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NEW SUFFICIENT CRITERIA FOR GLOBAL ROBUST STABILITY OF NEURAL NETWORKS WITH MULTIPLE TIME DELAYS

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ABSTRACT. In this paper, we study global robust asymptotic stability of the equilibrium point for neural networks with multiple time delays. By employing suitable Lyapunov functionals, we derive a set of delay independent sufficient conditions for global robust asymptotic stability of this class of neural networks. Some examples are constructed to compare the reported results with the related existing results. This comparison proves that our results establish a new set of robust stability criteria for delayed neural networks. It is also demonstrated that the reported results can be easily verified as they can be expressed in terms of the network parameters only.

Keywords: Neural networks, robust stability, delayed systems, Lyapunov functionals, interval matrices.

AMS Subject Classification: 83-02, 99A00

1. INTRODUCTION

In recent years, neural networks have been applied to various signal processing problems such as optimization, image processing and associative memory design. In such applications, it is important to know the convergence properties of the designed neural network. One of the key convergence properties of neural networks is the existence, uniqueness and global asymptotic stability of the equilibrium point. One may refer to [1]-[22] and the references therein for various stability results for different neural network models with or without delays. In the physical implementation of neural networks, the network parameters may be subject to some random errors. In such cases, it is important to investigate robust stability properties of neural networks. Recently, some important results concerning the equilibrium and stability properties of different classes of neural networks with time delays have been reported [23]-[34]. In the present paper, we will obtain new sufficient conditions for the global robust asymptotic stability of neural networks with multiple time delays.

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The delayed neural network model we consider is described by the following differential equations :

$$\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij})) + u_i, \quad i = 1,2,...,n \quad (1)$$

where n is the number of the neurons, $x_i(t)$ denotes the state of the neuron i at time t, $f_i(\cdot)$ denote activation functions, a_{ij} and b_{ij} denote the strengths of connectivity between neurons j and i at time t and $t-\tau_{ij}$, respectively; τ_{ij} represents the time delay required in transmitting a signal from the neuron j to the neuron i, u_i is the constant input to the neuron i, c_i is the charging rate for the neuron i.

For $\tau_{ij} = \tau_j$, neural network model (1) is of the form :

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_j)) + u_i, \ \ i = 1,2,...,n$$

which can be written in the compact form as follows

$$\dot{x}(t) = -Cx(t) + Af(x(t)) + Bf(x(t-\tau)) + u \quad (2)$$

where $x(t) = (x_1(t), x_2(t), ..., x_n(t))^T \in R^n$, $C = diag(c_i > 0)_{n\times n}$ $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $u = (u_1, u_2, ..., u_n)^T$ and $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), ..., f_n(x_n(t)))^T$ and $f(x(t-\tau)) = (f_1(x_1(t-\tau_1)), f_2(x_2(t-\tau_2)), ..., f_n(x_n(t-\tau_n)))^T$.

The parameters $A = (a_{ij})$, $B = (b_{ij})$ and $C = diag(c_i > 0)$ are assumed to be intervalized as follows

$$\begin{aligned} C_I &:= \{C = diag(c_i) : 0 < \underline{C} \le C \le \overline{C}, i.e., 0 < \underline{c}_i \le c_i \le \overline{c}_i, i = 1,2,...,n\} \\ A_I &:= \{A = (a_{ij}) : \underline{A} \le A \le \overline{A}, i.e., \underline{a}_{ij} \le a_{ij} \le \overline{a}_{ij}, i,j = 1,2,...,n\} \\ B_I &:= \{B = (b_{ij}) : \underline{B} \le B \le \overline{B}, i.e., \underline{b}_{ij} \le b_{ij} \le \overline{b}_{ij}, i,j = 1,2,...,n\} \end{aligned} \quad (3)$$

We will assume that the functions f_i satisfy the following condition :

There exist some positive constants μ_i such that

$$0 \le \frac{f_i(x) - f_i(y)}{x - y} \le \mu_i, \ \ i = 1,2,...,n, \ \ \forall x, y \in R, x \ne y$$

This class of functions will be denoted by $f \in \mathcal{K}$.

2. PRELIMINARIES

The concept of robust stability for neural networks is given by the following definition :

Definition 1 (31). *The neural network defined by (1) or (2) with the parameter ranges defined by (3) is globally asymptotically robust stable if the unique equilibrium point $x^* = (x_1^*, x_2^*, ..., x_n^*)^T$ of the neural system is globally asymptotically stable for all $C \in C_I$, $A \in A_I$ and $B \in B_I$.*

We view some basic facts about norms of vectors and matrices. Let $v = (v_1, v_2, ..., v_n)^T \in R^n$. The three commonly used vector norms are $||v||_1, ||v||_2, ||v||_\infty$ which are defined as :

$$||v||_1 = \sum_{i=1}^{n} |v_i|,\ \ ||v||_2 = \sqrt{\sum_{i=1}^{n} v_i^2},\ \ ||v||_\infty = \max_{1\leq i\leq n} |v_i|$$

If $Q = (q_{ij})_{n\times n}$, then $||Q||_1$, $||Q||_2$ and $||Q||_\infty$ are defined as follows :

$$||Q||_1 = \max_{1\leq i\leq n} \sum_{j=1}^{n} |q_{ji}|,\ \ ||Q||_2 = [\lambda_{\max}(Q^T Q)]^{1/2},\ \ ||Q||_\infty = \max_{1\leq i\leq n} \sum_{j=1}^{n} |q_{ij}|$$

Throughout this paper, for $v = (v_1, v_2, ..., v_n)^T$, $|v|$ will denote $|v| = (|v_1|, |v_2|, ..., |v_n|)^T$. For any real matrix $Q = (q_{ij})_{n\times n}$, the matrix $|Q|$ will denote $|Q| = (|q_{ij}|)_{n\times n}$, and $\lambda_m(Q)$ and $\lambda_M(Q)$ will denote the minimum and maximum eigenvalues of Q, respectively. If $Q = (q_{ij})_{n\times n}$ is a symmetric matrix, then, $Q > 0 (< 0)$ will imply that Q is positive (negative) definite,i.e., Q has all positive (negative) eigenvalues. For any two real matrices $P = (p_{ij})_{n\times n}$ and $Q = (q_{ij})_{n\times n}$, $P \leq Q$ will imply that $p_{ij} \leq q_{ij}$ for $i, j = 1, 2, ..., n$.

We will now express some key results that will play an important role in determining the sufficient conditions for the global robust stability of the equilibria of neural networks (1) and (2). Before proceeding any further, we will first observe the following :

Let define A as $A \in A_I := \{A = (a_{ij}) : \underline{A} \leq A \leq \overline{A}, i.e., \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, i, j = 1, 2, ..., n\}$. Then, a_{ij} can be expressed as follows :

$$\begin{aligned} a_{ij} &= \frac{1}{2}(\overline{a}_{ij} + \underline{a}_{ij}) + \frac{1}{2}\sigma_{ij}(\overline{a}_{ij} - \underline{a}_{ij}), \\ -1 &\leq \sigma_{ij} \leq 1,\ \ i, j = 1, 2, ..., n \end{aligned}$$

Define $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, and $\tilde{A} = (\tilde{a}_{ij})_{n\times n}$ with $\tilde{a}_{ij} = \frac{1}{2}\sigma_{ij}(\overline{a}_{ij} - \underline{a}_{ij})$, where $-1 \leq \sigma_{ij} \leq 1,\ \ i, j = 1, 2, ..., n$. In this case, A can be expressed as follows :

$$A = \frac{1}{2}(\overline{A} + \underline{A}) + \tilde{A} = A^* + \tilde{A}$$

We are now in a position to prove the following result :

Lemma 1. *Let* $x = (x_1, x_2, ..., x_n)^T \in R^n$. *If*

$$A \in A_I := \{A = (a_{ij}) : \underline{A} \leq A \leq \overline{A}, i.e., \underline{a}_{ij} \leq a_{ij} \leq \overline{a}_{ij}, i, j = 1, 2, ..., n\}$$

, then, the following inequality holds :

$$x^T(A + A^T)x \leq x^T(A^* + A^{*T} + ||A_* + A_*^T||_2 I)x$$

where $A^* = \frac{1}{2}\ (\overline{A} + \underline{A})$, $A_* = \frac{1}{2}\ (\overline{A} - \underline{A})$.

Proof. For $x = (x_1, x_2, ..., x_n)^T \in R^n$, we have

$$\begin{aligned} x^T(A + A^T)x &= x^T(A^* + A^{*T})x + x^T(\tilde{A} + \tilde{A}^T)x \\ &= x^T(A^* + A^{*T})x + x^T\tilde{A}x + x^T\tilde{A}^T x \\ &\leq x^T(A^* + A^{*T})x + |x^T||\tilde{A}||x| + |x^T||\tilde{A}^T||x| \end{aligned}$$

The fact that $|\tilde{a}_{ij}|\leq\frac{1}{2}(\overline{a}_{ij}-\underline{a}_{ij}),\ i,j=1,2,...,n$ implies that $|\tilde{A}|\leq A_*$. Hence, it directly follows that $|x^T||\tilde{A}||x|\leq|x^T|A_*|x|$. Hence, we can write

$$\begin{aligned} x^T(A+A^T)x &\leq x^T(A^*+A^{*T})x+|x^T|A_*|x|+|x^T|A_*^T|x| \\ &= x^T(A^*+A^{*T})x+|x^T|(A_*+A_*^T)|x| \\ &\leq x^T(A^*+A^{*T})x+||A_*+A_*^T||_2x^Tx \\ &= x^T(A^*+A^{*T}+||A_*+A_*^T||_2I)x \end{aligned}$$

implying that

$$x^T(A+A^T)x\leq x^T(A^*+A^{*T}+||A_*+A_*^T||_2I)x$$

□

We will also make use of the following lemmas :

Lemma 2 (31). *If $A\in A_I:=\{A=(a_{ij}):\underline{A}\leq A\leq\overline{A}, i.e., \underline{a}_{ij}\leq a_{ij}\leq\overline{a}_{ij}, i,j=1,2,...,n\}$, then, the following inequality holds :*

$$||A||_2\leq||A^*||_2+||A_*||_2$$

where $A^*=\frac{1}{2}(\overline{A}+\underline{A}),\ A_*=\frac{1}{2}(\overline{A}-\underline{A})$.

Lemma 3 (1). *If $H(x)\in C^0$ satisfies the following conditions*

(i) $H(x)\neq H(y)$ for all $x\neq y$,

(ii) $||H(x)||\to\infty$ as $||x||\to\infty$,

then, $H(x)$ is homeomorphism of R^n.

3. Global Robust Stability Analysis

In this section, we present new sufficient conditions under which the neural systems defined by (1) and (2) asymptotically converge to a unique equilibrium point with the parameter uncertainties given by (3). In order to simplify the proofs, we will need to shift the equilibrium point x^* of system (1) to the origin. If we let $z_i(\cdot)=x_i(\cdot)-x_i^*,\ \ i=1,2,...,n$, then, we note that the $z_i(\cdot)$ are governed by :

$$\dot{z}_i(t)=\ -c_iz_i(t)+\sum_{j=1}^{n}a_{ij}g_j(z_j(t))+\sum_{j=1}^{n}b_{ij}g_j(z_j(t-\tau_{ij})),\ \ i=1,2,...,n \tag{4}$$

where $g_i(z_i(\cdot))=f_i(z_i(\cdot)+x_i^*)-f_i(x_i^*),\ \ i=1,2,...,n$. It can easily be verified that the functions g_i satisfy the assumptions on f_i, i.e., $f\in\mathcal{K}$ implies that $g\in\mathcal{K}$. We also note that $g_i(0)=0, i=1,2,...,n$. It is thus sufficient to prove the stability of the origin of the transformed system (4) instead of considering the stability of x^* of system (1).

For $\tau_{ij}=\tau_j$, (4) takes form :

$$\dot{z}(t)=-Cz(t)+Ag(z(t))+Bg(z(t-\tau)) \tag{5}$$

where $z(t)=(z_1(t),z_2(t),...,z_n(t))^T,\ \ g(z(t))=(g_1(z_1(t)),\ g_2(z_2(t)),...,g_n(z_n(t)))^T$ and $g(z(t-\tau))=(g_1(z_1(t-\tau_1)),\ g_2(z_2(t-\tau_2)),...,g_n(z_n(t-\tau_n)))^T$.

We now proceed with following result :

Theorem 1. *Let $f \in \mathcal{K}$. Then, the neural network model (2) is globally asymptotically robust stable, if*

$$\Phi = \quad 2rI - A^* - A^{*T} - ||A_* + A_*^T||_2 I - 2(||B^*||_2 + ||B_*||_2)I > 0$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $B^* = \frac{1}{2}(\overline{B} + \underline{B})$, $B_* = \frac{1}{2}(\overline{B} - \underline{B})$ *and* $r = min(\frac{c_i}{\mu_i})$.

Proof. We first show that the conditions in Theorem 1 implies the existence and uniqueness of the equilibrium point. Define the following mapping associated with (2) :

$$H(x) = -Cx + Af(x) + Bf(x) + u \tag{6}$$

We know that if x^* is an equilibrium point of (2), then it satisfies

$$-Cx^* + Af(x^*) + Bf(x^*) + u = 0$$

It is clear that every solution of $H(x) = 0$ is an equilibrium point of (1). Therefore, we can conclude from Lemma 3 that, for the neural system defined by (2), there exists a unique equilibrium point for every input vector u if $H(x)$ is homeomorphism of R^n. In order to prove that $H(x)$ is a homeomorphism of R^n, we consider two vectors $x \in R^n$ and $y \in R^n$ such that $x \neq y$. For $H(x)$ defined by (6), we can write

$$H(x) - H(y) = -C(x-y) + A(f(x) - f(y)) + B(f(x) - f(y)) \tag{7}$$

For the functions belonging to class $\mathcal{K}$, $x \neq y$ implies that $f(x) - f(y) = 0$ or $f(x) - f(y) \neq 0$. For case where $x \neq y$ and $f(x) - f(y) = 0$, we obtain

$$H(x) - H(y) = -C(x-y)$$

in which $x - y \neq 0$ implies that $H(x) \neq H(y)$ since C is a positive diagonal matrix. Now consider the case where $x - y \neq 0$ and $f(x) - f(y) \neq 0$. Multiplying both sides of (7) by $2(f(x) - f(y))^T$ results in

$$\begin{aligned} 2(f(x) - f(y))^T(H(x) - H(y)) \quad = \quad & -2(f(x) - f(y))^T C(x-y) \\ & +2(f(x) - f(y))^T A(f(x) - f(y)) \\ & +2(f(x) - f(y))^T B(f(x) - f(y)) \end{aligned} \tag{8}$$

We can write the following :

$$2(f(x) - f(y))^T A(f(x) - f(y)) = (f(x) - f(y))^T (A + A^T)(f(x) - f(y))$$

From Lemma 1, we know that

$$\begin{aligned} (f(x) - f(y))^T(A + A^T)(f(x) - f(y)) \quad \leq \quad & (f(x) - f(y))^T(A^* + A^{*T})(f(x) - f(y)) \\ & +(f(x) - f(y))^T(||A_* + A_*^T||_2 I)(f(x) - f(y)) \end{aligned}$$

Hence, we obtain

$$\begin{aligned} (f(x) - f(y))^T(A + A^T)(f(x) - f(y)) \leq & (f(x) - f(y))^T(A^* + A^{*T})(f(x) - f(y)) \\ & +||A_* + A_*^T||_2 (f(x) - f(y))^T(f(x) - f(y)) \end{aligned} \tag{9}$$

We also know from Lemma 2 that $||B||_2 \leq (||B^*||_2 + ||B_*||_2)$. Therefore, we can write

$$\begin{aligned} 2(f(x) - f(y))^T B(f(x) - f(y)) \leq & 2||B||_2 ||f(x) - f(y)||_2^2 \\ \leq & 2(||B^*||_2 + ||B_*||_2)(f(x) - f(y))^T(f(x) - f(y)) \end{aligned} \tag{10}$$

We also note the following ,

$$\begin{aligned} -2(f(x)-f(y))^T C(x-y) &= -2\sum_{i=1}^{n} c_i(f_i(x_i)-f_i(y_i))(x_i-y_i) \\ &\leq -2\sum_{i=1}^{n} \frac{c_i}{\mu_i}(f_i(x_i)-f_i(y_i))^2 \\ &\leq -2r\sum_{i=1}^{n}(f_i(x_i)-f_i(y_i))^2 \\ &= -2r(f(x)-f(y))^T(f(x)-f(y)) \end{aligned} \tag{11}$$

where $r = min(\frac{c_i}{\mu_i})$.

Using (9)-(11) in (8) results in

$$\begin{aligned} 2(f(x)-f(y))^T(H(x)-H(y)) \leq & -2r(f(x)-f(y))^T(f(x)-f(y)) \\ & +(f(x)-f(y))^T(A^* + A^{*T})(f(x)-f(y)) \\ & +||A_* + A_*^T||_2(f(x)-f(y))^T(f(x)-f(y)) \\ & +2(||B^*||_2 + ||B_*||_2)(f(x)-f(y))^T(f(x)-f(y)) \end{aligned} \tag{12}$$

(12) is equivalent to the following :

$$2(f(x)-f(y))^T(H(x)-H(y)) \leq -(f(x)-f(y))^T\Phi(f(x)-f(y)) \tag{13}$$

If $f(x)-f(y) \neq 0$ and $\Phi > 0$, then the following inequality holds :

$$2(f(x)-f(y))^T(H(x)-H(y)) \quad < \quad 0$$

$f(x)-f(y) \neq 0$ implies that $H(x) \neq H(y)$. It directly follows that $H(x) \neq H(y)$ for all $x \neq y$.

For $y = 0$, (13) takes the form :

$$2(f(x)-f(0))^T(H(x)-H(0)) \leq -(f(x)-f(0))^T\Phi(f(x)-f(0))$$

from which one would obtain

$$|2(f(x)-f(0))^T(H(x)-H(0))| \geq (f(x)-f(0))^T\Phi(f(x)-f(0))$$

which yields

$$2||f(x)-f(0)||_\infty ||H(x)-H(0)||_1 > \lambda_m(\Phi)||f(x)-f(0)||_2^2$$

Using the facts that $||f(x)-f(0)||_\infty \leq ||f(x)-f(0)||_2$, $||H(x)-H(0)||_1 \leq ||H(x)||_1 + ||H(0)||_1$ and $||f(x)-f(0)||_2 \geq ||f(x)||_2 - ||f(0)||_2$, we obtain

$$||H(x)||_1 \quad > \quad \frac{\lambda_m(\Phi)||f(x)||_2 - \lambda_m(\Phi)||f(0)||_2 - 2||H(0)||_1}{2}$$

Since, $||H(0)||_1$ and $||f(0)||_2$ are finite, then $||H(x)|| \to \infty$ as $||f(x)|| \to \infty$. Thus, it follows from Lemma 3 that the map $H(x) : R^n \to R^n$ is homeomorphism of R^n, hence there exists a unique x^* such $H(x^*) = 0$ which is a solution of (1). Hence, the proof of the existence and uniqueness of the equilibrium point is completed.

In order to show that $\Phi > 0$ also implies the global asymptotic stability of the origin of system (5), we construct the following positive definite Lyapunov functional :

$$V(z(t)) = z^T(t)z(t) + 2\alpha \sum_{i=1}^{n} \int_0^{z_i(t)} g_i(s)ds + (\alpha\gamma + \beta) \sum_{i=1}^{n} \int_{t-\tau_i}^{t} g_i^2(z_i(\zeta))d\zeta$$

where α, β and γ are some positive constants to be determined later. The time derivative of the functional along the trajectories of system (5) is obtained as follows

$$\begin{aligned}\dot{V}(z(t)) &= -2z^T(t)Cz(t) + 2z^T(t)Ag(z(t)) + 2z^T(t)Bg(z(t-\tau)) \\ &\quad -2\alpha g^T(z(t))Cz(t) + 2\alpha g^T(z(t))Ag(z(t)) + 2\alpha g^T(z(t))Bg(z(t-\tau)) \\ &\quad +\alpha\gamma||g(z(t))||_2^2 - \alpha\gamma||g(z(t-\tau))||_2^2 + \beta||g(z(t))||_2^2 - \beta||g(z(t-\tau))||_2^2 \end{aligned} \tag{14}$$

We can write the following inequalities :

$$-z^T(t)Cz(t) + 2z^T(t)Ag(z(t)) \leq g^T(z(t))A^TC^{-1}Ag(z(t)) \leq ||A||_2^2||C^{-1}||_2||g(z(t))||_2^2 \tag{15}$$

$$\begin{aligned}-z^T(t)Cz(t) + 2z^T(t)Bg(z(t-\tau)) &\leq g^T(z(t-\tau))B^TC^{-1}Bg(z(t-\tau)) \\ &\leq ||B||_2^2||C^{-1}||_2||g(z(t-\tau))||_2^2\end{aligned} \tag{16}$$

$$\begin{aligned}2\alpha g^T(z(t))Bg(z(t-\tau)) &\leq 2\alpha||B||_2||g(z(t))||_2||g(z(t-\tau))||_2 \\ &\leq \alpha||B||_2||g(z(t))||_2^2 + \alpha||B||_2||g(z(t-\tau))||_2^2\end{aligned} \tag{17}$$

$$-2\alpha g^T(z(t))Cz(t) \leq -2\alpha r||g(z(t))||_2^2 \tag{18}$$

Using (15)-(18) in (14) results in :

$$\begin{aligned}\dot{V}(z(t)) &\leq ||A||_2^2||C^{-1}||_2||g(z(t))||_2^2 + ||B||_2^2||C^{-1}||_2||g(z(t-\tau))||_2^2 - 2\alpha r||g(z(t))||_2^2 \\ &\quad +\alpha g^T(z(t))(A + A^T)g(z(t)) + \alpha||B||_2||g(z(t))||_2^2 + \alpha||B||_2||g(z(t-\tau))||_2^2 \\ &\quad +\alpha\gamma||g(z(t))||_2^2 - \alpha\gamma||g(z(t-\tau))||_2^2 + \beta||g(z(t))||_2^2 - \beta||g(z(t-\tau))||_2^2\end{aligned}$$

From Lemma 1, we can write

$$g^T(z(t))(A + A^T)g(z(t)) \leq g^T(z(t))(A^* + A^{*T} + ||A_* + A_*^T||_2I)g(z(t))$$

From lemma 2, we can write the inequalities $||A||_2 \leq ||A^*||_2 + ||A_*||_2$ and $||B||_2 \leq ||B^*||_2 + ||B_*||_2$. We also have $||C^{-1}||_2 \leq ||\underline{C}^{-1}||_2$. Hence, $\dot{V}(z(t))$ can be written as follows :

$$\begin{aligned}\dot{V}(z(t)) &\leq (||A^*||_2 + ||A_*||_2)^2||\underline{C}^{-1}||_2||g(z(t))||_2^2 \\ &\quad +(||B^*||_2 + ||B_*||_2)^2||\underline{C}^{-1}||_2||g(z(t-\tau))||_2^2 - 2\alpha r||g(z(t))||_2^2 \\ &\quad +\alpha g^T(z(t))(A^* + A^{*T} + ||A_* + A_*^T||_2I)g(z(t)) \\ &\quad +\alpha||P||_2(||B^*||_2 + ||B_*||_2)||g(z(t))||_2^2 \\ &\quad +\alpha||P||_2(||B^*||_2 + ||B_*||_2)||g(z(t-\tau))||_2^2 \\ &\quad +\alpha\gamma||g(z(t))||_2^2 - \alpha\gamma||g(z(t-\tau))||_2^2 \\ &\quad +\beta||g(z(t))||_2^2 - \beta||g(z(t-\tau))||_2^2\end{aligned}$$

If we let $\beta = (||B^*||_2 + ||B_*||_2)^2||\underline{C}^{-1}||_2$ and $\gamma = (||B^*||_2 + ||B_*||_2)$, then, $\dot{V}(z(t))$ takes the form

$$\begin{aligned}\dot{V}(z(t)) &\leq ((||A^*||_2 + ||A_*||_2)^2 + (||B^*||_2 + ||B_*||_2)^2)||\underline{C}^{-1}||_2||g(z(t))||_2^2 - 2\alpha r||g(z(t))||_2^2 \\ &\quad +\alpha g^T(z(t))(A^* + A^{*T} + ||A_* + A_*^T||_2)g(z(t)) \\ &\quad +2\alpha(||B^*||_2 + ||B_*||_2)||g(z(t))||_2^2 \\ &= ((||A^*||_2 + ||A_*||_2)^2 + (||B^*||_2 + ||B_*||_2)^2)||\underline{C}^{-1}||_2||g(z(t))||_2^2 \\ &\quad -\alpha g^T(z(t))\Phi g(z(t)) \\ &\leq ((||A^*||_2 + ||A_*||_2)^2 + (||B^*||_2 + ||B_*||_2)^2)||\underline{C}^{-1}||_2||g(z(t))||_2^2 \\ &\quad -\alpha\lambda_m(\Phi)||g(z(t))||_2^2\end{aligned}$$

The choice

$$\alpha > \frac{||\underline{C}^{-1}||_2((||A^*||_2 + ||A_*||_2)^2 + (||B^*||_2 + ||B_*||_2)^2)}{\lambda_m(\Phi)}$$

ensures that $\dot{V}(z(t))$ is negative definite for all $g(z(t)) \neq 0$. (It should be noted here that $g(z(t)) \neq 0$ implies that $z(t) \neq 0$). Now consider the case where $g(z(t)) = 0$ and $z(t) \neq 0$. In this case, $\dot{V}(z(t))$ satisfies

$$\begin{aligned}\dot{V}(z(t)) &= -2z^T(t)Cz(t) + 2z^T(t)Bg(z(t-\tau)) - \beta g^T(z(t-\tau))g(z(t-\tau)) \\ &\quad -\alpha\gamma g^T(z(t-\tau))g(z(t-\tau)) \\ &\leq -2z^T(t)Cz(t) + 2z^T(t)Bg(z(t-\tau)) - \beta g^T(z(t-\tau))g(z(t-\tau))\end{aligned}$$

Since

$$-z^T(t)Cz(t) + 2z^T(t)Bg(z(t-\tau)) - \beta g^T(z(t-\tau))g(z(t-\tau)) \leq 0$$

then, $\dot{V}(z(t)) \leq -z^T(t)Cz(t)$. It follows that $\dot{V}(z(t))$ is negative definite for all $z(t) \neq 0$ with $g(z(t)) = 0$. Finally, consider the case where $g(z(t)) = 0$ and $z(t) = 0$. This case implies that

$$\dot{V}(z(t)) = -\beta g^T(z(t-\tau))g(z(t-\tau)) - \alpha\gamma g^T(z(t-\tau))g(z(t-\tau))$$

Obviously, $\dot{V}(z(t))$ is negative definite for all $g(z(t-\tau)) \neq 0$. Hence, it follows that $\dot{V}(z(t)) = 0$ if and only if $z(t) = g(z(t)) = g(z(t-\tau)) = 0$, otherwise $\dot{V}(z(t)) < 0$. Moreover, $V(z(t))$ is radially unbounded since $V(z(t)) \to \infty$ as $||z(t)|| \to \infty$. Thus, it can be concluded that the origin of system (5), or equivalently the equilibrium point of system (2) is globally asymptotically stable.

□

The other main result of this paper is given in the following :

Theorem 2. *Let $f \in \mathcal{K}$. Then, the neural network model (1) is globally asymptotically robust stable, if*

$$\Theta = 2rI - A^* - A^{*T} - ||A_* + A_*^T||_2 I - (||\hat{B}||_1 + ||\hat{B}||_\infty)I > 0$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $\hat{B} = (b_{ij}^*)_{n\times n}$, $b_{ij}^* = max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$, and $r = min(\frac{\underline{c}_i}{\mu_i})$.

Proof. We first note the following inequality

$$
\begin{aligned}
2(f(x)-f(y))^T B(f(x)-f(y)) &= \sum_{i=1}^{n}\sum_{j=1}^{n} 2b_{ij}(f_i(x_i)-f_i(y_i))(f_j(x_j)-f_j(y_j)) \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} 2|b_{ij}||f_i(x_i)-f_i(y_i)||f_j(x_j)-f_j(y_j)| \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}|((f_i(x_i)-f_i(y_i))^2+(f_j(x_j)-f_j(y_j))^2) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}|(f_i(x_i)-f_i(y_i))^2 \\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ji}|(f_i(x_i)-f_i(y_i))^2 \\
&\leq \sum_{i=1}^{n}\sum_{j=1}^{n} |b^*_{ij}|(f_i(x_i)-f_i(y_i))^2 \\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n} |b^*_{ji}|(f_i(x_i)-f_i(y_i))^2 \\
&\leq (||\hat{B}||_\infty+||\hat{B}||_1)(f(x)-f(y))^T(f(x)-f(y)) \qquad (19)
\end{aligned}
$$

Now, using (9), (11) and (19) in (8), we obtain

$$
\begin{aligned}
2(f(x)-f(y))^T(H(x)-H(y)) &\leq -2r(f(x)-f(y))^T(f(x)-f(y)) \\
&\quad +(f(x)-f(y))^T(A^*+A^{*T})(f(x)-f(y) \\
&\quad +||A_*+A_*^T||_2(f(x)-f(y))^T(f(x)-f(y)) \\
&\quad +(||\hat{B}||_\infty+||\hat{B}||_1)(f(x)-f(y))^T(f(x)-f(y)) \qquad (20)
\end{aligned}
$$

which is equivalent to

$$
2(f(x)-f(y))^T P(H(x)-H(y)) \leq -(f(x)-f(y))^T\Theta(f(x)-f(y)) \qquad (21)
$$

Note that (21) is exactly in the same form as (13). When replacing Φ in (13) by Θ yields (21). For (13), we have already proved that if $\Phi > 0$, then $||H(x)|| \to \infty$ as $||x|| \to \infty$, and $H(x) \neq H(y)$ for all $x \neq y$. Therefore, we can directly conclude that if $\Theta > 0$, then $||H(x)|| \to \infty$ as $||x|| \to \infty$, and $H(x) \neq H(y)$ for all $x \neq y$. Hence, the proof of the existence and uniqueness of the equilibrium point is complete.

We will now prove that $\Theta > 0$ is also a sufficient condition for global asymptotic stability of the origin of (5). To this end, we consider the following positive definite Lyapunov functional :

$$
\begin{aligned}
V(z(t)) &= \sum_{i=1}^{n} nz_i^2(t)+2\alpha\sum_{i=1}^{n}\int_0^{z_i(t)} g_i(s)ds+\sum_{i=1}^{n}\sum_{j=1}^{n}(\varepsilon+\alpha b^*_{ij})\int_{t-\tau_{ij}}^{t} g_j^2(z_j(\xi))d\xi \\
&\quad +\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\underline{c}_i}n^2(b^*_{ij})^2\int_{t-\tau_{ij}}^{t} g_j^2(z_j(\xi))d\xi
\end{aligned}
$$

where $b^*_{ij} = max\{|\underline{b}_{ij}|, |\overline{b}_{ij}|\}$, $a^*_{ij} = max\{|\underline{a}_{ij}|, |\overline{a}_{ij}|\}$, α and ε are positive constants to be determined later. The time derivative of the functional along the trajectories of system (5) is obtained as follows

$$\begin{aligned}
\dot{V}(z(t)) &= -2\sum_{i=1}^{n} nc_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} 2na_{ij} z_i(t) g_j(z_j(t)) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} 2nb_{ij} z_i(t) g_j(z_j(t-\tau_{ij})) - 2\alpha\sum_{i=1}^{n} c_i z_i(t) g_i(z_i(t)) \\
&+ \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} 2a_{ij} g_i(z_i(t)) g_j(z_j(t)) + \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} 2b_{ij} g_i(z_i(t)) g_j(z_j(t-\tau_{ij})) \\
&+ \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ij} g_j^2(z_j(t)) - \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ij} g_j^2(z_j(t-\tau_{ij})) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon g_j^2(z_j(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon g_j^2(z_j(t-\tau_{ij})) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b^*_{ij})^2 g_j^2(z_j(t)) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b^*_{ij})^2 g_j^2(z_j(t-\tau_{ij}))
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} 2na_{ij} z_i(t) g_j(z_j(t)) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} c_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{c_i} n^2 a_{ij}^2 g_j^2(z_j(t)) \\
&\leq \sum_{i=1}^{n} nc_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (a^*_{ij})^2 g_j^2(z_j(t))
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} 2nb_{ij} z_i(t) g_j(z_j(t-\tau_{ij})) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} c_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{c_i} n^2 b_{ij}^2 g_j^2(z_j(t-\tau_{ij})) \\
&\leq \sum_{i=1}^{n} nc_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b^*_{ij})^2 g_j^2(z_j(t-\tau_{ij}))
\end{aligned}$$

$$\begin{aligned}
\alpha\sum_{i=1}^{n}\sum_{j=1}^{n} 2b_{ij} g_i(z_i(t)) g_j(z_j(t-\tau_{ij})) &\leq \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}| g_i^2(z_i(t)) + \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}| g_j^2(z_j(t-\tau_{ij})) \\
&\leq \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ij} g_i^2(z_i(t)) + \alpha\sum_{i=1}^{n}\sum_{j=1}^{n} b^*_{ij} g_j^2(z_j(t-\tau_{ij}))
\end{aligned}$$

In the light above inequalities, $\dot{V}(z(t))$ can be written as follows

$$\begin{aligned}
\dot{V}(z(t)) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\underline{c}_i}n^2(a^*_{ij})^2g_j^2(z_j(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\underline{c}_i}n^2(b^*_{ij})^2g_j^2(z_j(t))\\
&\quad -2\alpha\sum_{i=1}^{n}c_iz_i(t)g_i(z_i(t)+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}2a_{ij}g_i(z_i(t))g_j(z_j(t))\\
&\quad +\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b^*_{ij}g_j^2(z_j(t))+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b^*_{ij}g_i^2(z_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon g_i^2(z_i(t))\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\underline{c}_i}n^2[(a^*_{ji})^2+(b^*_{ji})^2]g_i^2(z_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon g_i^2(z_i(t))\\
&\quad -2\alpha\sum_{i=1}^{n}c_iz_i(t)g_i(z_i(t))+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}2a_{ij}g_i(z_i(t))g_j(z_j(t))\\
&\quad +\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b^*_{ji}g_i^2(z_i(t))+\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b^*_{ij}g_i^2(z_i(t))
\end{aligned}$$

We also note the following inequalities

$$-2\alpha\sum_{i=1}^{n}c_iz_i(t)g_i(z_i(t))\leq -2\alpha r g^T(z(t))g(z(t))$$

$$\begin{aligned}
\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}2a_{ij}g_i(z_i(t))g_j(z_j(t)) &= \alpha g^T(z(t))(A+A^T)g(z(t))\\
&\leq \alpha g^T(z(t))(A^*+A^{*T})g(z(t))\\
&\quad +\alpha g^T(z(t))||A_*+A_*^T||_2g(z(t))
\end{aligned}$$

$$\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b^*_{ji}g_i^2(z_i(t)) \leq \alpha||\hat{B}||_1\sum_{i=1}^{n}g_i^2(z_i(t))=\alpha||\hat{B}||_1g^T(z(t))g(z(t))$$

$$\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}b^*_{ij}g_i^2(z_i(t)) \leq \alpha||\hat{B}||_\infty\sum_{i=1}^{n}g_i^2(z_i(t))=\alpha||\hat{B}||_\infty g^T(z(t))g(z(t))$$

Let

$$\delta = max(\frac{1}{\underline{c}_i}n^2[(a^*_{ji})^2+(b^*_{ji})^2])$$

Then, we have

$$\begin{aligned}
\dot{V}(z(t)) &\leq \sum_{i=1}^{n}\sum_{j=1}^{n}\delta g_i^2(z_i(t))+\sum_{i=1}^{n}\sum_{j=1}^{n}\varepsilon g_i^2(z_i(t))-2\alpha r g^T(z(t))g(z(t))\\
&\quad +\alpha g^T(z(t))(A^*+A^{*T})g(z(t))+\alpha g^T(z(t))||A_*+A_*^T||_2g(z(t))\\
&\quad +\alpha||\hat{B}||_1g^T(z(t))g(z(t))+\alpha||\hat{B}||_\infty g^T(z(t))g(z(t))\\
&= n(\delta+\varepsilon)||g(z(t))||_2^2-\alpha g^T(z(t))\Theta g(z(t))\\
&\leq n(\delta+\varepsilon)||g(z(t))||_2^2-\alpha\lambda_m(\Theta)||g(z(t))||_2^2\\
&= -(\alpha\lambda_m(\Theta)-n(\delta+\varepsilon))||g(z(t))||_2^2
\end{aligned}$$

in which $\alpha > \frac{n(\delta+\varepsilon)}{\lambda_m(\hat{\Theta})}$ implies that $\dot{V}(z(t))$ is negative definite for all $g(z(t)) \neq 0$. (We know that $g(z(t)) \neq 0$ implies that $z(t) \neq 0$). Now let $g(z(t)) = 0$. In this case $\dot{V}(z(t))$ satisfies

$$\begin{aligned}
\dot{V}(z(t)) &= -2\sum_{i=1}^{n} nc_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} 2nb_{ij} z_i(t) g_j(z_j(t-\tau_{ij})) \\
&\quad -\alpha \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^* g_j^2(z_j(t-\tau_{ij})) - \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon g_j^2(z_j(t-\tau_{ij})) \\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b_{ij}^*)^2 g_j^2(z_j(t-\tau_{ij})) \\
&\leq -2\sum_{i=1}^{n} nc_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} 2nb_{ij} z_i(t) g_j(z_j(t-\tau_{ij})) \\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b_{ij}^*)^2 g_j^2(z_j(t-\tau_{ij}))
\end{aligned}$$

Since

$$-\sum_{i=1}^{n} nc_i z_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n} 2nb_{ij} z_i(t) g_j(z_j(t-\tau_{ij})) - \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b_{ij}^*)^2 g_j^2(z_j(t-\tau_{ij})) \leq 0$$

we obtain

$$\dot{V}(z(t)) \leq -\sum_{i=1}^{n} nc_i z_i^2(t)$$

implying that $\dot{V}(z(t)) < 0$ for all $z(t) \neq 0$. Now consider the case where $g(z(t)) = z(t) = 0$. In this case, for $\dot{V}(z(t))$, we have

$$\begin{aligned}
\dot{V}(z(t)) &= -\alpha \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^* g_j^2(z_j(t-\tau_{ij})) - \sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon g_j^2(z_j(t-\tau_{ij})) \\
&\quad -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{1}{\underline{c}_i} n^2 (b_{ij}^*)^2 g_j^2(z_j(t-\tau_{ij})) \\
&\leq -\sum_{i=1}^{n}\sum_{j=1}^{n} \varepsilon g_j^2(z_j(t-\tau_{ij}))
\end{aligned}$$

in which $\dot{V}(z(t)) < 0$ if there exists at least one nonzero $g_j(z_j(t-\tau_{ij}))$. Hence, we can conclude that $\dot{V}(z(t)) = 0$ if and only if $g(z(t)) = z(t) = 0$ and $g_j(z_j(t-\tau_{ij})) = 0$ for all i, j, $\dot{V}(z(t)) < 0$ otherwise. In addition, $V(z(t))$ is radially unbounded since $V(z(t)) \to \infty$ as $||z(t)|| \to \infty$. Thus, the origin of system (5), or equivalently the equilibrium point of system (1) is globally asymptotically stable.

□

4. Comparison and Examples

In this section, we will compare our results with the previous results derived in the literature. In order to make the comparison precise, we first restate the previous results :

Theorem 3 (31). *Let $f \in \mathcal{K}$. Then, the neural network model (2) is globally asymptotically robust stable, if there exists a positive definite matrix $D > 0$ such that*

(i) *the symmetric matrix $S = (s_{ij})_{n\times n}$ is positive definite*

(ii) $(||B^*||_2 + ||B_*||_2)^2 \leq \dfrac{2r - ||D||_2}{||D^{-1}||_2}$

where $S = (s_{ij})_{n\times n}$ with $s_{ii} = -2\overline{a}_{ii}$, $s_{ij} = -max(|\overline{a}_{ij} + \overline{a}_{ji}|, |\underline{a}_{ij} + \underline{a}_{ji}|)$ for $i \neq j$, $B^* = \frac{1}{2}(\overline{B} + \underline{B})$, $B_* = \frac{1}{2}(\overline{B} - \underline{B})$ and $r = min(\frac{p_i \underline{c}_i}{\mu_i})$.

Theorem 4 (32). *Let $f \in \mathcal{K}$. Then, the neural network model (2) is globally asymptotically robust stable, if*

$$\Omega = 2rI + S - 2(||B^*||_2 + ||B_*||_2)I > 0$$

where $S = (s_{ij})_{n\times n}$ with $s_{ii} = -2\overline{a}_{ii}$, $s_{ij} = -max(|\overline{a}_{ij} + \overline{a}_{ji}|, |\underline{a}_{ij} + \underline{a}_{ji}|)$ for $i \neq j$, $B^* = \frac{1}{2}(\overline{B} + \underline{B})$, $B_* = \frac{1}{2}(\overline{B} - \underline{B})$ and $r = min(\frac{\underline{c}_i}{\mu_i})$.

Theorem 5 (33). *Let $f \in \mathcal{K}$. Then, the neural network model (2) is globally exponentially robust stable, if there exist positive constants α_i, $i = 1, 2, ..., n$ such that*

$$\alpha_i(\frac{\underline{c}_i}{\mu_i} - \overline{a}_{ii}) - \sum_{\substack{j=1\\ j\neq i}}^{n} \alpha_j a^*_{ji} - \sum_{j=1}^{n} \alpha_j b^*_{ji} > 0, \;\; i = 1, 2, ..., n$$

Theorem 6 (34). *Let $f \in \mathcal{K}$. Then, the neural network model (2) is globally asymptotically robust stable, if*

$$\Psi \;\; = \;\; 2rI - A^* - A^{*T} - 2||A_*||_2 - 2(||B^*||_2 + ||B_*||_2)I > 0$$

where $A^* = \frac{1}{2}(\overline{A} + \underline{A})$, $A_* = \frac{1}{2}(\overline{A} - \underline{A})$, $B^* = \frac{1}{2}(\overline{B} + \underline{B})$, $B_* = \frac{1}{2}(\overline{B} - \underline{B})$ and $r = min(\frac{\underline{c}_i}{\mu_i})$.

We will now consider the following examples :

Example 1. *Since Theorem 3 given in [31] requires that S be a positive definite matrix, we will first give an example where the results of Theorem 1 hold when S is not positive definite matrix. Assume that the network parameters of neural system (2) are given as follows :*

$$\underline{A} = \underline{B} = \begin{bmatrix} -a & -a & -a \\ -a & -a & -a \\ -a & -a & -a \end{bmatrix}, \overline{A} = \overline{B} = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix},$$

$$\underline{C} = C = \overline{C} = I, \mu_1 = \mu_2 = \mu_3 = 1$$

where $a > 0$ is real number. We obtain the following matrices:

$$A^* = B^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, A_* = B_* = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Then, $\Phi = 2rI - A^* - A^{*T} - ||A_* + A_*^T||_2 I - 2(||B^*||_2 + ||B_*||_2)I$ in Theorem 1 is obtained as

$$\Phi = \begin{bmatrix} 2-12a & 0 & 0 \\ 0 & 2-12a & 0 \\ 0 & 0 & 2-12a \end{bmatrix}$$

where $a < \frac{1}{6}$ implies that Φ is positive definite. Hence, this example prove the advantages of our results over the results of [31] when S is not positive definite.

Example 2. *Assume that the network parameters of neural system (2) are given as follows :*

$$\underline{A} = \begin{bmatrix} -a & -a & -a \\ 0 & -a & 0 \\ -2a & 0 & -a \end{bmatrix}, \overline{A} = \begin{bmatrix} a & a & a \\ 2a & a & 2a \\ 0 & 2a & a \end{bmatrix}$$

$$\underline{B} = \begin{bmatrix} -a & -a & -a \\ -a & -a & -a \\ -a & -a & -a \end{bmatrix}, \overline{B} = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix},$$

$$\underline{C} = C = \overline{C} = I, \mu_1 = \mu_2 = \mu_3 = 1$$

where $a > 0$ is real number. We obtain the following matrices:

$$A^* = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & a \\ -a & a & 0 \end{bmatrix}, A_* = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

$$B^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_* = \begin{bmatrix} a & a & a \\ a & a & a \\ a & a & a \end{bmatrix}$$

Then, $\Phi = 2rI - A^* - A^{*T} - ||A_* + A_*^T||_2 I - 2(||B^*||_2 + ||B_*||_2)I$ in Theorem 1 is obtained as

$$\Phi = \begin{bmatrix} 2-12a & -a & a \\ -a & 2-12a & -2a \\ a & -2a & 2-12a \end{bmatrix}$$

Let $a = \frac{5}{36}$. In this case, Φ is of the form :

$$\Phi = \frac{1}{36} \begin{bmatrix} 12 & -5 & 5 \\ -5 & 12 & -10 \\ 5 & -10 & 12 \end{bmatrix}$$

Since Φ is symmetric and its eigenvalues are all positive, it directly follows that Φ is positive definite, thus implying that the condition of Theorem 1 holds.
Now let us check the condition of Theorem 4 for the same network parameters when $a = \frac{5}{36}$. For the network parameters given in this example, we obtain

$$S = \begin{bmatrix} -2a & -3a & -3a \\ -3a & -2a & -4a \\ -3a & -4a & -2a \end{bmatrix}$$

Then, for $P = I$, $\Phi = 2rI + S - 2||P||_2(||B^*||_2 + ||B_*||_2)I > 0$ in Theorem 4 is obtained as

$$\Phi = \begin{bmatrix} 2-8a & -3a & -3a \\ -3a & 2-8a & -4a \\ -3a & -4a & 2-8a \end{bmatrix}$$

For $a = \frac{5}{36}$, Φ is of the form :

$$\Phi = \frac{1}{36} \begin{bmatrix} 32 & -15 & -15 \\ -15 & 32 & -20 \\ -15 & -20 & 32 \end{bmatrix}$$

It is easy to see that the determinant of Φ is negative, thus meaning that Φ is not positive definite. Therefore, the conditions Theorem 4 are not applicable to this example.

Example 3. *Assume that the network parameters of neural system (1) are given as follows:*

$$\underline{A} = \underline{B} = \begin{bmatrix} -a & -a & -a \\ -3a & -a & -a \\ -3a & -3a & -a \end{bmatrix}, \overline{A} = \overline{B} = \begin{bmatrix} 3a & 3a & 3a \\ a & 3a & 3a \\ a & a & 3a \end{bmatrix},$$

$$\underline{C} = C = \overline{C} = I, \mu_1 = \mu_2 = \mu_3 = 1$$

where $a > 0$ is real number. We obtain the following matrices:

$$A^* = B^* = \begin{bmatrix} a & a & a \\ -a & a & a \\ -a & -a & a \end{bmatrix}, A_* = B_* = \begin{bmatrix} 2a & 2a & 2a \\ 2a & 2a & 2a \\ 2a & 2a & 2a \end{bmatrix}, \hat{A} = \hat{B} = \begin{bmatrix} 3a & 3a & 3a \\ 3a & 3a & 3a \\ 3a & 3a & 3a \end{bmatrix}$$

Then, $\Theta = 2rI - A^* - A^{*T} - ||A_* + A_*^T||_2 I - (||\hat{B}||_1 + ||\hat{B}||_\infty)I$ in Theorem 2 is obtained as

$$\hat{\Theta} = \begin{bmatrix} 2-32a & 0 & 0 \\ 0 & 2-32a & 0 \\ 0 & 0 & 2-32a \end{bmatrix}$$

where $\Theta > 0$ if $a < \frac{1}{16}$. When checking the conditions of Theorems 5 for the network parameters given in this example, it is easy to see that Theorem 5 requires that the following matrix be a nonsingular M-matrix :

$$I - \hat{A} - \hat{B} = \begin{bmatrix} 1-6a & -6a & -6a \\ -6a & 1-6a & -6a \\ -6a & -6a & 1-6a \end{bmatrix}$$

Clearly, $I - \hat{A} - \hat{B}$ is nonsingular M-matrix if and only if $a < \frac{1}{18}$. Obviously, for $\frac{1}{18} \leq a < \frac{1}{16}$, our condition obtained in Theorem 2 is satisfied but the results of Theorem 5 do not hold.

Finally we will show that Theorem 6 is a special case of Theorem 1. Since $||A_* + A_*^T||_2 \leq 2||A_*||_2$, it follows that $\Phi \geq \Psi$, thus directly implying that Theorem 6 is a special case of Theorem 1.

5. CONCLUSIONS

We have studied the global robust asymptotic stability of the equilibrium point for delayed neural networks. By employing suitable Lyapunov functionals, we have derived some easily verifiable delay independent sufficient conditions for the global robust stability of the equilibrium point. We have also presented some numerical examples to illustrate the effectiveness and advantages of our results over the previous relevant robust stability results published in the literature. Since the obtained results proved to be a new set of robust stability criteria, they can be used to expand significantly the application domain of neural networks.

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