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GRACEFUL COLORING OF LADDER GRAPHS

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Abstract. A graceful k-coloring of a non-empty graph $G = (V, E)$ is a proper vertex coloring $f : V(G) \to \{1, 2, ..., k\}$, $k \geq 2$, which induces a proper edge coloring $f^* : E(G) \to \{1, 2, ..., k-1\}$ defined by $f^*(uv) = |f(u) - f(v)|$, where $u, v \in V(G)$. The minimum k for which G has a graceful k-coloring is called graceful chromatic number, $\chi_g(G)$. The graceful chromatic number for a few variants of ladder graphs are investigated in this article.

Keywords: Graceful chromatic number, ladder graphs.

AMS Subject Classification: 05C15, 05C78.

1. Introduction

All the graphs $G = (V, E)$ discussed in this paper are connected, simple and finite. Graph labeling introduced by Alexander Rosa in 1967 [10], is an assignment of integers to the vertices, edges (or both) of a graph G subject to certain conditions. Graph labeling and its types are extensively studied in the literature [4]. Among the various labelings, β-labeling is one of the prominent labeling. It is also referred as graceful labeling by Golomb [5], which was initiated to solve the famous Ringel conjecture [10]. Graceful labeling has an extensive range of applications in network addressing, coding theory, communication networks, X-ray crystallography, dental arch, etc.

Let $G = (V, E)$ be a graph with m edges. An injective function $f : V(G) \to \{0, 1, 2, ...m\}$ is a graceful labeling if it induces a bijective function $f^* : E(G) \to \{1, 2, ..., m\}$ with the property that for every edge $xy \in E(G)$, $f^*(xy) = |f(x) - f(y)|$. If there exists a graceful labeling for a graph G, then G is a graceful graph.

A proper coloring of a graph G is an assignment of colors to the vertices or edges of the graph such that every pair of adjacent vertices or edges receive distinct colors respectively. Chromatic number ($\chi(G)$) is the least number of colors required for proper coloring the vertices of the graph G, whereas the chromatic index ($\chi^{'}(G)$) is the least number of colors

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needed for proper coloring the edges of the graph. In [6], the existence of graceful graphs with arbitrarily large chromatic number was proved.

As an extension of graceful labeling, the concept of graceful chromatic number was introduced by Gary Chartrand in 2015 [2]. A graceful k-coloring of a non-empty graph $G = (V, E)$ is a proper vertex coloring $f : V(G) \to \{1, 2, ..., k\}$, $k \geq 2$, which induces a proper edge coloring $f^* : E(G) \to \{1, 2, ..., k-1\}$ defined by $f^*(uv) = |f(u) - f(v)|$, where $u, v \in V(G)$. The minimum k for which G has a graceful k-coloring is called graceful chromatic number, $\chi_g(G)$.

In the introductory paper [2] on graceful coloring, the graceful chromatic number for some well known graphs were computed.

Theorem 1.1. [2] *For a cycle C_n, $n \geq 4$,*

$$\chi_g(C_n) = \begin{cases} 4, & \text{if } n \neq 5 \\ 5, & \text{if } n = 5 \end{cases}$$

Theorem 1.2. [2] *For a path P_n, $n \geq 5$, $\chi_g(P_n) = 5$.*

Theorem 1.3. [2] *For a wheel graph W_n, $n \geq 6$, $\chi_g(W_n) = n$.*

Theorem 1.4. [2] *If T is a tree with maximum degree Δ, then $\chi_g(T) \leq \lceil \frac{5\Delta}{3} \rceil$.*

Theorem 1.5. [2] *If G is a complete bipartite graph of order $n \geq 3$, then $\chi_g(G) = n$.*

Theorem 1.6. [2] *If G is a r-regular graph, then $\chi_g(G) \geq r + 2$, where $r \geq 2$.*

Theorem 1.7. [2] *For a nontrivial connected graph G, $\chi_g(G) \geq \Delta + 1$.*

Theorem 1.8. [2] *For a subgraph G' of G, $\chi_g(G') \leq \chi_g(G)$.*

Theorem 1.9. [2] *Let $f : V(G) \to \{1, 2, ..., k\}$, $k \geq 2$ be a coloring of a nontrivial connected graph G. Then f is a graceful coloring of G if and only if*

(i) *for each vertex v of G, the vertices in the closed neighborhood $N[v]$ of v are assigned distinct colors by f and*
(ii) *for each path (x, y, z) of order 3 in G, $f(y) \neq \frac{f(x)+f(z)}{f(y)}$.*

Let $T_{\Delta,h}$ denote the rooted tree (root v) with every vertex at a distance less than the height h from v having degree Δ and the remaining vertices are at a distance h from v as leaves [3].

Theorem 1.10. [3] *For each integer $\Delta \geq 2$, $\chi_g(T_{\Delta,2}) = \lceil \frac{1}{2}(3\Delta + 1) \rceil$.*

Theorem 1.11. [3] *For each integer $\Delta \geq 2$, $\chi_g(T_{\Delta,3}) = \lceil \frac{1}{8}(13\Delta + 1) \rceil$.*

Theorem 1.12. [3] *For each integer $\Delta \geq 2$, $\chi_g(T_{\Delta,4}) = \lceil \frac{1}{32}(53\Delta + 1) \rceil$.*

Theorem 1.13. [3] *For $\Delta \geq 2$, $h \geq 2 + \lfloor \frac{1}{3}\Delta \rfloor$, $\chi_g(T_{\Delta,h}) = \lceil \frac{5}{3}\Delta \rceil$.*

The graceful chromatic number of caterpillars were investigated along with a characterization in [13]. The graceful chromatic number for some subclasses of the following graphs have been established in the literature: unicyclic graphs[1]; graphs with diameter at least 2 [7]; regular and irregular graphs [8].

2. Preliminaries

Denote $[a, b]$ as $\{a, a+1, ..., b\}$ and $[a]$ as $[1, a]$, where $a, b \in \mathbb{Z}^+$ such that $a < b$. A closed ladder L_n, $n \geq 2$ is a graph obtained from two paths P_n with $V(L_n) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(L_n) = \{x_i x_{i+1}, y_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\}$. An open ladder OL_n, $n \geq 2$ is a graph formed by removing the edges x_1y_1 and x_ny_n from the closed ladder L_n. A slanting ladder SL_n, $n \geq 2$ is a graph obtained from two paths P_n with $V(SL_n) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(SL_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1} : 1 \leq i \leq n-1\}$. A triangular ladder TL_n, $n \geq 2$ is a graph obtained from two paths P_n with $V(TL_n) = \{x_i, y_i : 1 \leq i \leq n\}$ and $E(TL_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i y_i : 1 \leq i \leq n\}$. An open triangular ladder $O(TL_n)$, $n \geq 2$ is a graph obtained by removing the edges x_1y_1 and x_ny_n from the triangular ladder TL_n. A diagonal ladder DL_n, $n \geq 2$ is a graph obtained by adding the edges $x_{i+1}y_i$, $1 \leq i \leq n-1$ in TL_n. An open diagonal ladder $O(DL_n)$, $n \geq 2$ is a graph formed by removing the edges x_1y_1 and x_ny_n from the diagonal ladder DL_n. A circular ladder graph CL_n, $n \geq 2$ is a graph obtained by adding the edges x_1x_n and y_1y_n in the closed ladder L_n. These variants of ladder graphs [12] are illustrated in Figure 1.

The cartesian product $G \square G'$ of two simple connected graphs G and G' is a graph with vertices $V(G \square G') = V(G) \times V(G')$ and two vertices (a, a') and (b, b') in $G \square G'$ are adjacent if the distance between a and b is 0; and a' and b' is 1 or the distance between a and b is 1; and a' and b' is 0. The strong product $G \boxtimes G'$ of two connected simple graphs G and G' is a graph with vertices $V(G \boxtimes G') = V(G) \times V(G')$ and two vertices (a, a') and (b, b') in $G \boxtimes G'$ are adjacent if the distance between a and b is 0; and a' and b' is 1 or the distance between a and b is 1; and a' and b' is 0 or the distance between both a and b; and a' and b' is 1 [11]. Note that, the cartesian product of P_n with P_2; and C_n with P_2 is equivalent to L_n and CL_n respectively. Also, the strong product of P_n with P_2 results in DL_n.

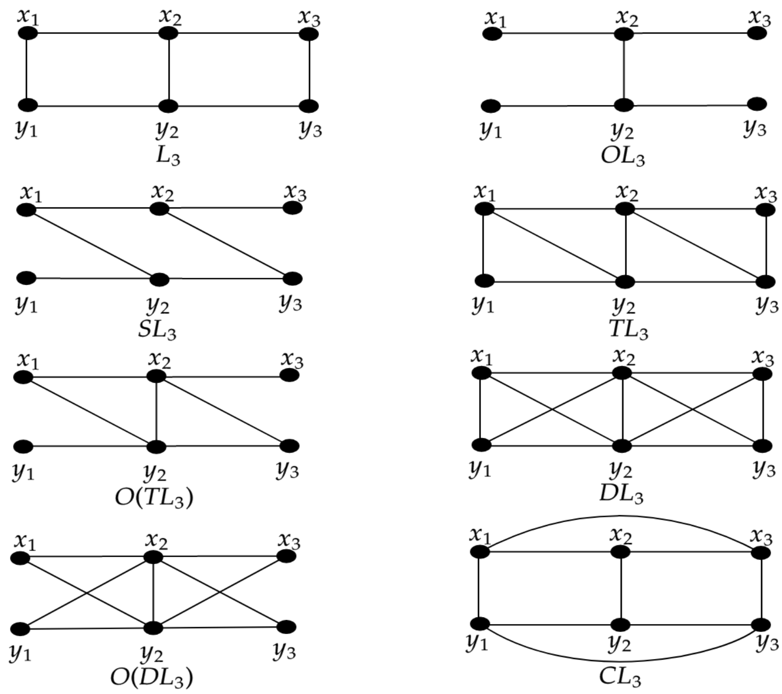

Figure 1

3. Main Results

Observation 3.1. If $[\Delta+i]$, $i \in \mathbb{Z}^+$ colors are applied in graceful coloring, then the vertex of maximum degree will receive the first and last i colors from $[\Delta + i]$.

Proof. Let w be a vertex of maximum degree and let $X = \{$ first i colors and last i colors $\}$. We prove $f(w) \in X$. Suppose on the contrary, let $f(w) = a$, where $a \notin X$. Then the Δ neighbours of w should be distinctly colored from $\{1, 2, ..., a-1, a+1, ..., \Delta+i\}$. Hence there exist at least two neighbours p and q of w such that $f(p) = a+u$ and $f(q) = a-u$, $u \in [1, \Delta-1]$, a contradiction to the proper edge coloring ($f^*(wp) = f^*(wq)$). Hence $f(w) \in X$. □

Theorem 3.2. $\chi_g(L_n) = \begin{cases} 4, & n = 2 \\ 5, & n \geq 3 \end{cases}$

Proof. Let $V(L_n) = \{x_i, y_i, 1 \leq i \leq n\}$ and $E(L_n) = \{x_ix_{i+1}, y_iy_{i+1}, 1 \leq i \leq n-1\} \cup \{x_iy_i, 1 \leq i \leq n\}$. Let $x_ix_{i+1} = e_i^{'}$, $y_iy_{i+1} = e_i^*$, $1 \leq i \leq n-1$ and $x_iy_i = e_i$, $1 \leq i \leq n$.
Case 1 ($n = 2$): Note that $L_2 = C_4$ and hence $\chi_g(L_2) = 4$, by Theorem 1.1.
Case 2 ($n = 3$): Since L_2 is a subgraph of L_3, $\chi_g(L_3) \geq \chi_g(L_2) = 4$, by the Theorem 1.8. We now show that $\chi_g(L_3) \neq 4$. Suppose that there exist a graceful 4-coloring of L_3. It is clear from the Observation 3.1, the vertices of maximum degree are colored using the colors 1 and 4. Without loss of generality, let $f(x_2) = 1$ and $f(y_2) = 4$. Then $f(x_1) = 3$ and $f(y_1) = 2$. Now, the vertices x_3 and y_3 can be colored using the colors which are at distance at least 3 from them. Thus, $f(x_3) = 2$, and hence $f(y_3) = 3$ which is a contradiction to the proper edge coloring ($f^*(x_3y_3) = 1 = f^*(y_2y_3)$). Hence $\chi_g(L_3) \geq 5$. In addition, we prove $\chi_g(L_3) \leq 5$.

Define a proper vertex coloring $f : V(L_3) \to [1, 5]$ as $f(v) = \begin{cases} 1, & \text{if } v = x_2 \\ 2, & \text{if } v = x_3, y_1 \\ 3, & \text{if } v = x_1 \\ 4, & \text{if } v = y_3 \\ 5, & \text{if } v = y_2 \end{cases}$

which induces a proper edge coloring $f^* : E(L_3) \to [1, 4]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = e_1, e_2^{'}, e_2^* \\ 2, & \text{if } e = e_1^{'}, e_3 \\ 3, & \text{if } e = e_1^* \\ 4, & \text{if } e = e_2 \end{cases}$$

Consequently, $\chi_g(L_3) = 5$.
Case 3 ($n > 3$): From the Theorem 1.8, $\chi_g(L_n) \geq \chi_g(L_3) = 5$, for $n > 3$. We show that $\chi_g(L_n) \leq 5$ by describing a proper vertex coloring $f : V(L_n) \to [1, 5]$ as

$$f(v) = \begin{cases} 1, & \text{if } v = x_i : i \equiv 2 (\text{mod } 4), y_j : j \equiv 0 (\text{mod } 4) : 1 \leq i, j \leq n \\ 2, & \text{if } v = x_i : i \equiv 3 (\text{mod } 4), y_j : j \equiv 1 (\text{mod } 4) : 1 \leq i, j \leq n \\ 3, & \text{if } v = x_1 \\ 4, & \text{if } v = x_i : i \equiv 0 (\text{mod } 4), y_j : j \equiv 2 (\text{mod } 4) : 1 \leq i, j \leq n \\ 5, & \text{if } v = x_i : i \equiv 1 (\text{mod } 4), y_j : j \equiv 3 (\text{mod } 4) : 1 \leq i, j \leq n \text{ and } i \neq 1 \end{cases}$$

which induces $f^* : E(L_n) \to [1, 4]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e_1\}, \{e_i^{'}\}, \{e_j^*\} : i, j \equiv 0 (\text{mod } 2) : 1 \leq i, j \leq n-1 \\ 2, & \text{if } e = \{e_1^{'}, e_i^{'}\} : i \equiv 3 (\text{mod } 4), \{e_j^*\} : j \equiv 1 (\text{mod } 4) : 1 \leq i, j \leq n-1 \\ 3, & \text{if } e = \{e_k\} : k \equiv 1 (\text{mod } 1) : 1 \leq k \leq n \\ 4, & \text{if } e = \{e_i^{'}\} : i \equiv 1 (\text{mod } 4), \{e_j^*\} : j \equiv 3 (\text{mod } 4) : 1 \leq i, j \leq n-1 \text{ and } i \neq 1 \end{cases}$$

Hence $\chi_g(L_n) = 5$, for $n > 3$. □

Corollary 3.1. $\chi_g(OL_n) = 5, n > 3$

Theorem 3.3. $\chi_g(SL_n) = 5, n \geq 4$.

Proof. Let SL_n be the slanting ladder with the vertex set $V(SL_n) = \{x_i, y_i, 1 \leq i \leq n\}$ and the edge set $E(SL_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1}, 1 \leq i \leq n-1\}$. Let $x_i x_{i+1} = e_i^{'}$, $y_i y_{i+1} = e_i^*$, $x_i y_{i+1} = e_i$. Clearly L_n is a subgraph of SL_n, $\chi_g(SL_n) \geq \chi_g(L_n) = 5$, by the Theorem 1.8. Define $f : V(SL_n) \to [1, 5]$ as

$$f(v) = \begin{cases} 1, & \text{if } v = x_i, y_j : i \equiv 1 (\text{mod } 4), j \equiv 0 (\text{mod } 4) : 1 \leq i, j \leq n \\ 2, & \text{if } v = x_i, y_j : i \equiv 3 (\text{mod } 4), j \equiv 2 (\text{mod } 4) : 1 \leq i, j \leq n \\ 4, & \text{if } v = x_i, y_j : i \equiv 2 (\text{mod } 4), j \equiv 1 (\text{mod } 4) : 1 \leq i, j \leq n \\ 5, & \text{if } v = x_i, y_j : i \equiv 0 (\text{mod } 4), j \equiv 3 (\text{mod } 4) : 1 \leq i, j \leq n \end{cases}$$

which induces $f^* : E(SL_n) \to [1, 4]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = e_i : 1 \leq i \leq n-1 \\ 2, & \text{if } e = e_j^{'}, e_k^* : j \equiv 2 (\text{mod } 4), k \equiv 1 (\text{mod } 4) : 1 \leq j, k \leq n-1 \\ 3, & \text{if } e = e_j^{'}, e_k^* : j \equiv 1 (\text{mod } 2), k \equiv 0 (\text{mod } 2) : 1 \leq j, k \leq n-1 \\ 4, & \text{if } e = e_j^{'}, e_k^* : j \equiv 0 (\text{mod } 4), k \equiv 3 (\text{mod } 4) : 1 \leq j, k \leq n-1 \end{cases}$$

Therefore, $\chi_g(SL_n) \leq 5$ implies $\chi_g(SL_n) = 5$, for $n \geq 4$.

□

Theorem 3.4. $\chi_g(TL_n) = \begin{cases} 6, & n = 3, 4 \\ 7, & n \geq 5 \end{cases}$

Proof. Let $V(TL_n) = \{x_i, y_i, 1 \leq i \leq n\}$ and $E(TL_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_{i+1}, 1 \leq i \leq n-1\} \cup \{x_i y_i, 1 \leq i \leq n\}$. Let $x_i x_{i+1} = e_i^{'}$, $y_i y_{i+1} = e_i^*$, $x_i y_{i+1} = a_i$, $1 \leq i \leq n-1$ and $x_i y_i = e_i$, $1 \leq i \leq n$.

Case 1 ($n = 3, 4$): Since the maximum degree of TL_n is 4, we get $\chi_g(TL_n) \geq 5$, by the Theorem 1.7. We claim that, $\chi_g(TL_n) \neq 5$. Suppose on the contrary, $\chi_g(TL_n) = 5$. It is clear that $f(w) \notin \{2, 3, 4\}$, where w is a vertex of maximum degree, by the Observation 3.1. For $n = 3$, without loss of generality, let $f(x_2) = 1$ and $f(y_2) = 5$. Obviously, $f(x_1) \notin \{1, 3, 5\}$ and hence $f(x_1) \in \{2, 4\}$. Without loss of generality, assume $f(x_1) = 2$, then the only choice of color for the vertex y_1 is 4. Now $f(y_3) \notin [1, 5]$ (by the Theorem 1.9), which is a contradiction to the assumption that $\chi_g(TL_n) = 5$. Same argument holds when $f(x_1) = 4$. For $n = 4$, an induced subgraph of maximum degree vertices of TL_n form a cycle of length 4 which can be gracefully colored with four distinct colors, by the Theorem 1.1. But we have only two colors $\{1, 5\}$, which is a contradiction to the assumption that $\chi_g(TL_n) = 5$. Hence, at least 6 colors are needed for graceful coloring of TL_n, for $n = 3, 4$.

Thus $\chi_g(TL_n) \geq 6$. Define $f : V(TL_n) \rightarrow [1,6]$ as

$$f(v) = \begin{cases} 1, & \text{if } v = y_2 \\ 2, & \text{if } v = x_2 \\ 3, & \text{if } v = y_1, y_4 \\ 4, & \text{if } v = x_1, x_4 \\ 5, & \text{if } v = y_3 \\ 6, & \text{if } v = x_3 \end{cases}$$

which induces $f^* : E(TL_n) \rightarrow [1,4]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = e_1, e_2, e_3, e_4 \\ 2, & \text{if } e = e_1^{'}, e_3^{'}, e_1^*, e_3^* \\ 3, & \text{if } e = a_1, a_2, a_3 \\ 4, & \text{if } e = e_2^{'}, e_2^* \end{cases}$$

Therefore, $\chi_g(TL_n) \leq 6$, implies $\chi_g(TL_n) = 6$, for $n = 3, 4$.

Case 2 ($n \geq 5$): Since TL_4 is a subgraph of TL_n, $\chi_g(TL_n) \geq \chi_g(TL_4) = 6$ (by the Theorem 1.8). We show $\chi_g(TL_n) \neq 6$. Assume the contrary that, $\chi_g(TL_n) = 6$. It is clear from the Observation 3.1, $f(w) \notin \{3, 4\}$, w is a vertex of maximum degree. Let H be an induced subgraph of maximum degree vertices in TL_n. Note that L_n, $n \geq 3$ is also a subgraph of H which cannot be gracefully colored using four colors $\{1, 2, 5, 6\}$, by the Theorem 3.2. Hence at least 7 colors are needed for graceful coloring of TL_n. Thus $\chi_g(TL_n) \geq 7$. We now define a graceful 7-coloring f of TL_n. Define $f : V(TL_n) \rightarrow [1,7]$ as

$$f(v) = \begin{cases} 1, & \text{if } v = y_j : j \equiv 0 (\text{mod } 3) : 1 \leq j \leq n \\ 2, & \text{if } v = x_i : i \equiv 1 (\text{mod } 3) : 1 \leq i \leq n \\ 3, & \text{if } v = x_i : i \equiv 2 (\text{mod } 3) : 1 \leq i \leq n \\ 4, & \text{if } v = y_1 \\ 5, & \text{if } v = y_j : j \equiv 1 (\text{mod } 3) : 1 \leq j \leq n \text{ and } j \neq 1 \\ 6, & \text{if } v = x_i : i \equiv 0 (\text{mod } 3) : 1 \leq i \leq n \\ 7, & \text{if } v = y_j : j \equiv 2 (\text{mod } 3) : 1 \leq j \leq n \end{cases}$$

which induces $f^* : E(TL_n) \rightarrow [1,6]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e_i^{'}\} : i \equiv 1 (\text{mod } 3), \{a_l\} : l \equiv 0 (\text{mod } 3) : 1 \leq i, l \leq n-1 \\ 2, & \text{if } e = \{e_1\}, \{e_j^*\}, : j \equiv 1 (\text{mod } 3), \{a_l\} : l \equiv 2 (\text{mod } 3) : 1 \leq l, j \leq n-1 \\ & \quad \text{and } j \neq 1 \\ 3, & \text{if } e = \{e_1^*\}, \{e_i^{'}\} : i \equiv 2 (\text{mod } 3), \{e_k\} : k \equiv 1 (\text{mod } 3) : 1 \leq i \leq n-1, \\ & 4 \leq k \leq n \\ 4, & \text{if } e = \{e_i^{'}\} : i \equiv 0 (\text{mod } 3), \{e_j^*\} : j \equiv 0 (\text{mod } 3), \{e_k\} : k \equiv 2 (\text{mod } 3) : \\ & 1 \leq i, j \leq n-1, 1 \leq k \leq n \\ 5, & \text{if } e = \{e_k\} : k \equiv 0 (\text{mod } 3), \{a_l\} : l \equiv 1 (\text{mod } 3) : 1 \leq k \leq n, 1 \leq l \leq n-1 \\ 6, & \text{if } e = \{e_j^*\} : j \equiv 2 (\text{mod } 3) : 1 \leq j \leq n-1 \end{cases}$$

Therefore, $\chi_g(TL_n) \leq 7$, implies $\chi_g(TL_n) = 7$, for $n \geq 5$. □

Corollary 3.2. $\chi_g(O(TL_n)) = 7, n \geq 5$

Theorem 3.5. $\chi_g(DL_n) = \begin{cases} 8, & n = 5, 6 \\ 9, & n \geq 7 \end{cases}$

Proof. Consider the diagonal ladder DL_n with the vertex and the edge set as follows: $V(DL_n) = \{x_i, y_i, 1 \leq i \leq n\}$, $E(DL_n) = \{x_ix_{i+1}, y_iy_{i+1}, x_iy_{i+1}, y_ix_{i+1}, 1 \leq i \leq n-1\} \cup \{x_iy_i, 1 \leq i \leq n\}$. Let $x_ix_{i+1} = e_i^{'}$, $y_iy_{i+1} = e_i^*$, $x_iy_{i+1} = a_i^{'}$, $y_ix_{i+1} = a_i^*$, $1 \leq i \leq n-1$ and $x_iy_i = e_i$, $1 \leq i \leq n$

Case 1 ($n = 5, 6$): TL_n is a subgraph of DL_n, $\chi_g(DL_n) \geq \chi_g(TL_n) = 7$ (by the Theorem 1.8). We now show that $\chi_g(DL_n) \neq 7$. Suppose on the contrary, $\chi_g(DL_n) = 7$. Observe that $f(w) \notin \{3, 4, 5\}$, where w is a vertex of maximum degree (by the Observation 3.1). Let H be an induced subgraph of maximum degree vertices in DL_n. Also TL_n $(n = 3, 4)$ is a subgraph of H which cannot be gracefully colored with 4 colors $\{1, 2, 6, 7\}$ (by the Theorem 3.4), which implies that our assumption $\chi_g(DL_n) = 7$, for $n = 5, 6$ is wrong. Hence at least 8 colors are needed for graceful coloring of DL_n. Therefore, $\chi_g(DL_n) \geq 8$. Define $f : V(DL_n) \to [1, 8]$ as

$$f(v) = \begin{cases} 1, & \text{if } v = x_4 \\ 2, & \text{if } v = y_2, y_5 \\ 3, & \text{if } v = x_3 \\ 4, & \text{if } v = y_1, y_6 \\ 5, & \text{if } v = x_1, x_6 \\ 6, & \text{if } v = y_4 \\ 7, & \text{if } v = x_2, x_5 \\ 8, & \text{if } v = y_3 \end{cases}$$

which induces $f^* : E(DL_n) \to [1, 7]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e_1, e_6\}, \{a_2^{'}, a_4^{'}\}, \{a_2^*, a_4^*\} \\ 2, & \text{if } e = \{e_1^{'}, e_3^{'}, e_5^{'}\}, \{e_1^*, e_3^*, e_5^*\} \\ 3, & \text{if } e = \{a_1^{'}, a_3^{'}, a_5^{'}\}, \{a_1^*, a_5^*\} \\ 4, & \text{if } e = \{e_2^{'}\}, \{e_4^*\} \\ 5, & \text{if } e = \{e_2, e_3, e_4, e_5\} \\ 6, & \text{if } e = \{e_4^{'}\}, \{e_2^*\} \\ 7, & \text{if } e = \{a_3^*\} \end{cases}$$

Thus, $\chi_g(DL_n) \leq 8$. Consequently, $\chi_g(DL_n) = 8$, for $n = 5, 6$.

Case 2 ($n \geq 7$): Obviously, $\chi_g(DL_n) \geq \chi_g(DL_6) = 8$. We show that, graceful coloring of DL_n need at least 9 colors. Assume the contrary that, $\chi_g(DL_n) = 8$. It can be seen that $f(w) \notin \{4, 5\}$, where w is a vertex of maximum degree (by the Observation 3.1). Let H be an induced subgraph of maximum degree vertices in DL_n. Indeed, TL_n, $n \geq 5$ is a subgraph of H. By the Theorem 3.4, the colors $[1, 3] \cup [6, 8]$ are inadequate for graceful coloring of DL_n. Hence, at least 9 colors are required for graceful coloring of DL_n. Thus,

$\chi_g(DL_n) \geq 9$. It remains to show $\chi_g(DL_n) \leq 9$ by describing $f : V(DL_n) \to [1, 9]$ as

$$f(v) = \begin{cases} 1, & \text{if } v = x_i : i \equiv 2(\text{mod } 4) : 1 \leq i \leq n \\ 2, & \text{if } v = y_j : j \equiv 2(\text{mod } 4) : 1 \leq j \leq n \\ 3, & \text{if } v = y_j : j \equiv 0(\text{mod } 4) : 1 \leq j \leq n \\ 4, & \text{if } v = x_i : i \equiv 0(\text{mod } 4) : 1 \leq i \leq n \\ 5, & \text{if } v = y_1 \\ 6, & \text{if } v = y_j : j \equiv 1(\text{mod } 4) : 5 \leq j \leq n \\ 7, & \text{if } v = x_i : i \equiv 1(\text{mod } 4) : 1 \leq i \leq n \\ 8, & \text{if } v = x_i : i \equiv 3(\text{mod } 4) : 1 \leq i \leq n \\ 9, & \text{if } v = y_j : j \equiv 3(\text{mod } 4) : 1 \leq j \leq n \end{cases}$$

which induces $f^* : E(DL_n) \to [1, 8]$ as

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e_k\} : k \equiv 1(\text{mod } 1) : 1 \leq k \leq n \\ 2, & \text{if } e = \{e_1\}, \{a_l^{'}\} : l \equiv 0(\text{mod } 4) : 1 \leq l \leq n-1 \\ 3, & \text{if } e = \{e_i^{'}\} : i \equiv 0(\text{mod } 4), \{e_1^*, e_j^*\} : j \equiv 0(\text{mod } 4) 1 \leq i, j \leq n-1 \\ 4, & \text{if } e = \{e_i^{'}\} : i \equiv 3(\text{mod } 4), \{e_j^*\} : j \equiv 1(\text{mod } 4), \{a_1^*, a_m^*\} : m \equiv 0(\text{mod } 4) : \\ & 1 \leq i, j, m \leq n-1 \text{ and } j \neq \{1\} \\ 5, & \text{if } e = \{a_l^{'}\} : l \equiv 1(\text{mod } 2), \{a_m^*\} : m \equiv 1(\text{mod } 2) : m \neq 1, 1 \leq l, m \leq n-1 \\ 6, & \text{if } e = \{e_i^{'}\} : i \equiv 1(\text{mod } 4), \{e_j^*\} : j \equiv 3(\text{mod } 4), \{a_2^*, a_m^*\} : m \equiv 2(\text{mod } 4) : \\ & 1 \leq i, j, m \leq n-1 \text{ and } m \neq 2 \\ 7, & \text{if } e = \{e_i^{'}\} : i \equiv 2(\text{mod } 4), \{e_j^*\} : j \equiv 2(\text{mod } 4) : 1 \leq i, j \leq n-1 \\ 8, & \text{if } e = \{a_l^{'}\} : l \equiv 2(\text{mod } 4) : 1 \leq l \leq n-1 \end{cases}$$

Hence, $\chi_g(DL_n) = 9$, for $n \geq 7$. □

Corollary 3.3. $\chi_g(O(DL_n)) = 9, n \geq 7$.

Theorem 3.6. For $n \geq 4$, $\chi_g(CL_n) = \begin{cases} 5, & n \equiv 0(\text{mod } 4) \\ 6, & \textit{otherwise} \end{cases}$

Proof. A circular ladder CL_n is formed by adding two edges x_1x_n and y_1y_n in the closed ladder L_n.

Case 1 ($n \equiv 0(\text{mod } 4)$): Since CL_n is a 3-regular graph, $\chi_g(CL_n) \geq 5$ (by the Theorem 1.6). We claim that $\chi_g(CL_n) \leq 5$ by defining $f : V(CL_n) \to [1, 5]$ as follows. For $n = 4b + 4$, where $b \in \{0, 1, 2, 3, ...\}$

$$f(v) = \begin{cases} 1, & \text{if } v = \{x_i\} : i \equiv 1(\text{mod } 4), \{y_j\} : j \equiv 3(\text{mod } 4) : 1 \leq i, j \leq n \\ 2, & \text{if } v = \{x_i\} : i \equiv 2(\text{mod } 4), \{y_j\} : j \equiv 0(\text{mod } 4) : 1 \leq i, j \leq n \\ 4, & \text{if } v = \{x_i\} : i \equiv 0(\text{mod } 4), \{y_j\} : j \equiv 2(\text{mod } 4) : 1 \leq i, j \leq n \\ 5, & \text{if } v = \{x_i\} : i \equiv 3(\text{mod } 4), \{y_j\} : j \equiv 1(\text{mod } 4) : 1 \leq i, j \leq n \end{cases}$$

which induces $f^* : E(CL_n) \to [1, 4]$

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e_i^{'}, e_j^*\} : i, j \equiv 1(\text{mod } 2) : 1 \leq i, j \leq n \\ 2, & \text{if } e = \{e_k\} : k \equiv 0(\text{mod } 2) : 1 \leq k \leq n \\ 3, & \text{if } e = \{e_i^{'}, e_j^*\} : i, j \equiv 0(\text{mod } 2) : 1 \leq i, j \leq n \\ 4, & \text{if } e = \{e_k\} : k \equiv 1(\text{mod } 2) : 1 \leq k \leq n \end{cases}$$

Case 2 ($n \not\equiv 0(\text{mod } 4)$): Obviously, $\chi_g(CL_n) \geq \chi_g(L_n) = 5$, for $n \geq 3$ (by the Theorem 1.8). It is also clear that, the vertices of CL_n are colored using $\{1, 2, 4, 5\}$ (by the Observation 3.1). We claim that, $\chi_g(L_n) \neq 5$. Consider a proper vertex coloring f of L_n as $(1, 2, 5, 4, 1, 2, 5, 4, ...)$ for the vertices in the upper path and $(5, 4, 1, 2, 5, 4, 1, 2, ...)$ for the vertices in the lower path; which induces the edge coloring $(1, 3, 1, 3, 1, 3, 1, 3, ...)$ and $(1, 3, 1, 3, 1, 3, 1, 3, ...)$ respectively. Note that, the vertex x_n will not receive the color 4 in CL_n ($n \not\equiv 0(\text{mod } 4)$).
If $f(x_n) = 1$, then $f^*(x_1x_n) = 0$
If $f(x_n) = 2$, then $f^*(x_1x_n) = 1 = f^*(x_1x_2)$
If $f(x_n) = 5$, then $f^*(x_1x_n) = 4 = f^*(x_1y_1)$
Note that all the above cases leads to a contradiction to the proper edge coloring. Thus $\chi_g(CL_n) \geq 6$. In addition, we prove that $\chi_g(CL_n) \leq 6$ by defining $f : V(CL_n) \to [1, 6]$ as follows. For $n = 4b + 5$, where $b \in \{0, 1, 2, 3, ...\}$,

$$f(v) = \begin{cases} 1, & \text{if } v = \{x_i\} : i \equiv 1(\text{mod } 4) \text{ and } i \neq n, \{y_j\} : j \equiv 3(\text{mod } 4) : 1 \leq i, j \leq n \\ 2, & \text{if } v = \{x_i\} : i \equiv 2(\text{mod } 4), \{y_n, y_j\} : j \equiv 0(\text{mod } 4) \text{ and } j \neq n-1 : \\ & 1 \leq i, j \leq n \\ 3, & \text{if } v = x_{n-1} \\ 4, & \text{if } v = \{x_n, x_i\} : i \equiv 0(\text{mod } 4) \text{ and } i \neq n-1, \{y_j\} : j \equiv 2(\text{mod } 4) : \\ & 1 \leq i, j \leq n \\ 5, & \text{if } v = \{x_i\} : i \equiv 3(\text{mod } 4), \{y_j\} : j \equiv 1(\text{mod } 4) \text{ and } j \neq n : \\ & 1 \leq i, j \leq n \\ 6, & \text{if } v = y_{n-1} \end{cases}$$

which induces $f^* : E(CL_n) \to [1, 5]$

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e'_{n-1}, e'_i\} : i \equiv 1(\text{mod } 2) \text{ and } i \neq n-2, \{e^*_j\} : j \equiv 1(\text{mod } 2) : \\ & 1 \leq i \leq n, 1 \leq j \leq n-4 \\ 2, & \text{if } e = \{e'_{n-2}, e_n\}, \{e_k\} : k \equiv 0(\text{mod } 2) \text{ and } k \neq n-1 : 1 \leq k \leq n-2 \\ 3, & \text{if } e = \{e'_n, e'_i\} : i \equiv 0(\text{mod } 2) \text{ and } i \neq n-1, \{e^*_n, e^*_j\} : j \equiv 0(\text{mod } 2) \\ & \text{ and } j \neq n-1, e_{n-1} : 1 \leq i, j \leq n \\ 4, & \text{if } e = e^*_{n-1}, \{e_k\} : k \equiv 1(\text{mod } 2) \text{ and } k \neq n : 1 \leq k \leq n-1 \\ 5, & \text{if } e = e^*_{n-2} \end{cases}$$

For $n = 4b + 6$, where $b \in \{0, 1, 2, 3, ...\}$,

$$f(v) = \begin{cases} 1, & \text{if } v = \{x_1, x_4, x_i\} : i \equiv 3(\text{mod } 4), \{y_6, y_9, y_j\} : j \equiv 1(\text{mod } 4) \text{ and } n \neq 6 : \\ & 11 \leq i \leq n, 13 \leq j \leq n \\ 2, & \text{if } v = \{x_2, x_7\}, x_5 : n = 6, y_{10} \\ 3, & \text{if } v = x_5 : n \neq 6, \{x_i\} : i \equiv 0(\text{mod } 4), y_6 : n = 6, \{y_3, y_j\} : j \equiv 2(\text{mod } 4) : \\ & 8 \leq i \leq n, 14 \leq j \leq n \\ 4, & \text{if } v = x_6 : n = 6, \{x_3, x_i\} : i \equiv 2(\text{mod } 4), y_5 : n \neq 6, \{y_j\} : j \equiv 0(\text{mod } 4) : \\ & 10 \leq i \leq n, 12 \leq j \leq b \\ 5, & \text{if } v = y_5 : n \neq 6, \{y_2, y_7\} \\ 6, & \text{if } v = \{x_6, x_i\} : i \equiv 1(\text{mod } 4), \{y_1, y_4, y_8, y_j\} : j \equiv 3(\text{mod } 4) : \\ & 13 \leq i \leq n, 11 \leq j \leq n \end{cases}$$

which induces $f^* : E(CL_n) \to [1,5]$

$$f^*(e) = \begin{cases} 1, & \text{if } e = e_4^{'} : n = 6, \{e_1^{'}, e_7^{'}, e_9^{'}\}, e_4^* : n = 6, \{e_1^*, , e_7^*, e_9^*\}, e_6 : n = 6, \\ & \{e_3, e_5, e_k\} : k \equiv 0 (\text{mod } 2) : 12 \leq k \leq n \\ 2, & \text{if } e = e_5^{'} : n = 6, \{e_2^{'}, e_4^{'}, e_8^{'}, e_i^{'}\} : i \equiv 1 (\text{mod } 2), e_5^* : n = 6, \{e_2^*, e_4^*, e_j^*\} : \\ & j \equiv 1 (\text{mod } 2), e_{10} : 11 \leq i, j \leq n \\ 3, & \text{if } e = e_6^{'} : n = 6, \{e_3^{'}, e_5^{'}, e_i^{'}\} : i \equiv 0 (\text{mod } 2), e_6^* : n = 6, \{e_3^*, e_5^*, e_j^*\} : \\ & j \equiv 0 (\text{mod } 2), e_5 : n = 6, \{e_2, e_7, e_8\} : 10 \leq i \leq n, 12 \leq j \leq n \\ 4, & \text{if } e = e_6^{'} : n \neq 6, e_6^* : n \neq 6, e_{10}^*, e_9 : n \neq 6 \\ 5, & \text{if } e = e_8^*, e_6 : n \neq 6, \{e_1, e_4, e_k\} : k \equiv 1 (\text{mod } 2), 11 \leq k \leq n \end{cases}$$

For $n = 4b + 7$, where $b \in \{0, 1, 2, 3, ...\}$,

$$f(v) = \begin{cases} 1, & \text{if } v = \{x_i\} : i \equiv 1 (\text{mod } 4), \{y_j\} : j \equiv 3 (\text{mod } 4) : j \neq n : 1 \leq i, j \leq n \\ 2, & \text{if } v = \{x_i\} : i \equiv 2 (\text{mod } 4) \text{ and } i \neq n-1, \{y_n, y_j\} : j \equiv 0 (\text{mod } 4) : \\ & 1 \leq i, j \leq n \\ 3, & \text{if } v = x_{n-1} \\ 4, & \text{if } v = \{x_n, x_i\} : i \equiv 0 (\text{mod } 4), \{y_j\} : j \equiv 2 (\text{mod } 4) \text{ and } j \neq n-1 : \\ & 1 \leq i, j \leq n \\ 5, & \text{if } v = \{x_i\} : i \equiv 3 (\text{mod } 4) \text{ and } i \neq n, \{y_j\} : j \equiv 1 (\text{mod } 4) : \\ & 1 \leq i, j \leq n \\ 6, & \text{if } v = y_{n-1} \end{cases}$$

which induces $f^* : E(CL_n) \to [1,5]$

$$f^*(e) = \begin{cases} 1, & \text{if } e = \{e_{n-1}^{'}, e_i^{'}\} : i \equiv 1 (\text{mod } 2) \text{ and } i \neq n-2, \{e_j^*\} : j \equiv 1 (\text{mod } 2), \\ & 1 \leq i, j \leq n \\ 2, & \text{if } e = e_{n-2}^{'}, e_n, \{e_k\} : k \equiv 0 (\text{mod } 2) \text{ and } k \neq n-1, 1 \leq k \leq n \\ 3, & \text{if } e = \{e_n^{'}, e_i^{'}\} : i \equiv 0 (\text{mod } 2) \text{ and } i \neq n-1, \{e_j^*\} : j \equiv 0 (\text{mod } 2) \\ & \text{and } j \neq n-1 : 1 \leq i, j \leq n \\ 4, & \text{if } e = e_{n-1}^*, \{e_k\} : k \equiv 1 (\text{mod } 2) \text{ and } k \neq n, 1 \leq k \leq n \end{cases}$$

Hence, $\chi_g(CL_n) = 6$, for $n \not\equiv 0 (\text{mod } 4)$. □

4. Conclusion

The graceful coloring of many graph classes like bipartite graphs, complete graphs, regular graphs, forbidden graphs, etc. are still unexplored as the study on this concept began only during 2018. We investigate the graceful coloring of a few variants of ladder graphs. It is also interesting to work on graph operations like strong product, tensor product, cartesian product, lexicographic product and corona product of some graph classes with respect to graceful coloring which are still open.

References

[1] Alfarisi, R., Dafik., Prihandini, R. M., Adawiyah, R., Albirri, E. R., Agustin, I. H., (2019), Graceful Chromatic Number of Unicyclic Graphs, J.Phys, Conf. Ser, 1306, 012039.

[2] Bi, Z., Byers, A., English, S., Laforge, E., Zhang, P., (2017), Graceful colorings of graphs, J. Combin. Math. Combin. Comput, 101, pp. 101-119.

[3] English, S., Zhang, P., (2017), On graceful colorings of trees, Mathematica Bohemica, 142(1), pp. 57-73.

[4] Gallian, J. A., (2009), A dynamic survey of graph labeling, Electronic J. of Combinatorics, 16, pp. 1-219.
[5] Golomb, S. W., (1972), How to number a graph, Graph theory and computing, Academic Press, pp. 23-37.
[6] Mahmoody, A., (2009), A note on graceful graphs with large chromatic numbers, Ars Combinatoria, 90, pp. 423-424.
[7] Mincu, R., Obreja, C., Popa, A., (2019), The graceful chromatic number for some particular classes of graphs, 21st International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, pp. 09-115.
[8] Obreja, C., (2020), Results on graceful chromatic number for particular graphs. 22nd International Symposium on Symbolic and Numeric Algorithms for Scientific Computing, pp. 109-116.
[9] Ringel, G., (1964), Theory of Graphs and its Applications, Proc. Symposium Smolenice, pp. 171-234.
[10] Rosa, A., (1967), On certain valuations of the vertices of a graph, Theory of Graphs, Proc. Internat. Symposium, pp. 349-355.
[11] Sabidussi, G., (1959), Graph multiplication, Math. Zeitschrift, 72, pp. 446–457.
[12] Sumathi, P., Rathi, A., (2018), Quotient labeling of Some Ladder Graphs, American J. of Engineering Research, 1(3), pp. 80-85.
[13] Zhang, P., (2016), A Kaleidoscopic View of Graph Colorings, Springer publication, ISSN: 2191-8198, pp. 35-51.

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