation method it is shown that the balance between the weak nonlinearity and dispersion yields a coupled nonlinear Schrödinger (CNLS) equation for the slowly varying amplitudes of the fundamental wave and its fifth harmonic component. The nonlinearity effects of the materials and the ratio of layers' thicknesses on the linear instabilities of solutions and the existence of solitary waves are examined.

Keywords: Nonlinear elasticity, nonlinear waves, solitary waves, perturbation methods

AMS Subject Classification: 74J30, 74B20.

1. INTRODUCTION

A special class of problems on wave propagation is the nonlinear wave interaction which has been studied extensively as a consequence of its importance in many different fields such as plasma physics [1], [2], atmospheric science [3], [4] and biophysics [5]. In addition, investigation of nonlinear elastic wave interaction in terms of different elastic wave guides and wave polarization has been the subject of many studies, [6]-[13].

In [6], a system of six semi-linear hyperbolic partial differential equations was derived for the nonlinear interaction of two co-directional quasi-harmonic Rayleigh surface waves on an isotropic solid. Kalyanasundaram derived a CNLS system for the nonlinear mode coupling between monochromatic Rayleigh and Love waves on a half-space of homogeneous isotropic elastic solid covered by a thin layer [7]. In [8], it was shown that nonlinear interaction between SH surface and Rayleigh waves on an elastic medium was governed asymptotically

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by a Zakharov type system. Teymur also showed that the nonlinear interaction of two co-directional surface shear horizontal waves in a layered elastic half-space was governed by a CNLS system asymptotically, see, [9]. Another such example is the work [10] where the generation of the second harmonic for finite-amplitude waves in a homogeneous isotropic elastic plate was discussed. In addition, the interaction of two Rayleigh-Lamb modes of the same nature in isotropic weakly nonlinear elastic plates is investigated in [11]. In [12], possibility of nonlinear elastic wave interactions for an isotropic solid defined by three material constants of the third order was studied by a different wave polarization. Ahmetolan and Demirci investigated the nonlinear interaction of co-directional SH waves in a two-layered plate of uniform thickness, and showed that the first order slowly varying amplitudes of interacting waves were governed asymptotically by a CNLS system, see, [13].

In some of the aforementioned works above, different types of CNLS systems were obtained for complex wave amplitudes of modulated waves by balancing the nonlinearity and dispersion by using an appropriate asymptotic perturbation method [14], [15]. One common feature of these works is the assumption of the nonexistence of harmonic resonance. Note that for the same phase velocities of the fundamental wave, c_c and its mth harmonic component at a critical wave number, k_c, the mth harmonic resonance occurs between the interacting waves. Investigation of such resonance in the interaction of nonlinear waves has been the subject of few articles [16], [17]. For example, Nayfeh investigated the temporal and spatial variation of the amplitudes and phases of capillary-gravity waves in a deep water near the third harmonic resonant wave-number in [16]. Also, Teymur investigated the fifth harmonic resonance of Love waves on a neo-Hookean layered half space in [17].

When the phase velocity of the fundamental wave coincides with its mth harmonic at the critical wave number k_c, the uniform validity of the asymptotic expansion ceases. In this case, since the (k_c, ω_c) and $(mk_c, m\omega_c)$ pairs simultaneously satisfy the linear dispersion relation, mth order amplitude functions tend to be infinite as $k \to k_c$. When this happens, energy transfer occurs between the fundamental wave and its mth harmonic component, that is, nonlinear resonant interaction occurs between them. In [18], the propagation of nonlinear SH waves in a two-layered compressible elastic medium with materials in both layers with different material properties is studied by using an appropriate asymptotic perturbation method. In that work, it has also been observed that mth harmonic resonance emerges at some critical wave number-phase velocity pairs for specific linear material parameters and thickness values of layers where $m \in \{3, 5, 7, 9, ...\}$. The asymptotic expansion used in [18] is not uniformly valid for (k_c, c_c), and thus an appropriate uniformly valid asymptotic expansion is required for this investigation.

In the present work, the mth, harmonic resonance phenomena of nonlinear SH waves which propagate in a medium given in [18] is investigated. An mth harmonic resonance may exist where $m \in \{3, 5, 7, 9, ...\}$, if (k_c, ω_c) and $(mk_c, m\omega_c)$ simultaneously satisfy the linear dispersion relation.

The outline of the article is as follows: Formulation of the problem is given in section 2, and a nonlinear boundary value problem characterizing SH wave propagation in a two-layered incompressible elastic medium is obtained. The relations for the critical wave number-phase velocity pair corresponding to the third and fifth harmonic resonances are derived. In Section 3, the mth harmonic resonance of slowly varying amplitudes of weakly nonlinear SH waves in such a medium is investigated by employing a multiple scale perturbation method [14]. For $m = 5$, it is shown that the balance between the weak nonlinearity and dispersion yields a CNLS system for the slowly varying amplitudes of the fundamental wave and its fifth harmonic component. In Section 4, discussions on the effects of linear and nonlinear material properties of the medium and the ratio of the layers' thickness on the linear stability of the plane wave solutions as well as the existence of solitary envelope solutions are performed [20]. Some concluding remarks are presented in the final section.

2. Formulation of Problem

Let (X, Y, Z) and (x, y, z) be material and spatial coordinate systems, respectively. Consider a plate of uniform thickness which is composed of two layers occupying the regions

$$\begin{aligned} P_1 &= \{(X, Y, Z) | Y \in (0, h_1), X, Z \in (-\infty, \infty)\}, \\ P_2 &= \{(X, Y, Z) | Y \in (-h_2, 0), X, Z \in (-\infty, \infty)\} \end{aligned} \tag{1}$$

where P_1 and P_2 denote the upper and lower layers with thickness h_1 and h_2, respectively. The free boundaries, $Y = h_1$ and $-h_2$, are assumed to be free of traction and stresses, and displacements are continuous at $Y = 0$. SH deformation of a particle which is an anti-plane shear motion is defined by

$$x = X,\ y = Y,\ z = Z + u_\nu(X, Y, t),\ \nu = 1, 2 \tag{2}$$

where t is the time, and u_ν is the particle's displacement in P_ν in the Z-direction due to the polarization of waves [21]. The constituent materials of the layers are assumed to be incompressible, homogeneous, isotropic, elastic, and their strain energy functions are considered to be in the form: $\Sigma_\nu = \Sigma_\nu(I_\nu)$ where I_ν are the first invariant of the Green's deformation tensor $C_{KL} = x_{kK} x_{kL}$ [21]. These materials are called generalized neo-Hookean and the wave motion described above can exist in the layered plate made of such materials in the absence of body forces acting in the (X, Y) plane. For problems involving anti-plane shear motions in nonlinear elastic media [22], generally, an anti-plane shear motion without any restrictions on stress constitutive equations can't be maintained in a medium in the absence of body forces acting in a plane perpendicular to the polarization direction. Then replacing the displacement of a particle in the upper layer, u_1 by $u = u(X, Y, t)$ and in the lower layer, u_2 by $v = v(X, Y, t)$, the governing equation of motion and boundary conditions of the problem involving terms not higher than the third degree in the deformation gradients are obtained;

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c_1^2\Big(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2}\Big) &= n_1\Big[\frac{\partial}{\partial X}\Big(\frac{\partial u}{\partial X}\mathcal{Q}(u)\Big) + \frac{\partial}{\partial Y}\Big(\frac{\partial u}{\partial Y}\mathcal{Q}(u)\Big)\Big] \text{ in } P_1, \\ \frac{\partial^2 v}{\partial t^2} - c_2^2\Big(\frac{\partial^2 v}{\partial X^2} + \frac{\partial^2 v}{\partial Y^2}\Big) &= n_2\Big[\frac{\partial}{\partial X}\Big(\frac{\partial v}{\partial X}\mathcal{Q}(v)\Big) + \frac{\partial}{\partial Y}\Big(\frac{\partial v}{\partial Y}\mathcal{Q}(v)\Big)\Big] \text{ in } P_2, \end{aligned} \tag{3}$$

$$\begin{gathered} \frac{\partial u}{\partial Y} = 0 \quad \text{on} \quad Y = h_1, \qquad \frac{\partial v}{\partial Y} = 0 \quad \text{on} \quad Y = -h_2\ , \\ u = v \quad \text{and} \quad \frac{\partial u}{\partial Y} - \gamma\frac{\partial v}{\partial Y} = \gamma q_2\frac{\partial v}{\partial Y}\mathcal{Q}(v) - q_1\frac{\partial u}{\partial Y}\mathcal{Q}(u) \quad \text{on} \quad Y = 0, \end{gathered} \tag{4}$$

where $\mathcal{Q}(\psi) = (\frac{\partial\psi}{\partial X})^2 + (\frac{\partial\psi}{\partial Y})^2$. c_1 and c_2 are linear shear wave velocities in the layers defined by $c_\nu^2 = \mu_\nu/\rho_\nu$, $\nu = 1, 2$, where $\mu_\nu = 2\Sigma'_\nu(3)$ are linear shear modules of the layer materials, and ρ_ν are densities of the layers in the initial state. Since the wave motion is isochoric, ρ_ν remains constant during the motion. The constants γ and q_ν are defined by

$$\gamma = \mu_2/\mu_1 \qquad q_\nu = n_\nu/c_\nu^2\ ,\ \nu = 1, 2, \tag{5}$$

where $n_\nu = \frac{2}{\rho_\nu}\Sigma_\nu''(3)$ exhibits the nonlinearity of the materials. If $n_\nu > 0$, it exhibits hardening behavior in shear, whereas softening behavior for $n_\nu < 0$ [21],[23] .

3. mTH HARMONIC RESONANCE PHENOMENA

Let the solutions of linear equations be in the following form

$$u(X,Y,t) = U(Y)e^{i(kX-\omega t)} + c.c. \ , \ v(X,Y,t) = V(Y)e^{i(kX-\omega t)} + c.c. \tag{6}$$

Here k and ω are the wave numbers and angular frequency, respectively, "c.c." represents the complex conjugate of the preceding terms and the phase velocity of waves is $c = \omega/k$. It is known that $c_1 < c_2 \leq c$ yields the following

$$U(Y) = Ae^{ikp_1Y} + Be^{-ikp_1Y} \ , \ V(Y) = Ce^{ikp_2Y} + De^{-ikp_2Y}. \tag{7}$$

Here, $p_i = (c^2/c_i^2 - 1)^{1/2}$, $i = 1, 2$, and A, B, C and D are constants. If (6) together with (7) are used in the homogeneous boundary conditions, one obtains the following

$$\mathbf{WU} = \mathbf{0} \tag{8}$$

where amplitude vector is $\mathbf{U} = [A \ B \ C \ D]^T$, and the dispersion matrix is

$$\mathbf{W} = \begin{bmatrix} ikp_1e^{ikh_1p_1} & -ikp_1e^{-ikh_1p_1} & 0 & 0 \\ 1 & 1 & -1 & -1 \\ ikp_1 & -ikp_1 & -i\gamma kp_2 & i\gamma kp_2 \\ 0 & 0 & ikp_2e^{-ikh_2p_2} & -ikp_2e^{ikh_2p_2} \end{bmatrix}. \tag{9}$$

Existence of nontrivial solutions requires

$$\det \mathbf{W} = 0 \ \Rightarrow \tan(kh_1p_1) + \gamma\frac{p_2}{p_1}\tan(kh_2p_2) = 0. \tag{10}$$

One gets the following dispersion relation for $c_1 < c \leq c_2$ by substituting $p_2 = i\nu_2$ in (10);

$$\det \mathbf{W} = 0 \ \Rightarrow \tan(kh_1p_1) - \gamma\frac{\nu_2}{p_1}\tanh(kh_2\nu_2) = 0. \tag{11}$$

In the following analysis, the pairs (k, c) and (mk, c) (or (k, ω) and $(mk, m\omega)$), $m \in \{3, 5, 7, 9, ...\}$, which satisfy the dispersion relation at the same point will be found.

Case 1 ($c_1 < c_2 \leq c$): Let (k, c) and (mk, c) satisfy (10). The periodicity of tangent function yields

$$\tan(mkh_ip_i) = \tan(kh_ip_i) = \tan(kh_ip_i + n\pi). \tag{12}$$

Using the fact that m is an odd integer ($m = 2s + 1$), one gets

$$kh_1p_1 = \frac{n\pi}{2s}, \ kh_2p_2 = \frac{l\pi}{2s}, \ l, n = 0, 1, 2, .. \tag{13}$$

for $s \neq 0$. (13) gives

$$lp_1 = nhp_2. \tag{14}$$

where $h = h_2/h_1$. Since (14) is independent of s and consequently of m, then (10) can be expressed as follows

$$\tan(\frac{n\pi}{2s}) = -\gamma\frac{p_2}{p_1}\tan(\frac{n\pi}{2s}\frac{p_2}{p_1}h). \tag{15}$$

The dispersion relation (10) for $m = 3$ (s=1) is

$$\tan\frac{n\pi}{2} = -\gamma\frac{p_2}{p_1}\tan\frac{l\pi}{2}. \tag{16}$$

If $n = 2r$, then either $c = c_2$ or l is an even integer. If $n = 2r + 1$, then $c \neq c_1$ and consequently l is an odd integer since $p_1 \neq 0$.

The dispersion relation (10) for $m = 5$ (s=2) is

$$\tan\frac{n\pi}{4} = -\gamma\frac{p_2}{p_1}\tan\frac{l\pi}{4}. \tag{17}$$

If $n = 4r + 1$, the equation (17) reduces to the following for $r = 0, 1, 2, ...$

$$1 = -\gamma \frac{p_2}{p_1} \tan \frac{l\pi}{4}. \tag{18}$$

For $l = 2d$, $d = 1, 2, ...$, (18) yields $c = c_2$. Thus, the interaction of the fundamental wave and its fifth harmonic component occurs when $c = c_2$. The equation can be expressed in the following form for $l = 4d - 1$, $d = 1, 2, ...$

$$p_1 = \gamma p_2. \tag{19}$$

Since (k, c) and $(5k, c)$ satisfy the dispersion relation at the same time for $m = 5$, then

$$5p_1^2 \tan^4 kh_1p_1 - 3(p_1^2 + \gamma^2 p_2^2) \tan^2 kh_1p_1 + 5\gamma^2 p_2^2 = 0. \tag{20}$$

Case 2 ($c_1 < c \leq c_2$) : Using $m = 2s + 1$, equation (12) gives the following

$$kh_2p_2 = kh_1p_1 \frac{\nu_2}{p_1} h, \; h = \frac{h_2}{h_1}.$$

Then, the dispersion equation (11) can be expressed as follows

$$\tan \frac{n\pi}{2s} = \gamma \frac{\nu_2}{p_1} \tanh \left(\frac{n\pi}{2s} \frac{\nu_2}{p_1} h\right), \; n = 0, 1, .., s = 1, 2, ... \tag{21}$$

For the third harmonic resonance ($m = 3$) if n is an odd integer ($n = 2r + 1$, $r = 0, 1, ...$), then the tangent function is unbounded whereas the hyperbolic tangent function is bounded. Hence, c approaches c_1. If n is an even integer ($n = 2r$, $r = 0, 1, ...$), then either $c = c_2$ or $c \to c_1$. Thus, for $c_1 < c \leq c_2$, the interaction of the fundamental wave and its 3rd harmonic component occurs for $c = c_2$.

The dispersion relation (11) for the fifth harmonic resonance phenomenon ($m = 5$) is

$$\tan \frac{n\pi}{4} = \gamma \frac{\nu_2}{p_1} \tanh \left(\frac{n\pi}{4} \frac{\nu_2}{p_1} h\right). \tag{22}$$

If $n = 4r$, $r = 1, 2, ...$, then either $c \to c_1$ or $c = c_2$. If $n = 4r \pm 1$, $r = 0, 1, ...$, then equation (22) yields

$$1 = \gamma \frac{\nu_2}{p_1} \tanh \left(\frac{n\pi}{4} \frac{\nu_2}{p_1} h\right). \tag{23}$$

Since the dispersion relation (11) is satisfied for both (k, c) and $(5k, c)$ simultaneously, the fifth harmonic resonance occurs for the pair (k, c) which satisfies

$$5p_1^2 \tan^4(kh_1p_1) - 3(p_1^2 - \nu_2^2\gamma^2) \tan^2(kh_1p_1) - 5n_2^2\gamma^2 = 0. \tag{24}$$

Therefore, the interaction between the fundamental wave and its fifth harmonic component emerges at the wave numbers k for which $c = c_2$, or which satisfty (23) or (24).

4. Analysis of the propagation of nonlinear SH waves at the wave numbers where the mth harmonic resonance occurs

In this section, the mth harmonic resonance of slowly varying amplitudes of weakly nonlinear SH waves in a two layered elastic plate is investigated. The multiple scales method [14] is employed and thus the following new independent variables are introduced

$$x_i = \epsilon^i X \; , \; y = Y \; , \; t_i = \epsilon^i t \; , \; i = 0, 1, 2 \tag{25}$$

where $\epsilon > 0$ is a small parameter measuring the degree of nonlinearity. The variables $\{x_1, x_2, t_1, t_2\}$ are the slow variables introduced to specify the slow variations of the amplitude whereas $\{x_0, y, t_0\}$ are the fast variables. It is assumed that the displacement functions u and v are functions of new stretched variables, $u=u(x_0,x_1,x_2,y,t_0,t_1,t_2)$ and $v=v(x_0,x_1,x_2,y,t_0,t_1,t_2)$ and they are expanded in the following asymptotic series of ϵ

$$u=\sum_{n=1}^{\infty}\epsilon^n u_n(x_0,x_1,x_2,y,t_0,t_1,t_2),\ v=\sum_{n=1}^{\infty}\epsilon^n v_n(x_0,x_1,x_2,y,t_0,t_1,t_2). \tag{26}$$

Employing (25) and (26) into (3) together with (4), and arranging in like powers of ϵ, a hierarchy of equations is obtained. The equations up to third order in ϵ are as follows

$$O(\epsilon):\qquad \mathcal{L}_0^{(1)}u_1=0 \text{ in } P_1 \qquad\text{and}\qquad \mathcal{L}_0^{(2)}v_1=0 \text{ in } P_2, \tag{27}$$

$$\begin{aligned}&\frac{\partial u_1}{\partial y}=0 \quad\text{on}\quad y=h_1, \qquad \frac{\partial v_1}{\partial y}=0 \quad\text{on}\quad y=-h_2\ ,\\ &u_1=v_1 \quad\text{and}\quad \frac{\partial u_1}{\partial y}-\gamma\frac{\partial v_1}{\partial y}=0 \quad\text{on}\quad y=0,\end{aligned} \tag{28}$$

$$O(\varepsilon^2):\qquad \mathcal{L}_0^{(1)}u_2=\mathcal{L}_1^{(1)}u_1 \text{ in } P_1 \qquad\text{and}\qquad \mathcal{L}_0^{(2)}v_2=\mathcal{L}_1^{(2)}v_1 \text{ in } P_2 \tag{29}$$

$$\begin{aligned}&\frac{\partial u_2}{\partial y}=0 \quad\text{on}\quad y=h_1, \qquad \frac{\partial v_2}{\partial y}=0 \quad\text{on}\quad y=-h_2\ ,\\ &u_2=v_2 \quad\text{and}\quad \frac{\partial u_2}{\partial y}-\gamma\frac{\partial v_2}{\partial y}=0 \quad\text{on}\quad y=0,\end{aligned} \tag{30}$$

$$O(\varepsilon^3):\qquad \begin{aligned}&\mathcal{L}_0^{(1)}u_3=\mathcal{L}_1^{(1)}u_2+\mathcal{L}_2^{(1)}u_1+n_1\mathcal{N}_0(u_1) \text{ in } P_1\\ &\mathcal{L}_0^{(2)}v_3=\mathcal{L}_1^{(2)}v_2+\mathcal{L}_2^{(2)}v_1+n_2\mathcal{N}_0(v_1) \text{ in } P_2\end{aligned} \tag{31}$$

$$\begin{aligned}&\frac{\partial u_3}{\partial y}=0 \quad\text{on}\quad y=h_1, \qquad \frac{\partial v_3}{\partial y}=0 \quad\text{on}\quad y=-h_2\ ,\\ &u_3=v_3 \quad\text{and}\quad \frac{\partial u_3}{\partial y}-\gamma\frac{\partial v_3}{\partial y}=\mathcal{B}(u_1,v_1) \quad\text{on}\quad y=0\ ,\end{aligned} \tag{32}$$

where the linear operators $\mathcal{L}_0^{(\nu)}$, $\mathcal{L}_1^{(\nu)}$ and $\mathcal{L}_2^{(\nu)}$, $\nu=1,2$, and the nonlinear operators $\mathcal{B}$ and $\mathcal{N}_0$ are

$$\begin{aligned}&\mathcal{L}_0^{(\nu)}\psi=\frac{\partial^2\psi}{\partial t_0^2}-c_\nu^2\Big(\frac{\partial^2\psi}{\partial x_0^2}+\frac{\partial^2\psi}{\partial y^2}\Big)\ ,\ \mathcal{L}_1^{(\nu)}\psi=-2\frac{\partial^2\psi}{\partial t_0\partial t_1}+2c_\nu^2\frac{\partial^2\psi}{\partial x_0\partial x_1}\ ,\\ &\mathcal{L}_2^{(\nu)}\psi=-\frac{\partial^2\psi}{\partial t_1^2}-2\frac{\partial^2\psi}{\partial t_0\partial t_2}+c_\nu^2\Big(\frac{\partial^2\psi}{\partial x_1^2}+2\frac{\partial^2\psi}{\partial x_0\partial x_2}\Big)\ ,\\ &\mathcal{N}_0(\psi)=\frac{\partial}{\partial x_0}\Big(\frac{\partial\psi}{\partial x_0}\mathcal{Q}_0(\psi)\Big)+\frac{\partial}{\partial y}\Big(\frac{\partial\psi}{\partial y}\mathcal{Q}_0(\psi)\Big)\ ,\quad \mathcal{Q}_0(\psi)=\Big(\frac{\partial\psi}{\partial x_0}\Big)^2+\Big(\frac{\partial\psi}{\partial y}\Big)^2,\\ &\mathcal{B}(\phi_1,\phi_2)=\gamma\beta_2\frac{\partial\phi_2}{\partial y}\mathcal{Q}_0(\phi_2)-\beta_1\frac{\partial\phi_1}{\partial y}\mathcal{Q}_0(\phi_1).\end{aligned} \tag{33}$$

The superscripts in linear operators refer to the layers. Here $c_1<c_2$ is assumed. For the existence of an SH wave, the phase velocity c must satisfy $c_1<c_2\le c$ or $c_1<c\le c_2$. The problem for the first condition will be investigated. The analysis shows that only the dependence on fast variables $\{x_0,y,t_0\}$ can be determined explicitly by the first order problem whereas the dependence on slow variables can be determined by the higher order perturbation problems. We are examining the interaction of the fundamental wave and its mth harmonic component for which both (k,ω) and $(mk,m\omega)$, or (k,c) and (mk,c) simultaneously satisfy the linear dispersion relation. The analysis is based on these

assumptions. The solutions of (27) are obtained as the sum of the fundamental wave and its mth harmonic mods

$$u_1 = \sum_{l=1,m} \mathcal{A}_1^{(l)}[R_1^{(l)} e^{ik_l p_1 y} + R_2^{(l)} e^{-ik_l p_1 y}] e^{i\phi_l} + c.c.$$
$$v_1 = \sum_{l=1,m} \mathcal{A}_1^{(l)}[R_3^{(l)} e^{ik_l p_2 y} + R_4^{(l)} e^{-ik_l p_2 y}] e^{i\phi_l} + c.c. \tag{34}$$

where $p_i = (\frac{c^2}{c_i^2} - 1)^{1/2}$, $i = 1,2$, $\phi_l = (k_l x_0 - \omega_l t_0)$, $l = 1, m$. Here $k_l = lk$ and $\omega_l = l\omega$. The complex functions of the slow variables $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_1^{(m)}$ represent the first order slowly varying amplitudes of the fundamental wave and its mth harmonic component, respectively. $\mathbf{R}^{(l)}$ is the column vector satisfying $\mathbf{W}^{(l)}\mathbf{R}^{(l)} = \mathbf{0}$ where

$$\mathbf{W}^{(l)} = \begin{bmatrix} ik_l p_1 e^{ik_l h_1 p_1} & -ik_l p_1 e^{-ik_l h_1 p_1} & 0 & 0 \\ 1 & 1 & -1 & -1 \\ ik_l p_1 & -ik_l p_1 & -i\gamma k_l p_2 & i\gamma k_l p_2 \\ 0 & 0 & ik_l p_2 e^{-ik_l h_2 p_2} & -ik_l p_2 e^{ik_l h_2 p_2} \end{bmatrix}. \tag{35}$$

To determine the first order solution of the nonlinear problem, it is sufficient to find $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_1^{(m)}$. Thus the higher order perturbation problems need to be examined.

To examine the behaviour of the solutions, the interaction of the fundamental wave and its mth harmonic component is discussed here. To obtain the second order solutions, the first order solutions are introduced into the governing equations in (29), and then nonhomogeneous equations are obtained for u_2 and v_2. Solutions of these equations are categorized into two groups $u_2 = \bar{u}_2 + \tilde{u}_2$, and $v_2 = \bar{v}_2 + \tilde{v}_2$ such that $\bar{u}_2$ and $\bar{v}_2$ are the particular solutions of (29). $\tilde{u}_2$ and $\tilde{v}_2$ are the solutions of the corresponding homogeneous equations and the nonhomogeneous boundary conditions obtained from (30). From the nonhomogeneous boundary conditions, the following nonhomogeneous algebraic system of equations for the second order amplitude functions is obtained

$$\mathbf{W}^{(l)}\mathbf{U}_2^{(l)} = \mathbf{b}_2^{(l)} \quad l = 1, m \tag{36}$$

where

$$\mathbf{b}_2^{(l)} = -i\Big(\frac{\partial \mathcal{A}_1^{(l)}}{\partial t_1}\frac{\partial \mathbf{W}^{(l)}}{\partial \omega_l} - \frac{\partial \mathcal{A}_1^{(l)}}{\partial x_1}\frac{\partial \mathbf{W}^{(l)}}{\partial k_l}\Big)\mathbf{R}^{(l)}. \tag{37}$$

Since $\det \mathbf{W}^{(l)} = 0$ and $\mathbf{b_2}^{(l)} \neq \mathbf{0}$, the compatibility condition

$$\mathbf{L}^{(l)} \cdot \mathbf{b}_2^{(l)} = \mathbf{0} \tag{38}$$

must be satisfied. Here, $\mathbf{L}^{(l)}$ are row vectors defined by $\mathbf{L}^{(l)}\mathbf{W}^{(l)} = \mathbf{0}$. From the compatibility condition (38), it can easily be shown that $\mathcal{A}_1^{(l)}$ remains constant in a frame of reference moving with a group velocity $V_g^{(l)}$ of the waves; that is, $\mathcal{A}_1^{(l)} = \mathcal{A}_1^{(l)}(x_1 - V_g^{(l)}t_1, x_2, t_2)$ where the group velocity of the waves $V_g^{(l)} = d\omega_l/dk_l$ is as follows $V_g^{(l)} = -(\mathbf{L}^{(l)}\frac{\partial \mathbf{W}^{(l)}}{\partial k_l}\mathbf{R}^{(l)})/(\mathbf{L}^{(l)}\frac{\partial \mathbf{W}^{(l)}}{\partial \omega_l}\mathbf{R}^{(l)})$. Then, one obtains the solution of the equation (36) as

$$\mathbf{U}_2^{(l)} = \mathcal{A}_2^{(l)}\mathbf{R}^{(l)} - i\frac{\mathcal{A}_1^{(l)}}{\partial x_1}\mathbf{T}^{(l)}, \ \mathbf{T}^{(l)} = \Big(\frac{\partial \mathbf{R}^{(l)}}{\partial k_l} + V_g^{(l)}\frac{\partial \mathbf{R}^{(l)}}{\partial \omega_l}\Big) \tag{39}$$

where $\mathcal{A}_2^{(l)}$ denotes the second order slowly varying amplitude of the waves, and if necessary, they can be determined from higher order perturbation problems. Since this work focuses on the mth harmonic resonant interaction of weakly nonlinear SH waves, $\mathcal{A}_2^{(l)}$ need not be calculated explicitly. In order to complete the first order solutions, the third order problem is considered. Introducing the first and second order solutions into (31) gives

$$\begin{aligned}\mathcal{L}_0^{(1)}(u_3)=&\alpha_1 e^{i\phi_1}+\alpha_3 e^{3i\phi_1}+\alpha_m e^{i\phi_m}+\alpha_{-21m}e^{i(-2\phi_1+\phi_m)}+\alpha_{21m}e^{i(2\phi_1+\phi_m)}\\&+\alpha_{-12m}e^{i(-\phi_1+2\phi_m)}+\alpha_{12m}e^{i(\phi_1+2\phi_m)}+\alpha_{3m}e^{3i\phi_m}+c.c.\end{aligned}\tag{40}$$

$$\begin{aligned}\mathcal{L}_0^{(1)}(v_3)=&\beta_1 e^{i\phi_1}+\beta_3 e^{3i\phi_1}+\beta_m e^{i\phi_m}+\beta_{-21m}e^{i(-2\phi_1+\phi_m)}+\beta_{21m}e^{i(2\phi_1+\phi_m)}\\&+\beta_{-12m}e^{i(-\phi_1+2\phi_m)}+\beta_{12m}e^{i(\phi_1+2\phi_m)}+\beta_{3m}e^{3i\phi_m}+c.c.\end{aligned}\tag{41}$$

where α_i and β_i are functions of $x_1, x_2, y, t_1, t_2\}$. Forms of these functions are extremely lengthy and thus explicit forms are excluded here. For $m \in \{3, 5, ...\}$, one gets

$$\begin{aligned}&2\phi_1+\phi_m=(2+m)\phi_1\ ,\ \phi_1+2\phi_m=(2m+1)\phi_1,\\&\phi_m-2\phi_1=(m-2)\phi_1\ ,\ 2\phi_m-\phi_1=(2m-1)\phi_1.\end{aligned}\tag{42}$$

Depending on the odd values of m and considering the relations in (42), the nature of the right hand side terms of the equations (40)-(41) changes. For $m = 5$, the system of equations characterizing the change in the amplitude functions $\mathcal{A}_1^{(1)}$ and $\mathcal{A}_1^{(5)}$ is a system of CNLS equations. However, for $m = 3$, the structure of the couple system of equations characterizing the amplitude functions is completely different [16], [19].

Fifth Harmonic Resonance Case: In this case, only the terms in the coefficients of $e^{\pm\phi_1}$ and $e^{\pm\phi_5}$ contribute each other. Thus the right hand sides of the equations (40) are

$$\begin{aligned}&\mathcal{L}_0^{(1)}(u_3)=\alpha_1 e^{i\phi_1}+\alpha_{15}e^{i\phi_5}+\text{the terms in}\{e^{i\phi_3},e^{i\phi_7},e^{i\phi_9},e^{i\phi_{11}},e^{15i\phi_{15}}\}+c.c.\\&\mathcal{L}_0^{(1)}(v_3)=\beta_1 e^{i\phi_1}+\beta_{15}e^{i\phi_5}+\text{the terms in}\{e^{i\phi_3},e^{i\phi_7},e^{i\phi_9},e^{i\phi_{11}},e^{15i\phi_{15}}\}+c.c.\end{aligned}\tag{43}$$

where the explicit forms of the coefficients α_1, α_{15}, β_1 and β_{15} are given by (54)-(57) in Appendix, respectively.

As in the second order problem, u_3 and v_3 are decomposed as $u_3 = \bar{u}_3 + \tilde{u}_3$ and $\bar{v}_3 = \bar{v}_3 + \tilde{v}_3$ where $\bar{u}_3$ and $\bar{v}_3$ are the particular solutions of the equations (43). $\tilde{u}_3$ and $\tilde{v}_3$ are the solutions of the corresponding homogeneous equations under the nonhomogeneous boundary conditions obtained from (32) of the third order problem by employing the decomposition of u_3 and v_3. The particular solutions $\bar{u}_3$ and $\bar{v}_3$ are expressed as

$$\begin{aligned}\bar{u}_3=&\{(X_1+X_2y)ye^{ikp_1y}+(X_3+X_4y)ye^{-ikp_1y}+X_5e^{3ikp_1y}+X_6e^{-3ikp_1y}+X_7e^{9ikp_1y}\\&+X_8e^{-9ikp_1y}+X_9e^{11ikp_1y}+X_{10}e^{-11ikp_1y}\}e^{i\phi_1}+\{Y_1e^{3ikp_1y}+Y_2e^{-3ikp_1y}\\&+(Y_3+Y_4y)ye^{5ikp_1y}+(Y_5+Y_6y)ye^{-5ikp_1y}+Y_7e^{7ikp_1y}+Y_8e^{-7ikp_1y}\\&+Y_9e^{15ikp_1y}+Y_{10}e^{-15ikp_1y}\}e^{i\phi_5}+\text{the terms in}\{e^{\phi_3},e^{i\phi_7},e^{i\phi_9},e^{i\phi_{11}},e^{i\phi_{15}}\}+c.c.\\\bar{v}_3=&\{(Z_1+Z_2y)ye^{ikp_2y}+(Z_3+Z_4y)ye^{-ikp_2y}+Z_5e^{3ikp_2y}+Z_6e^{-3ikp_2y}+Z_7e^{9ikp_2y}\\&+Z_8e^{-9ikp_2y}+Z_9e^{11ikp_2y}+Z_{10}e^{-11ikp_2y}\}e^{i\phi_1}+\{V_1e^{3ikp_2y}+V_2e^{-3ikp_2y}\\&+(V_3+V_4y)ye^{5ikp_2y}+(V_5+V_6y)ye^{-5ikp_2y}+V_7e^{7ikp_2y}+V_8e^{-7ikp_2y}\\&+V_9e^{15ikp_2y}+V_{10}e^{-15ikp_2y}\}e^{i\phi_5}+\text{the terms in}\{e^{\phi_3},e^{i\phi_7},e^{i\phi_9},e^{i\phi_{11}},e^{i\phi_{15}}\}+c.c.\end{aligned}\tag{44}$$

Substituting (44) into (43), one can find X_j, Y_j, Z_j and V_j, $j = 1, ..., 10$. They are functions of the variables $\{x_1, x_2, t_1, t_2\}$ and their explicit structures are not given here.

Because of the effect of nonlinearity, the terms on the right hand side of (44) need to be taken into consideration when constructing the solutions of $\tilde{u}_3$ and $\tilde{v}_3$

$$\begin{aligned}\tilde{u}_3 &= \sum_l \{A_3^{(l)} e^{ik_l p_1 y} + B_3^{(l)} e^{-ik_l p_1 y}\} e^{i\phi_l} + \sum_L \{A_3^{(L)} e^{ik_L p_1 y} + B_3^{(L)} e^{-ik_L p_1 y}\} e^{i\phi_L} + c.c.\\ \tilde{v}_3 &= \sum_l \{C_3^{(l)} e^{ik_l p_2 y} + D_3^{(l)} e^{-ik_l p_2 y}\} e^{i\phi_l} + \sum_L \{C_3^{(L)} e^{ik_L p_2 y} + D_3^{(L)} e^{-ik_L p_2 y}\} e^{i\phi_L} + c.c.\end{aligned} \tag{45}$$

where $l = 1, 5$ and $L = 3, 7, 9, 11, 15$. Using these solutions with the solutions u_1, v_1, u_2, v_2, $\bar{u}_3$ and $\bar{v}_3$ in the boundary conditions (33) yields

$$\mathbf{W}^{(l)} \mathbf{U}_3^{(l)} = \mathbf{b}_3^{(l)}, \qquad \mathbf{W}^{(L)} \mathbf{U}_3^{(L)} = \mathbf{b}_3^{(L)} \tag{46}$$

where $\mathbf{U}_3^{(l)} = (A_3^{(l)},\ B_3^{(l)},\ C_3^{(l)},\ D_3^{(l)})^T$ and $\mathbf{U}_3^{(L)} = (A_3^{(L)},\ B_3^{(L)},\ C_3^{(L)},\ D_3^{(L)})^T$. The explicit form of $\mathbf{b}_3^{(L)}$ is not given here, since in the sequel this will not be required whereas $\mathbf{b}_3^{(l)}$ can be written as in the following form by using a lengthy but straightforward algebra

$$\begin{aligned}\mathbf{b}_3^{(1)} &= \mathbf{B}_3^{(1)} + \mathbf{F}_{11} |\mathcal{A}_1^{(1)}|^2 \mathcal{A}_1^{(1)} + \mathbf{F}_{51} |\mathcal{A}_1^{(5)}|^2 \mathcal{A}_1^{(1)}\ ,\\ \mathbf{b}_3^{(5)} &= \mathbf{B}_3^{(5)} + \mathbf{F}_{55} |\mathcal{A}_1^{(5)}|^2 \mathcal{A}_1^{(5)} + \mathbf{F}_{15} |\mathcal{A}_1^{(1)}|^2 \mathcal{A}_1^{(5)}\ .\end{aligned} \tag{47}$$

$\mathbf{B}_3^{(l)}$ depends on linear material properties and wave numbers. On the other hand, $\mathbf{F}_{ll}$ and $\mathbf{F}_{nl}$ depend on both linear and nonlinear material properties and wave numbers, and the formulations of $\mathbf{F}_{ll}$ and $\mathbf{F}_{nl}$ are extremely lengthy and thus, the explicit forms are excluded here. If the nonlinear properties are assumed to be zero, then both $\mathbf{F}_{ll}$ and $\mathbf{F}_{nl}$ vanish. For L, the solution of the second equation in (46) is obtained as $\mathbf{U}_3^{(L)} = {\mathbf{W}^{(L)}}^{-1} \cdot \mathbf{b}_3^{(L)}$. In order that the first equation in (46) is algebraically solvable for $\mathbf{U}_3^{(1)}$, the compatibility condition

$$\mathbf{L}^{(l)} \cdot \mathbf{b}_3^{(l)} = \mathbf{0}\ ,\ \ l = 1, 5 \tag{48}$$

must be satisfied. If it is assumed that $\mathcal{A}_2^{(l)}$ depends on x_1 and t_1 through the combination $x_1 - V_{l_g} t_1$ as $\mathcal{A}_1^{(l)}$, and if the following nondimensional variables and constants are defined,

$$\begin{aligned}\tau = \omega_1 t_2\ ,\ \xi = k_1(x_1 - V_{1_g} t_1) = k_1 \epsilon^{-1}(x_2 - V_{1_g} t_2)\ ,\ \mathcal{A}_l = k_1 \mathcal{A}_1^{(l)}\\ \Gamma_l = k_1^2 \tilde{\Gamma}_l\ ,\ \Delta_{nl} = \frac{\tilde{\Delta}_{nl}}{\omega_1 k_1^2}\ ,\ \Lambda = \frac{k_1}{\epsilon \omega_1}(V_{5_g} - V_{1_g})\ ,\end{aligned} \tag{49}$$

$$\tilde{\Gamma}_l = \frac{1}{2} \frac{d^2 \omega_l}{d k_l^2}\ ,\ \tilde{\Delta}_{ll} = -\frac{\mathbf{L}^{(l)} \cdot \mathbf{F}_{ll}}{\mathbf{L}^{(l)} \frac{\partial \mathbf{W}^{(l)}}{\partial \omega_l} \mathbf{R}^{(l)}}\ ,\ \tilde{\Delta}_{nl} = -\frac{\mathbf{L}^{(l)} \cdot \mathbf{F}_{nl}}{\mathbf{L}^{(l)} \frac{\partial \mathbf{W}^{(l)}}{\partial \omega_l} \mathbf{R}^{(l)}}$$

then, from the compatibility condition (48), the following coupled equations for the nondimensional amplitude functions $\mathcal{A}_1$ and $\mathcal{A}_5$ are obtained

$$\begin{aligned}&i\frac{\partial \mathcal{A}_1}{\partial \tau} + \Gamma_1 \frac{\partial^2 \mathcal{A}_1}{\partial \xi^2} + \Delta_{11} |\mathcal{A}_1|^2 A_1 + \Delta_{51} |\mathcal{A}_5|^2 \mathcal{A}_1 = 0,\\ &i\frac{\partial \mathcal{A}_5}{\partial \tau} + i\Lambda \frac{\partial \mathcal{A}_5}{\partial \xi} + \Gamma_5 \frac{\partial^2 \mathcal{A}_5}{\partial \xi^2} + \Delta_{55} |\mathcal{A}_5|^2 \mathcal{A}_5 + \Delta_{15} |\mathcal{A}_1|^2 \mathcal{A}_5 = 0.\end{aligned} \tag{50}$$

Here, Γ_1 and Γ_5 are linear dispersion coefficients, Δ_{11} and Δ_{55} are nonlinear coefficients which describe the self modulation of wave packets, and Δ_{15} and Δ_{51} are nonlinear coupling coefficients of the cross modulation between two wave packets. It should be noted that (50) can be reduced to the following CNLS system when the depending variable transformation for $\mathcal{A}_5$, $\mathcal{A}_5 \to \mathcal{A}_5 e^{-i(\xi\frac{\Lambda}{2\Gamma_5}-\tau\frac{\Lambda^2}{4\Gamma_5})}$ is used

$$
\begin{aligned}
&i\frac{\partial \mathcal{A}_1}{\partial \tau}+\Gamma_1\frac{\partial^2\mathcal{A}_1}{\partial \xi^2}+\Delta_{11}|\mathcal{A}_1|^2 A_1+\Delta_{51}|\mathcal{A}_5|^2\mathcal{A}_1=0\\
&i\frac{\partial \mathcal{A}_5}{\partial \tau}+\Gamma_5\frac{\partial^2\mathcal{A}_5}{\partial \xi^2}+\Delta_{55}|\mathcal{A}_5|^2 \mathcal{A}_5+\Delta_{15}|\mathcal{A}_1|^2\mathcal{A}_5=0\ .
\end{aligned}
\tag{51}
$$

Thus once solutions for $\mathcal{A}_1$ and $\mathcal{A}_5$ are derived from (51) for given initial values of the form $\mathcal{A}_1(\xi,0)=\mathcal{A}_{1_0}(\xi)$ and $\mathcal{A}_5(\xi,0)=\mathcal{A}_{5_0}(\xi)$, then the first order solutions u_1 and v_1 can be constructed by (34). The system (51) is also derived in different physical fields such as nonlinear optics, geophysical fluid dynamics, Rossby waves, etc. [3], [4], [5], [13], [24]-[28]. As stated in [28], there are relationships between the dispersion coefficient, nonlinear self-interaction and nonlinear cross interaction terms of (51) which are derived in nonlinear optics. However, there is no relationship between the coefficients of the CNLS system in geophysical applications, as was the case in our study.

The CNLS equations (51) was solved by Manakov [29] for the special choice of coefficients where $\Gamma_1=\Gamma_5=1$, and $\Delta_{11}=\Delta_{55}=\Delta_{15}=\Delta_{51}=$constant. The system is known as integrable (in inverse scattering sense) only for $\Gamma_1=\Gamma_5$ and $\Delta_{11}=\Delta_{55}=\Delta_{15}=\Delta_{51}$, or, for $\Gamma_1=-\Gamma_5$ and $\Delta_{11}=\Delta_{55}=-\Delta_{15}=-\Delta_{51}$ [30]. Different types of solutions of (51) were obtained for special choices of the coefficients, including travelling wave solutions in terms of elliptic Jacobian functions [31]-[33] and in terms of the Weierstrass elliptic function [27], [33].

5. The Effect of Nonlinearity and the Existence of Solitary Waves

In the present study, the focus is on the effects of linear and nonlinear material properties of the layered medium on the linear stability of the plane wave solutions, and the existence of solitary envelope solutions of the system (51) by following the work of Roskes [20].

According to the linearized stability analysis for the plane wave solutions of the CNLS equations in (51), it is known that these waves are unstable if $S_1=\Gamma_1\Gamma_2 T<0$ where $T=(\Delta_{11}\Delta_{55}-\Delta_{15}\Delta_{51})$. Solitary envelope solutions of the system (51) also exist. The following form of these solutions

$$
\mathcal{A}_1=\eta_1 e^{i\theta_1\tau}\mathrm{sech}(\delta\xi),\qquad \mathcal{A}_5=\eta_2 e^{i\theta_2\tau}\mathrm{sech}(\delta\xi),
\tag{52}
$$

provided that [20];

$$
S_2=(\Gamma_1\Delta_{55}-\Gamma_5\Delta_{51})T>0,\ \ S_3=(\Gamma_5\Delta_{11}-\Gamma_1\Delta_{15})T>0.
\tag{53}
$$

Here, η_1, η_2, θ_1 and θ_2 satisfy the conditions

$$
\delta^2=\frac{1}{2}\frac{T}{S_2}\eta_1^2=\frac{1}{2}\frac{T}{S_3}\eta_2^2,\ \ \theta_1=\Gamma_1\delta^2,\ \ \theta_2=\Gamma_5\delta^2.
$$

Considering the effects of the coefficients of (51) on the existence of the solutions of the CNLS system, the coefficients of the linear terms Γ_1 and Γ_5, and the nonlinear terms Δ_{11}, Δ_{55}, Δ_{15} and Δ_{51} are substituted in S_1, S_2 and S_3. Then, fictive linear and nonlinear parameter values, and thickness ratio are replaced in S_1, S_2 and S_3 to observe the effects of nonlinear material parameters and thickness' ratio of the layers on the stability of the solution. S_1, S_2 and S_3 will be examined at some wave number-phase velocity pairs where the fifth harmonic resonance emerges. The linear material properties are chosen as $\gamma=\mu_2/\mu_1=2.159$, $\rho=\rho_2/\rho_1=1.28007$.

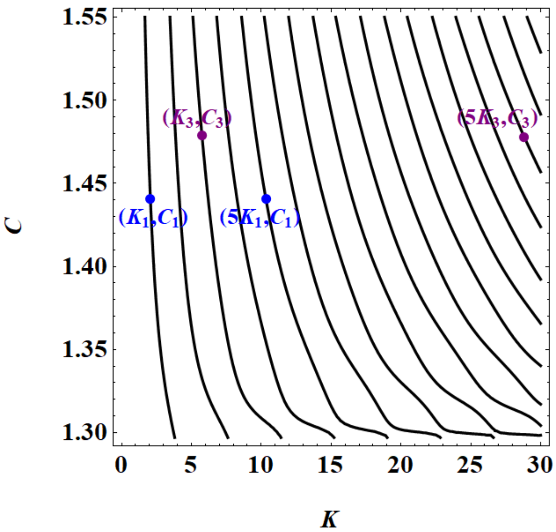

FIGURE 1. The chosen non-dimensional wave number-phase velocity pairs (K_1, C_1) and (K_3, C_3) satisfying (19) for $h = 1$

Non-dimensional variables are defined as $K = kh_1$ and $C = c/c_1$. In addition, for the values of the nonlinear material parameters q_i, the relevant medium exhibits hardening behavior (H) in shear when $q_i > 0$, and softening behavior (S) when $q_i < 0$. A finite region defined by $q_1 \in [-3, 3]$ and $q_2 \in [-3, 3]$ is chosen in order to observe the effect of the change in the nonlinear material parameter values on the interaction of the fundamental wave and its fifth harmonic component. It should be pointed out that for the appropriate non-dimensional wave number-phase velocity pair (K_j, C_j) for which the fifth harmonic resonance emerges, $S_i = S_i(q_1, q_2, h)$. Here S_1 is a quadratic polynomial whereas S_2 and S_3 are cubic polynomials of q_1 and q_2.

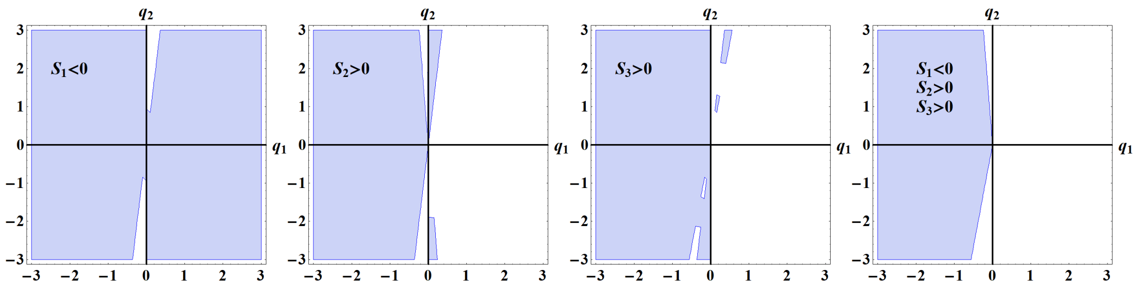

FIGURE 2. Common region for (K_1, C_1) satisfying (19) for $h = 1$

The non-dimensional wave number-phase velocity pair satisfying (19) can be chosen as (K_1, C_1) =(2.06995, 1.44081) and (K_2, C_2) =(1.82678, 1.44081) for $h = 1$ and 5, respectively. Similarly, for $l = 4$ and $n = 8$, the pair satisfying (14) are chosen as (K_3, C_3) = (5.76652, 1.47893) and (K_4, C_4) =(7.51844, 1.30323) for $h = 1$ and 5, respectively.

For $h = 1$, the positions of the pairs (K_1, C_1) and (K_3, C_3) on the branches of the dispersion relation are shown in Fig.1. For $h = 5$, a similar analysis can be made for (K_2, C_2) and (K_4, C_4). As can be seen in Fig.2, for (K_1, C_1), there exist solitary wave solutions for the region where S_1 is negative, and S_2 and S_3 are positive.

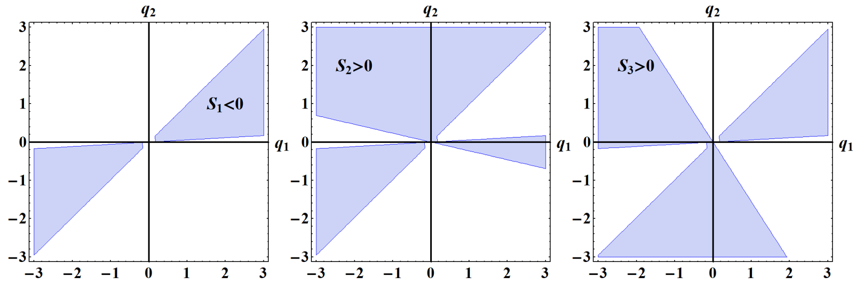

FIGURE 3. Signs of S_1, S_2 and S_3 for (K_2, C_2) satisfying (19) for $h = 5$

Fig.3 shows that the existence of solitary waves can not be stated with any certainty for the chosen (q_1, q_2) values and (K_2, C_2) =(1.82678, 1.44081).

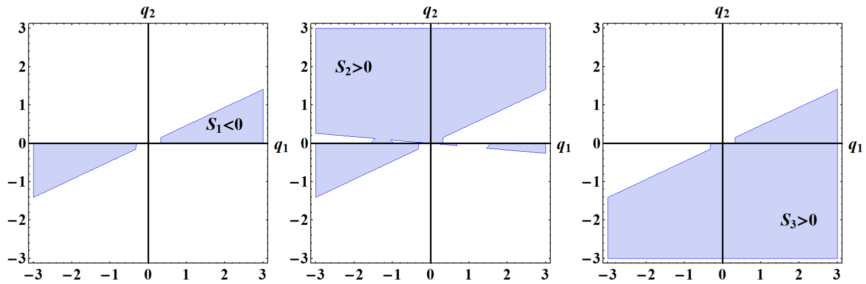

FIGURE 4. Signs of S_1, S_2 and S_3 for (K_3, C_3) satisfying (14) for $h = 1$

For (K_3, C_3) and (K_4, C_4), graphs illustrating the change in the sign of S_i on q_1q_2-plane are given in Fig.4 and 5, respectively. As can be seen in Figure 4, the existence of solitary waves of the CNLS equation can not be stated with any certainty for the chosen (q_1, q_2) values. However, Fig.5 shows that there exist solitary waves for $q_1 > 0$ and $q_2 < 0$; that is, the upper layer is a hardening material whereas the lower layer is a softening material.

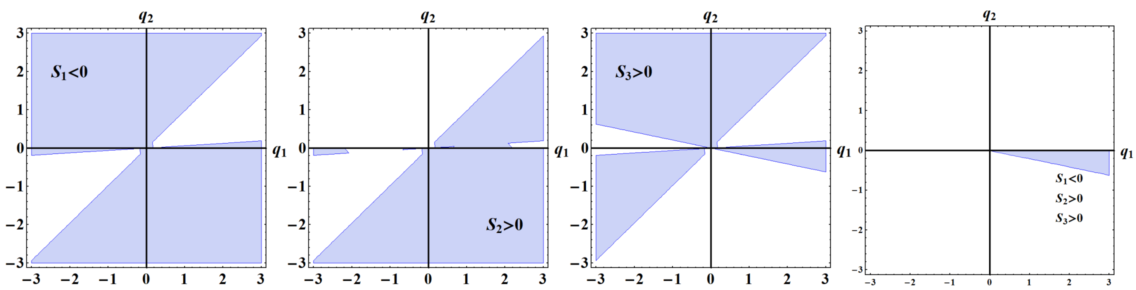

FIGURE 5. Common region for (K_4, C_4) satisfying (14) for $h = 5$

If requested, the coefficients of the CNLS equation can be calculated by using the selected linear and nonlinear material parameters and the (K_1, C_1) and (K_4, C_4) pairs where soliton solutions exist for the selected thickness ratios. With these coefficients and appropriate initial amplitude values, the explicit form of soliton solutions given in (52) can be determined for a chosen dimensionless time value. However, since only the existence of soliton solutions was examined in this study, this calculation was not included here.

6. Conclusion

In this work, the multiple scale method is used to investigate the interaction of the mth harmonic resonance of SH waves propagating in a two layered elastic medium with uniform thickness. The constituent materials of the layers are assumed to be incompressible, homogeneous, isotropic. The dispersion relations of linear waves are derived for the third and fifth harmonic resonances. Then, the fifth harmonic resonant interaction of weakly nonlinear SH waves is investigated in a two layered medium with uniform thickness and with nonlinear, isotropic and homogeneous materials in both layers having different material properties. It is shown that the first order slowly varying amplitudes of interacting waves were characterized by a CNLS equation asymptotically. The effects on the CNLS system's solitary wave solutions of the layers' thickness and of the nonlinear properties of the materials are examined for the different wave number-phase velocity pairs satisfying the criteria of the fifth harmonic resonance.

The outline of the article is as follows: Formulation of the problem is given in section 2, and a nonlinear boundary value problem characterizing SH wave propagation in a two-layered incompressible elastic medium is obtained. The relations for the critical wave number-phase velocity pair corresponding to the third and fifth harmonic resonances are derived. In Section 3, the mth harmonic resonance of slowly varying amplitudes of weakly nonlinear SH waves in such a medium is investigated by employing a multiple scale perturbation method [14]. For $m = 5$, it is shown that the balance between the weak nonlinearity and dispersion yields a CNLS system for the slowly varying amplitudes of the fundamental wave and its fifth harmonic component. In Section 4, discussions on the effects of linear and nonlinear material properties of the medium and the ratio of the layers' thickness on the linear stability of the plane wave solutions as well as the existence of solitary envelope solutions are performed [20]. Some concluding remarks are presented in the final section.

ACKNOWLEDGMENTS

We would like to thank the referees for their construtive comments and recommendations.

7. Appendix

$$
\begin{aligned}
\alpha_1 = {} & a_{3l}e^{-ikp_1y} + a_{1l}e^{ikp_1y} + a_{4l}e^{-ikp_1y}y + a_{2l}e^{ikp_1y}y + \\
& \left(a_{3nA}e^{-ikp_1y} + a_{1nA}e^{ikp_1y} + a_{6nA}e^{-3ikp_1y} + a_{5nA}e^{3ikp_1y}\right)\mathcal{A}_1|\mathcal{A}_1|^2 + \\
& (a_{3nB}e^{-ikp_1y} + a_{1nB}e^{ikp_1y} + a_{10nB}e^{-9ikp_1y} + a_{9nB}e^{9ikp_1y} \\
& + a_{8nB}e^{-11ikp_1y} + a_{7nB}e^{11ikp_1y})\mathcal{A}_1|\mathcal{A}_5|^2
\end{aligned}
\tag{54}
$$

$$
\begin{aligned}
\alpha_{15} = {} & a_{47l}e^{-5ikp_1y} + a_{45l}e^{5ikp_1y} + a_{48l}e^{-5ikp_1y}y + a_{46l}e^{5ikp_1y}y + \\
& (a_{44nB}e^{-3ikp_1y} + a_{43nB}e^{3ikp_1y} + a_{47nB}e^{-5ikp_1y} + a_{45nB}e^{5ikp_1y} + \\
& a_{50nB}e^{-7ikp_1y} + a_{49nB}e^{7ikp_1y})|\mathcal{A}_1|^2\mathcal{A}_5 + \\
& (a_{47nC}e^{-5ikp_1y} + a_{45nC}e^{5ikp_1y} + a_{52nC}e^{-15ikp_1y} + a_{51nC}e^{15ikp_1y})\mathcal{A}_5|\mathcal{A}_5|^2
\end{aligned}
\tag{55}
$$

$$\begin{aligned}\beta_1 &= b_{3l}e^{-ikp_2y} + b_{1l}e^{ikp_2y} + b_{4l}e^{-ikp_2y}y + b_{2l}e^{ikp_2y}y + \\ &\left(b_{3nA}e^{-ikp_2y} + b_{1nA}e^{ikp_2y} + b_{6nA}e^{-3ikp_2y} + b_{5nA}e^{3ikp_2y}\right)\mathcal{A}_1|\mathcal{A}_1|^2 + \\ &(b_{3nB}e^{-ikp_2y} + b_{1nB}e^{ikp_2y} + b_{10nB}e^{-9ikp_2y} + b_{9nB}e^{9ikp_2y} \\ &+ b_{8nB}e^{-11ikp_2y} + b_{7nB}e^{11ikp_2y})\mathcal{A}_1|\mathcal{A}_5|^2\end{aligned} \tag{56}$$

$$\begin{aligned}\beta_{15} &= b_{47l}e^{-5ikp_2y} + b_{45l}e^{5ikp_2y} + b_{48l}e^{-5ikp_2y}y + b_{46l}e^{5ikp_2y}y + \\ &(b_{44nB}e^{-3ikp_2y} + b_{43nB}e^{3ikp_2y} + b_{47nB}e^{-5ikp_2y} + b_{45nB}e^{5ikp_2y} + \\ &b_{50nB}e^{-7ikp_2y} + b_{49nB}e^{7ikp_2y})|\mathcal{A}_1|^2\mathcal{A}_5 + \\ &(b_{47nC}e^{-5ikp_2y} + b_{45nC}e^{5ikp_2y} + a_{52nC}e^{-15ikp_2y} + b_{51nC}e^{15ikp_2y})\mathcal{A}_5|\mathcal{A}_5|^2\end{aligned} \tag{57}$$

The coefficients a_{in} and b_{in} depend on linear and non-linear material parameters. The explicit form of the coefficients a_{il} and b_{il} are given as;

$$a_{2l} = -\frac{2iR_1}{kc_1^2p_1}\left(\omega^2\frac{\partial^2\mathcal{A}_1}{\partial t_1^2} + kc_1^2\left(2\omega\frac{\partial^2\mathcal{A}_1}{\partial x_1\partial t_1} + kc_1^2\frac{\partial^2\mathcal{A}_1}{\partial x_1^2}\right)\right) \tag{58}$$

$$\begin{aligned}a_{3l} &= 2iR_2\left(\omega\frac{\partial\mathcal{A}_1}{\partial t_2} + kc_1^2\frac{\partial\mathcal{A}_1}{\partial x_2}\right) + 2iR_2\left(\omega\frac{\partial\mathcal{A}_g}{\partial t_1} + kc_1^2\frac{\partial\mathcal{A}_g}{\partial x_1}\right) \\ &+ R_2\left(-\frac{\partial^2\mathcal{A}_1}{\partial t_1^2} + c_1^2\frac{\partial^2\mathcal{A}_1}{\partial x_1^2}\right) + 2T_2\left(\omega\frac{\partial\mathcal{A}_1}{\partial x_1\partial t_1} + kc_1^2\frac{\partial\mathcal{A}_1}{\partial x_1}\right)\end{aligned} \tag{59}$$

$$a_{4l} = \frac{2iR_2}{kc_1^2p_1}\left(\omega^2\frac{\partial^2\mathcal{A}_1}{\partial t_1^2} + kc_1^2\left(2\omega\frac{\partial^2\mathcal{A}_1}{\partial x_1\partial t_1} + kc_1^2\frac{\partial^2\mathcal{A}_1}{\partial x_1^2}\right)\right) \tag{60}$$

$$b_{2l} = -\frac{2iR_3}{kc_2^2p_2}\left(\omega^2\frac{\partial^2\mathcal{A}_1}{\partial t_1^2} + kc_2^2\left(2\omega\frac{\partial^2\mathcal{A}_1}{\partial x_1\partial t_1} + kc_2^2\frac{\partial^2\mathcal{A}_1}{\partial x_1^2}\right)\right) \tag{61}$$

$$\begin{aligned}b_{3l} &= 2iR_4\left(\omega\frac{\partial\mathcal{A}_1}{\partial t_2} + kc_2^2\frac{\partial\mathcal{A}_1}{\partial x_2}\right) + 2iR_4\left(\omega\frac{\partial\mathcal{A}_g}{\partial t_1} + kc_2^2\frac{\partial\mathcal{A}_g}{\partial x_1}\right) \\ &+ R_4\left(-\frac{\partial^2\mathcal{A}_1}{\partial t_1^2} + c_2^2\frac{\partial^2\mathcal{A}_1}{\partial x_1^2}\right) + 2T_4\left(\omega\frac{\partial\mathcal{A}_1}{\partial x_1\partial t_1} + kc_2^2\frac{\partial\mathcal{A}_1}{\partial x_1}\right)\end{aligned} \tag{62}$$

$$b_{4l} = \frac{2iR_4}{kc_2^2p_2}\left(\omega^2\frac{\partial^2\mathcal{A}_1}{\partial t_1^2} + kc_2^2\left(2\omega\frac{\partial^2\mathcal{A}_1}{\partial x_1\partial t_1} + kc_2^2\frac{\partial^2\mathcal{A}_1}{\partial x_1^2}\right)\right) \tag{63}$$

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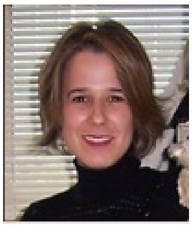

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