NEW RESULTS ON GENERALIZATION OF JORDAN CENTRALIZERS OVER MATRIX RINGS

ARINDAM GHOSH[1], OM PRAKASH[2*], SUSHMA SINGH[3], §

ABSTRACT. This paper presents a study on Jordan maps over matrix rings with some functional equations related to additive maps on these rings. We first show that every Jordan left (right) centralizer over a matrix ring is a left (right) centralizer. Moreover, every two-sided centralizer over the matrix ring is of a particular form. Further, we prove that any additive map satisfying functional equations over matrix rings becomes a two-sided centralizer. Finally, we conclude our work with some results on the Jordan left $\star$- centralizer over matrix rings and establish some results on functional equations that arise for the $\star$-centralizer.

Keywords: Prime and semiprime associative rings, Rings with involution, Jordan structure, Matrix rings, General theory of functional equations.

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1. INTRODUCTION

Throughout, R represents an associative ring and $Z(R)$ is its centre. Recall that a ring R is said to be *prime* if $xRy = 0$ for some $x, y \in R$ implies either $x = 0$ or $y = 0$ and *semiprime* if $xRx = 0$ for some $x \in R$ implies $x = 0$. The ring R is called *n-torsion free* if there exists $x \in R$ such that $nx = 0$ implies $x = 0$, where $n \geq 2$ is an integer. A map $T : R \to R$ is said to be a *left (right) centralizer* if $T(x + y) = T(x) + T(y)$ and $T(xy) = T(x)y$ $(T(xy) = xT(y))$, for all $x, y \in R$. It is well known that if R has an identity element $1 \neq 0$ and $T : R \to R$ is a left (right) centralizer, then $T(x) = T(1)x$ $(T(x) = xT(1))$, for all $x \in R$. The map T is *two-sided centralizer* if it is additive and $T(xy) = T(x)y = xT(y)$, for all $x, y \in R$. Also, the map $T : R \to R$ is said to be a *Jordan left (right) centralizer* if it is additive and $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$, for all $x \in R$.

[1] Government Polytechnic Kishanganj, Thakurganj, Kishanganj 855 116, India.
e-mail: arindam.rkmrc@gmail.com; ORCID: https://orcid.org/0000-0002-9833-3940.
[2] Department of Mathematics, Indian Institute of Technology Patna, Patna 801 106, India.
e-mail: om@iitp.ac.in; ORCID: https://orcid.org/0000-0002-6512-4229.
[3] Department of Mathematics, School of Applied Sciences, KIIT Bhubaneswar, 751 024, India.
e-mail: sushma.singhfma@kiit.ac.in; ORCID: https://orcid.org/0000-0002-2227-6468.
* Corresponding author.
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We can easily see that every left centralizer over a ring is a Jordan left centralizer, but the converse need not be valid (Example 2.1). In 1952, Wendel [17], introduced left centralizer over complex group algebra. After that, Johnson [10] formally introduced left (right) centralizer over an associative semi-group in 1964. Jordan left centralizer over some rings to be the left centralizer was seen in the past few years by researchers. Interestingly, every left centralizer over a ring is a Jordan left centralizer, but the converse need not be true. Later, in 1992, Brešar and Zalar [6] proved any Jordan left (right) centralizer over prime rings of characteristic not equal to two is a left (right) centralizer. Also, Zalar proved the same result for semiprime rings [18]. Motivated by the above result of Zalar, we show every Jordan left (right) centralizer over any matrix ring is a left (right) centralizer. The importance of the work is that many matrix rings are not semiprime rings. Further, [5, Theorem 2.3.2] motivates to prove that every two-sided centralizer over matrix ring is of a particular form. In 2008 and 2010, Vukman [15, 16], introduced (m,n)-Jordan derivation and (m,n)-Jordan centralizer, respectively. Few more works on (m,n)-Jordan derivation and (m,n)-Jordan centralizer are available in [8, 3, 9]. On the other hand, in 1999, Vukman [11] proved that any additive map T over 2-torsion-free semiprime ring R with the condition $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, becomes a two-sided centralizer. Under the same condition, we prove the result for the matrix ring over any 2-torsion free ring. In 2001, Vukman [12] proved that an additive map T on a 2-torsion free semiprime ring R with $T(xyx) = xT(y)x$, for all $x, y \in R$, becomes a two-sided centralizer. We prove the same result for matrix ring $M_r(R)$ ($r \geq 2$ is an integer) over an arbitrary ring R. Again, in 2003, Vukman and Ulbl [13] proved that an additive map T on a 2-torsion free semiprime ring R with $2T(xyx) = T(x)yx + xyT(x)$, for all $x, y \in R$, becomes a two-sided centralizer. We prove the result for matrix ring $M_r(R)$ over the ring R. In the same year, Vukman and Ulbl [14] proved that an additive map T on a 2-torsion free semiprime ring R with $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$, for all $x, y \in R$, becomes a two-sided centralizer. We prove the result for matrix ring $M_r(R)$ over 2-torsion free ring R. Note that the result is not true for 2-torsion rings.

An involution $\star$ over a ring R is an additive map satisfying $(xy)^\star = y^\star x^\star$ and $(x^\star)^\star = x$, for all $x, y \in R$. Also, an additive map $T : R \to R$ is a left (right) $\star$- centralizer if $T(xy) = T(x)y^\star$ ($T(xy) = x^\star T(y)$), for all $x, y \in R$. An additive map $T : R \to R$ is said to be a $\star$- centralizer if $T(xy) = T(x)y^\star = x^\star T(y)$, for all $x, y \in R$. An additive map $T : R \to R$ is said to be a Jordan left (right) $\star$- centralizer if $T(x^2) = T(x)x^\star$ ($T(x^2) = x^\star T(x)$), for all $x \in R$. An additive map $T : R \to R$ is said to be a reverse left (right) $\star$- centralizer if $T(xy) = T(y)x^\star$ ($T(xy) = y^\star T(x)$), for all $x, y \in R$. An additive map $T : R \to R$ is a reverse $\star$-centralizer if $T(xy) = T(y)x^\star = y^\star T(x)$, for all $x, y \in R$. In 2013, Ali et al. [4] proved that every Jordan left $\star$-centralizer on a semiprime ring with involution $\star$ and of characteristic different from two is a reverse left $\star$-centralizer. We prove some results based on the Jordan left $\star$-centralizer over matrix rings and some function equations arising from $\star$-centralizer.

2. Jordan centralizers over matrix rings

Let R be a ring with unity $1 \neq 0$, $M_r(R)$, $r \geq 2$ be the ring of $r \times r$ matrices over R and e_{ij} be the $r \times r$ matrix with 1 at (i, j)-th place and 0 elsewhere. We know that every left centralizer is a Jordan left centralizer, but the converse is invalid. Towards this, we have the following example.

Example 2.1. *Let $\mathbb{R}$ be the field of real numbers and $S = M_n(\mathbb{R})$. Then $F = S \times S \times S$ is a ring under componentwise addition and for any $(x_1, y_1, z_1), (x_2, y_2, z_2) \in F$, multiplication is defined by*

$$(x_1, y_1, z_1).(x_2, y_2, z_2) = (0, 0, x_1 y_2 - x_2 y_1).$$

Note that $X^2 = 0$ for all $X \in F$. Let $P = (1, 0, 0)$ and $Q = (0, 1, 0)$. Then $PQ = (0, 0, 1)$. Suppose $R' = \left\{ \begin{pmatrix} 0 & A & B \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix} \mid A, B \in F \right\}$ and define $T : R' \to R'$ by

$$T \begin{pmatrix} 0 & A & B \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It can be easily proved that T is a Jordan left centralizer. Now, consider

$$\tilde{A} = \begin{pmatrix} 0 & P & 0 \\ 0 & 0 & P \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \tilde{B} = \begin{pmatrix} 0 & Q & 0 \\ 0 & 0 & Q \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $T(\tilde{A}\tilde{B}) \neq T(\tilde{A})\tilde{B}$. Hence, T is not a left centralizer.

Proposition 2.1. *If R is a ring and $T : R \to R$ is a Jordan left centralizer, then $T(xy + yx) = T(x)y + T(y)x$ for all $x, y \in R$.*

Proof. Substituting $x + y$ for x in $T(x^2) = T(x)x$, we get the result. $\qquad\square$

We frequently use this proposition in the proof of Theorem 2.1.

Theorem 2.1. *Let R be a ring. Then every Jordan left (right) centralizer $T : M_r(R) \to M_r(R)$ is a left (right) centralizer.*

Proof. Let T be a Jordan left centralizer on $M_r(R)$ and for all $i, j \in \{1, 2, \ldots, r\}$,

$$T(e_{ij}) = \sum_{k=1}^{r} \sum_{l=1}^{r} a_{kl}^{(ij)} e_{kl} \ , \ \text{for } a_{kl}^{(ij)} \in R. \tag{1}$$

Since $e_{ii}^2 = e_{ii}$ and T is a Jordan left centralizer on $M_r(R)$, we have

$$T(e_{ii}) = \sum_{k=1}^{r} a_{ki}^{(ii)} e_{ki} \ , \ \text{for all } i \in \{1, 2, \ldots, r\}. \tag{2}$$

Also, $e_{ij} = e_{ii}e_{ij} + e_{ij}e_{ii}$, for $i \neq j$. Therefore,

$$T(e_{ij}) = \sum_{k=1}^{r} a_{ki}^{(ii)} e_{kj} + \sum_{k=1}^{r} a_{ki}^{(ij)} e_{kj}. \tag{3}$$

Again, we have $e_{ij} = e_{ij}e_{jj} + e_{jj}e_{ij}$. Hence, by applying (2) and (3), we have

$$T(e_{ij}) = \sum_{k=1}^{r} a_{ki}^{(ii)} e_{kj}. \tag{4}$$

Now, let $s \in R$ and for all $i, j \in \{1, 2, \ldots, r\}$,

$$T(se_{ij}) = \sum_{k=1}^{r} \sum_{l=1}^{r} a_{kl}^{s(ij)} e_{kl} \ , \ \text{for } a_{kl}^{s(ij)} \in R. \tag{5}$$

We know $se_{ij} = (se_{ij})e_{jj} + e_{jj}(se_{ij})$, for $i \neq j$. Since, T is a Jordan left centralizer on $M_r(R)$, applying (2) and (5),

$$T(se_{ij}) = \sum_{k=1}^{r} a_{kj}^{s(ij)} e_{kj}. \tag{6}$$

Similarly, by $se_{ij} = e_{ii}(se_{ij}) + (se_{ij})e_{ii}$, (2) and (6), we have

$$T(se_{ij}) = \sum_{k=1}^{r} a_{ki}^{(ii)} se_{kj}. \tag{7}$$

Now, $2se_{ii} = e_{ii}(se_{ii}) + (se_{ii})e_{ii}$, so by (2) and (5), we have

$$2T(se_{ii}) = T(e_{ii})(se_{ii}) + T(se_{ii})e_{ii} \implies a_{ki}^{s(ii)} = a_{ki}^{(ii)}s, \text{ for all } k = 1, 2, \ldots, r. \tag{8}$$

For all $i \neq j$, $(se_{ii})(e_{jj}) + (e_{jj})(se_{ii}) = 0$, by (2), we have

$$0 = T(se_{ii})e_{jj} + T(e_{jj})se_{ii} \implies a_{kj}^{s(ii)} = 0, \text{ for all } k = 1, 2, \ldots, r. \tag{9}$$

From (8) and (9),

$$T(se_{ii}) = \sum_{k=1}^{r} a_{ki}^{(ii)} se_{ki}. \tag{10}$$

By the relation (2), (4), (7) and (10), we have $T(se_{ij}) = T(e_{ij})s$, for all $s \in R$.

Now, by (2), $T(1) = \sum_{i=1}^{r} \sum_{k=1}^{r} a_{ki}^{(ii)} e_{ki}$.

Let $A = T(1)$ and $X = \sum_{i=1}^{r} \sum_{j=1}^{r} x_{ij} e_{ij}$ be any element in $M_r(R)$ (Here 1 denotes the identity matrix in $M_r(R)$).

$$\begin{aligned}
T(X) &= T\left(\sum_{i=1}^{r} \sum_{j=1}^{r} x_{ij} e_{ij}\right) = \sum_{i=1}^{r} \sum_{j=1}^{r} T(e_{ij}) x_{ij} \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} \left(\sum_{k=1}^{r} a_{ki}^{(ii)} e_{kj}\right) x_{ij} \\
&= \left(\sum_{i=1}^{r} \sum_{k=1}^{r} a_{ki}^{(ii)} e_{ki}\right) \left(\sum_{i=1}^{r} \sum_{j=1}^{r} x_{ij} e_{ij}\right) = AX.
\end{aligned}$$

Hence T is a left centralizer.

Due to symmetry, every Jordan right centralizer over $M_r(R)$ is a right centralizer. $\square$

Lemma 2.1. *Let R be a ring and $T : M_r(R) \to M_r(R)$ be a two-sided centralizer. Then there exists an $\alpha \in Z(R)$ such that $T(X) = \alpha X$, for all $X \in M_r(R)$.*

Proof. Let 1 be the identity matrix in $M_r(R)$. Since T is a two-sided centralizer, $T(X) = T(1)X = XT(1)$, for all $X \in M_r(R)$. Then $T(1)$ commutes with every element of $M_r(R)$. It is well-known that the center of a matrix ring coincides with the center of a ring (embedded diagonally into the matrix ring). Hence, $T(X) = \alpha X$ for all $X \in M_r(R)$ where $T(1) = \alpha.1$. $\square$

Theorem 2.2. *Let $m \geq 1$, $n \geq 1$ and m, $n \in \mathbb{Z}$, R be a ring with $n(m+n)^3$-torsion free. If $T : M_r(R) \to M_r(R)$ be an additive mapping such that there exists a two-sided centralizer $T_0 : M_r(R) \to M_r(R)$ satisfying*

$$(m+n)T(x^2) = mT(x)x + nxT_0(x), \text{ for all } x \in M_r(R), \tag{11}$$

then T becomes a two-sided centralizer. In fact, $T = T_0$.

Proof. Since T satisfies (11) and T_0 is a two sided centralizer on $M_r(R)$, by Lemma 2.1, we have

$$(m+n)T(x^2) = mT(x)x + n\alpha x^2, \text{ for all } x \in M_r(R) \text{ for some } \alpha \in Z(R). \qquad (12)$$

Replacing x by $x + y$ in (12),

$$(m+n)T(xy+yx) = n\alpha(xy+yx) + mT(x)y + mT(y)x, \text{ for all } x,y \in M_r(R). \qquad (13)$$

Let $T(e_{ij})$ be of the form (1). Since $e_{ii}^2 = e_{ii}$, using (1) and (12),

$$(m+n)T(e_{ii}) = m\sum_{\substack{k=1 \\ k \neq i}}^{r} a_{ki}^{(ii)} e_{ki} + (ma_{ii}^{(ii)} + n\alpha)e_{ii}. \qquad (14)$$

Now, $e_{ij} = e_{ii}e_{ij} + e_{ij}e_{ii}$, for $i \neq j$. By (13) and (14),

$$(m+n)^2 T(e_{ij})$$
$$= m^2 \sum_{\substack{k=1 \\ k \neq i}}^{r} a_{ki}^{(ii)} e_{kj} + m(ma_{ii}^{(ii)} + n\alpha)e_{ij} + m(m+n)\sum_{k=1}^{r} a_{ki}^{(ij)} e_{ki} + (m+n)n\alpha e_{ij}. \qquad (15)$$

Also, we have $e_{ij} = e_{ij}e_{jj} + e_{jj}e_{ij}$. Applying (14) and (15),

$$(m+n)^3 T(e_{ij})$$
$$= m^3 \sum_{\substack{k=1 \\ k \neq i}}^{r} a_{ki}^{(ii)} e_{kj} + m^2(ma_{ii}^{(ii)} + n\alpha)e_{ij} + m(m+n)n\alpha e_{ij} + (m+n)^2 n\alpha e_{ij}. \qquad (16)$$

Let $1 \neq 0$ be the identity element in $M_r(R)$. Since $1^2 = 1$, $1 = \sum_{k=1}^{r} e_{kk}$, and R is n-torsion free,

$$(m+n)\sum_{k=1}^{r} T(e_{kk}) = (m+n)\alpha \left(\sum_{k=1}^{r} e_{kk} \right). \qquad (17)$$

Applying (14) to (17), we have

$$a_{ij}^{(kk)} = 0 \text{ and } a_{kk}^{(kk)} = \alpha, \forall\, i,j,k \text{ with } i \neq j \text{ (Since } R \text{ is } (m+n)\text{-trosion free).} \qquad (18)$$

Again, by (18), (14) and (16),

$$T(e_{ij}) = \alpha e_{ij}, \text{ for any } i,j \text{ (Since } R \text{ is } (m+n)^3\text{-trosion free).} \qquad (19)$$

Therefore, by (19), $T = T_0$, and hence T is a two-sided centralizer. $\qquad \square$

3. SOME FUNCTIONAL EQUATIONS OVER MATRIX RINGS

The primary purpose of this section is to investigate a map satisfying some equations to become a two-sided centralizer.

Theorem 3.1. *Let R be a ring with 2-torsion free. If T is an additive function on $M_r(R)$ which satisfies*

$$2T(x^2) = T(x)x + xT(x), \text{ for all } x \in M_r(R), \qquad (20)$$

then T become a two-sided centralizer. In particular, $T(x) = \alpha x$, for all $x \in M_r(R)$ and for an $\alpha \in Z(R)$.

Proof. Let $T(e_{ij})$ and $T(se_{ij})$ be of the form (1) and (5), respectively for $s \in R$. Since $e_{ii}^2 = e_{ii}$, R is 2-torsion free and satisfies (20), we have

$$T(e_{ii}) = a_{ii}^{(ii)}e_{ii} \ , \ \text{for all } i \in \{1,2,\ldots,r\}. \tag{21}$$

Substituting $x + y$ for x in (20),

$$2T(xy + yx) = T(x)y + T(y)x + xT(y) + yT(x) \ , \ \text{for all } x,y \in M_r(R). \tag{22}$$

For $i \neq j$, applying (22) on $e_{ij} = e_{ii}e_{ij} + e_{ij}e_{ii}$, all the other coefficients of $T(e_{ij})$ except e_{ij} become zero and

$$a_{ij}^{(ij)} = a_{ii}^{(ii)}, \ \text{for all } i,j \in \{1,2,\ldots,r\}. \tag{23}$$

Hence, we have

$$T(e_{ij}) = a_{ij}^{(ij)}e_{ij} = a_{ii}^{(ii)}e_{ij}, \ \text{for all } i,j \in \{1,2,\ldots,r\}. \tag{24}$$

Again, applying (22) on $e_{ij} = e_{ij}e_{jj} + e_{jj}e_{ij}$, we have

$$a_{ii}^{(ii)} = a_{jj}^{(jj)}, \ \text{for all } i,j \in \{1,2,\ldots,r\}. \tag{25}$$

Also,

$$T(e_{ij}) = a_{11}^{(11)}e_{ij}, \ \forall \ i,j \in \{1,2,\ldots,r\}. \tag{26}$$

For $i \neq j$ and $s \in R$, applying (22) on $se_{ij} = (se_{ij})e_{jj} + e_{jj}(se_{ij})$, all the other coefficients of $T(se_{ij})$ except e_{ij} become zero and

$$a_{ij}^{s(ij)} = sa_{11}^{(11)}, \ \forall \ i, \ j. \tag{27}$$

For $i \neq j$ and $s \in R$,

$$T(se_{ij}) = sa_{11}^{(11)}e_{ij}, \ \text{for all } i, \ j. \tag{28}$$

Applying (22), $2se_{ii} = (se_{ii})e_{ii} + e_{ii}(se_{ii})$,

$$2a_{ii}^{s(ii)} = a_{11}^{(11)}s + sa_{11}^{(11)}, \ \text{for all } i \in \{1,2,\ldots,r\}. \tag{29}$$

For all $i \neq j$, $(se_{ii})(e_{jj}) + (e_{jj})(se_{ii}) = 0$, by (22),

$$a_{kj}^{s(ii)} = 0 \text{ and } a_{jk}^{s(ii)} = 0, \ \text{for all } k = 1,2,\ldots,r \text{ (Since } R \text{ is 2-torsion free)}. \tag{30}$$

Applying (22), $se_{ij} = (se_{ij})e_{ii} + e_{ii}(se_{ij})$,

$$a_{11}^{(11)}s = sa_{11}^{(11)}. \tag{31}$$

Again, we have

$$T(se_{ii}) = sa_{11}^{(11)}e_{ii}, \ \text{for all } i \in \{1,2,\ldots,r\}. \tag{32}$$

Let $X = \sum_{i=1}^{r}\sum_{j=1}^{r} x_{ij}e_{ij}$, $x_{ij} \in R$. Let $\alpha = a_{11}^{(11)}$. Then by using (28) and (32), we get

$$T(x) = a_{11}^{(11)}x = \alpha x, \ \text{for all } x \in M_r(R). \tag{33}$$

Thus, by (31) and (33), T is a two-sided centralizer. $\qquad\square$

Example 3.1. *Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Now, we provide an example of an additive map T on $M_r(R)$ with $2T(X^2) = T(X)X + XT(X)$, for all $X \in M_r(R)$, but it is not a two-sided centralizer.*
Let $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M_2(\mathbb{Z}_2)$ and $T : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ is defined by

$$T(X) = \begin{bmatrix} x+y+z+t & 0 \\ 0 & x+y+z+t \end{bmatrix}.$$

Then T satisfies $2T(X^2) = T(X)X + XT(X)$, for all $X \in M_2(\mathbb{Z}_2)$. Also, for $X = e_{11}$ and $Y = e_{12}$, $T(XY) \neq T(X)Y$. Thus, T is not a two-sided centralizer.

Theorem 3.2. *Let R be a 2-torsion free ring. If T is an additive mapping on $M_r(R)$ satisfying*

$$2T(x^2) = T(x)x + xT_0(x), \text{ for all } x \in M_r(R), \tag{34}$$

where T_0 is an additive mapping on $M_r(R)$ satisfying

$$2T_0(x^2) = T_0(x)x + xT_0(x), \text{ for all } x \in M_r(R), \tag{35}$$

then T is a two-sided centralizer.

Proof. By Theorem 3.1 and Lemma 2.1, T_0 is a two-sided centralizer and $T_0(x) = \alpha x$ for some $\alpha \in Z(R)$ and $x \in M_r(R)$. Therefore, we get

$$2T(x^2) = T(x)x + \alpha x^2, \text{ for all } x \in M_r(R). \tag{36}$$

Again, let $T(e_{ij})$ and $T(se_{ij})$ be of the form (1) and (5), respectively for $s \in R$. Applying (36) on $e_{ii}^2 = e_{ii}$, all the coefficients of $T(e_{ii})$ except e_{ii} become zero and

$$a_{ii}^{(ii)} = \alpha, \text{ for all } i \in \{1, 2, \ldots, r\}. \tag{37}$$

We have

$$T(e_{ii}) = \alpha e_{ii}, \text{ for all } i \in \{1, 2, \ldots, r\}. \tag{38}$$

Linearizing (36),

$$2T(xy + yx) = T(x)y + T(y)x + \alpha(xy + yx), \text{ for all } x, y \in M_r(R). \tag{39}$$

For $i \neq j$, applying (39) on $e_{ij} = e_{ii}e_{ij} + e_{ij}e_{ii}$, each coefficient of $T(e_{ij})$ other than e_{ij} is zero and

$$a_{ij}^{(ij)} = \alpha, \text{ for all } i, j \in \{1, 2, \ldots, r\}. \tag{40}$$

For $i \neq j$,

$$T(e_{ij}) = \alpha e_{ij}, \forall i, j. \tag{41}$$

For $i \neq j$ and $s \in R$, applying (39) on $se_{ij} = (se_{ij})e_{jj} + e_{jj}(se_{ij})$, all the coefficients of $T(se_{ij})$ except e_{ij} are zero and

$$a_{ij}^{s(ij)} = \alpha s, \text{ for all } i, j \in \{1, 2, \ldots, r\}. \tag{42}$$

For $i \neq j$ and $s \in R$,

$$T(se_{ij}) = \alpha se_{ij}, \text{ for all } i, j \in \{1, 2, \ldots, r\}. \tag{43}$$

Applying (39) on $2se_{ii} = (se_{ii})e_{ii} + e_{ii}(se_{ii})$, each coefficient of $T(se_{ii})$, except e_{ii}, is zero and

$$a_{ii}^{s(ii)} = \alpha s, \text{ for all } i \in \{1, 2, \ldots, r\}. \tag{44}$$

For all $s \in R$,

$$T(se_{ii}) = \alpha se_{ii}, \text{ for all } i \in \{1, 2, \ldots, r\}. \tag{45}$$

By (43) and (45), we conclude that

$$T = T_0. \tag{46}$$

Thus, T is a two-sided centralizer. $\qquad\square$

Now, we give an example which shows that $M_r(R)$ is not always a semiprime ring. Hence, Theorem 3.1 is not a consequence of any result of Vukman [11].

Example 3.2. *Let $\mathbb{Z}_9$ be the ring of residue classes of integers modulo 9. Then $\mathbb{Z}_9$ is 2-torsion free with unity $1 \neq 0$. It is easy to compute that $(3e_{11})M_2(\mathbb{Z}_9)(3e_{11}) = 0$. But $3e_{11} \neq 0$. Therefore, $M_2(\mathbb{Z}_9)$ is not a semiprime ring (since in a semiprime ring R, $aRa = 0 \implies a = 0$).*

Theorem 3.3. *Let R be a ring. If T is an additive map on $M_r(R)$ with*

$$T(xyx) = xT(y)x, \text{ for all } x, y \in M_r(R), \tag{47}$$

then T becomes a two-sided centralizer. In particular, $T(x) = \alpha x$ for all $x \in M_r(R)$ and for an $\alpha \in Z(R)$.

Proof. Put $x = 0$, $y = 1$ (1 and 0 denote the identity and zero matrix, respectively) in (47), we have $T(0) = 0$. Let $T(e_{ij})$ and $T(se_{ij})$ be of the form (1) and (5), respectively for $s \in R$. Since $e_{ii}^3 = e_{ii}$ and $x.1.x = x^2$, using (47),

$$T(e_{ii}) = a_{ii}^{(ii)}e_{ii},$$
$$\text{(hence) } T(1) = \sum_{k=1}^{r} a_{kk}^{(kk)}e_{kk}, \tag{48}$$
$$T(x^2) = xT(1)x \text{ for all } x, y \in M_r(R).$$

Put $x = x + y$ in (48),

$$T(xy + yx) = xT(1)y + yT(1)x, \text{ for all } x, y \in M_r(R). \tag{49}$$

Put $x = e_{ii}$ and $y = e_{ij}$ $(i \neq j)$ in (49),

$$T(e_{ij}) = a_{ii}^{(ii)}e_{ij}$$
$$\text{(also putting } x = e_{ij}, \ y = e_{jj} \text{ in (49)), } T(e_{ij}) = a_{jj}^{(jj)}e_{ij}$$
$$\text{(hence) } a_{ii}^{(ii)} = a_{jj}^{(jj)} = \alpha \text{ (say)} \tag{50}$$
$$T(e_{ij}) = \alpha e_{ij}.$$

Since $se_{ij} = (se_{ij})e_{jj} + e_{jj}(se_{ij})$, using (50),

$$T(se_{ij}) = s\alpha e_{ij}$$
$$\text{(similarly), } T(se_{ij}) = \alpha se_{ij} \tag{51}$$
$$\text{(hence) } \alpha s = s\alpha \ (s \in R).$$

Put $x = x + z$ in (47),

$$T(xyz + zyx) = xT(y)z + zT(y)x, \text{ for all } x, y, z \in M_r(R). \tag{52}$$

Put $x = se_{ii}$, $y = e_{ji}$ and $z = e_{ii}$ in (52),

$$T(se_{ii}) = s\alpha e_{ii} = \alpha se_{ii}. \tag{53}$$

From (51) and (53), $T(x) = \alpha x$ for all $x \in M_r(R)$. $\qquad\square$

Theorem 3.4. *Let R be a ring. If T is an additive map on $M_r(R)$ with*

$$2T(xyx) = T(x)yx + xyT(x), \text{ for all } x, y \in M_r(R), \tag{54}$$

then T is a two-sided centralizer. In particular, there exists an $\alpha \in Z(R)$ such that $T(x) = \alpha x, \ \forall \ x \in M_r(R)$.

Proof. Put $x = 0$, $y = 0$ in (54), we have $2T(0) = 0$. Let $T(e_{ij})$ and $T(se_{ij})$ be of the form (1) and (5), respectively for $s \in R$. Putting $x = e_{ii}$, $y = e_{ii}$ in (54),

$$a_{il}^{(ii)} = 0 \text{ for all } l \in \{1, 2, \ldots, i-1, i+1, \ldots, r\}. \tag{55}$$

Putting $x = e_{ii}$, $y = e_{ik}$ $(k \neq i)$ in (54),

$$a_{kl}^{(ii)} = 0 \text{ for all } l \in \{1, 2, \ldots, r\} \text{ and hence } T(e_{ii}) = a_{ii}^{(ii)} e_{ii}. \tag{56}$$

Putting $x = e_{ij}$, $y = e_{ji}$ $(j \neq i)$ in (54),

$$a_{im}^{(ij)} = 0 \text{ for all } m \in \{1, 2, \ldots, j-1, j+1, \ldots, r\}. \tag{57}$$

Putting $x = e_{ij}$, $y = e_{jk}$ $(j \neq i, i \neq k)$ in (54),

$$a_{km}^{(ij)} = 0 \text{ for all } m \in \{1, 2, \ldots, r\} \text{ and hence } T(e_{ij}) = a_{ij}^{(ij)} e_{ij}. \tag{58}$$

Putting $x = 1$ and $y = e_{ij}$ in (54), $2a_{ij}^{(ij)} = a_{ii}^{(ii)} + a_{jj}^{(jj)}$. Also, putting $x = e_{ij} + e_{ii}$ and $y = e_{ji}$ in (54), $a_{ii}^{(ii)} = a_{ij}^{(ij)}$. Therefore, $a_{ij}^{(ij)} = a_{ii}^{(ii)} = a_{jj}^{(jj)} = \alpha$ (say).

$$\text{Hence, } T(e_{kl}) = \alpha e_{kl} \text{ for all } k, \ l. \tag{59}$$

Putting $x = e_{ii}$ and $y = se_{ii}$ in (54) and using (59),

$$2T(se_{ii}) = (\alpha s + s\alpha) e_{ii}$$
$$2a_{ii}^{s(ii)} = \alpha s + s\alpha. \tag{60}$$

Put $x = x + z$ in (54),

$$2T(xyz + zyx) = T(x)yz + xyT(z) + T(z)yx + zyT(x), \text{ for all } x, y, z. \tag{61}$$

Putting $x = se_{ii}$, $y = e_{ij}$ and $z = e_{ji}$ $(j \neq i)$ in (61) and using (59) and (60),

$$a_{ii}^{s(ii)} = s\alpha = \alpha s \text{ and } a_{jl}^{s(ii)} = 0 \text{ for } l \in \{1, 2, \ldots, r\}. \tag{62}$$

Put $x = e_{ij}$, $y = e_{ji}$ and $z = se_{ii}$ in (61),

$$a_{il}^{s(ii)} = 0 \text{ for } l \in \{1, 2, \ldots, i-1, i+1, \ldots, r\} \text{ and hence } T(se_{ii}) = \alpha se_{ii} \ (s \in R, \alpha \in Z(R)). \tag{63}$$

Putting $x = se_{ij}$, $y = e_{jj}$ and $z = e_{jj}$ in (61),

$$a_{ij}^{s(ij)} = s\alpha \text{ and } a_{mj}^{s(ij)} = 0 \text{ for } m \in \{1, 2, \ldots, i-1, i+1, \ldots, r\}. \tag{64}$$

Putting $x = se_{ij}$, $y = e_{kk}$ and $z = e_{kk}$ $(k \neq j)$ in (61),

$$a_{mk}^{s(ij)} = 0 \text{ for } m \in \{1, 2, \ldots, r\} \text{ and hence } T(se_{ij}) = \alpha se_{ij}. \tag{65}$$

Thus, $T(x) = \alpha x$ for all $x \in M_r(R)$.

$\square$

Theorem 3.5. *Let R be a 2-torsion free ring. If T is an additive map on $M_r(R)$ with*

$$3T(xyx) = T(x)yx + xT(y)x + xyT(x), \text{ for all } x, y \in M_r(R), \tag{66}$$

then T is a two-sided centralizer. In particular, there exists an $\alpha \in Z(R)$ such that $T(x) = \alpha x$, for all $x \in M_r(R)$.

Proof. Let $T(e_{ij})$ and $T(se_{ij})$ be of the form (1) and (5), respectively for $s \in R$. Since $e_{ij}e_{ji}e_{ij} = e_{ij}$ and the torsion condition on R, using (66),

$$
\begin{aligned}
&a_{1j}^{(ij)} = \cdots = a_{i-1,j}^{(ij)} = a_{i+1,j}^{(ij)} = \cdots = a_{rj}^{(ij)} = 0, \\
&a_{i1}^{(ij)} = \cdots = a_{i,j-1}^{(ij)} = a_{i,j+1}^{(ij)} = \cdots = a_{ir}^{(ij)} = 0, \\
&a_{ij}^{(ij)} = a_{ji}^{(ji)} \text{ for all } i,j \in \{1,2,\ldots,r\}.
\end{aligned}
\tag{67}
$$

Substituting 1 and 0 for x and y in (66), respectively, we have $T(0) = 0$. Let $k \neq i$ and $l \neq j$ and since $e_{lk}e_{ij}e_{lk} = 0$, using (66),

$$
a_{kl}^{(ij)} = 0.
$$
$$
\text{Hence, } T(e_{ij}) = a_{ij}^{(ij)}e_{ij}.
\tag{68}
$$

Linearizing (66),

$$
\begin{aligned}
3T(xyz + zyx) &= T(x)yz + T(z)yx + xT(y)z \\
&+ zT(y)x + xyT(z) + zyT(x), \text{ for all } x,y,z \in M_r(R).
\end{aligned}
\tag{69}
$$

Substituting e_{ii}, e_{ij} and e_{jj} for x, y and z respectively in (69),

$$
2a_{ij}^{(ij)} = a_{ii}^{(ii)} + a_{jj}^{(jj)} \text{ for all } i \neq j.
\tag{70}
$$

Substituting e_{ji}, e_{ij} and e_{jj} for x, y and z, respectively in (69),

$$
\begin{aligned}
&\text{(by (67))} \ a_{jj}^{(jj)} = a_{ij}^{(ij)} = a_{ji}^{(ji)}, \\
&\text{(by above and (67))} \ a_{ii}^{(ii)} = a_{ij}^{(ij)} = a_{ji}^{(ji)} = a_{jj}^{(jj)}, \text{ for all } i \neq j. \\
&\text{Hence } T(e_{ij}) = \alpha e_{ij} \text{ (letting, } \alpha = a_{11}^{(11)}\text{), for all } i,j.
\end{aligned}
\tag{71}
$$

Let $s \in R$. Substituting 1 and se_{ij} for x and y, respectively in (66) and using (71),

$$
\begin{aligned}
&2T(se_{ij}) = (\alpha s + s\alpha)e_{ij}, \\
&2a_{ij}^{s(ij)} = \alpha s + s\alpha, \\
&T(se_{ij}) = a_{ij}^{s(ij)}e_{ij}, \text{ for all } i,j.
\end{aligned}
\tag{72}
$$

Substituting se_{ii}, e_{ij} and e_{jj} for x, y and z, respectively in (69) and using (71),

$$
a_{ij}^{s(ij)} = s\alpha = \alpha s \text{ (using (72)), for all } i \neq j.
\tag{73}
$$

Therefore, $T(x) = \alpha x$, for all $x \in M_r(R)$. $\qquad\square$

Example 3.3. *Now, we give an example of an additive mapping T on $M_r(R)$ satisfying (66), but it is not a two-sided centralizer. Let $\mathbb{Z}_2$ be the ring of residue classes of integers modulo 2, which is not 2-torsion free. Let $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M_2(\mathbb{Z}_2)$ and $T : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ be defined by*

$$
T\left(\begin{bmatrix} x & y \\ z & t \end{bmatrix} \right) = \begin{bmatrix} y & 0 \\ x & 0 \end{bmatrix}.
$$

Then T satisfies (66). In this case, for $X = e_{11}$ and $Y = e_{12}$, $T(XY) = e_{11} \neq e_{22} = T(X)Y$. Thus, T is not a two-sided centralizer.

4. JORDAN $\star$-CENTRALIZERS OVER RINGS

It is easy to prove that every reverse left $\star$-centralizer is a Jordan left $\star$-centralizer, but the converse is not generally valid. Hence, it is a genuine attempt to classify rings and algebras over which Jordan left $\star$-centralizer becomes reverse left $\star$-centralizer.

Example 4.1. *Let $S = \mathbb{Z}_2[x, y]$ with $x^2 = y^2 = 0$ and R_1 be the subring of S generated by x and y. One can easily recognize that every element of R_1 is of square zero and R_1 is commutative.*

Let $R_2 = \left\{ \begin{pmatrix} 0 & A & B \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix} \mid A, B \in R_1 \right\}$ and $\begin{pmatrix} 0 & A & B \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix}^{\star} = \begin{pmatrix} 0 & A & B \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix}$. Then R_2 is a ring with involution $\star$.

Define $T : R_2 \to R_2$ as,

$$T \begin{pmatrix} 0 & A & B \\ 0 & 0 & A \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ *It is easy to check that T is a Jordan left $\star$-centralizer.*

Let $\tilde{X} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$ and $\tilde{Y} = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$.

Then $T(\tilde{X}\tilde{Y}) = \begin{pmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \neq 0 = T(\tilde{Y})\tilde{X}^{\star}$. Hence, T is not a reverse left $\star$-centralizer.

Let $M_r(R)$ be a matrix ring with involution $\star$.

Theorem 4.1. *Every Jordan left $\star$-centralizer over $M_r(R)$ is a reverse left $\star$-centralizer.*

Proof. Let T be a Jordan left $\star$-centralizer over $M_r(R)$.

$$T(x^2) = T(x)x^{\star}$$
$$\implies [T(x^2)]^{\star} = [T(x)x^{\star}]^{\star} = (x^{\star})^{\star}(T(x))^{\star} = x(T(x))^{\star}$$

Now, let $S(x) = (T(x))^{\star}$ and

$$S(x^2) = (T(x^2))^{\star} = x(T(x))^{\star} = xS(x).$$

Then S is a Jordan right centralizer. Hence, by Theorem 2.1, S is a right centralizer. Also,

$$S(xy) = xS(y)$$
$$\implies (T(xy))^{\star} = x(T(y))^{\star}$$
$$\implies ((T(xy))^{\star})^{\star} = (x(T(y))^{\star})^{\star}$$
$$\implies T(xy) = T(y)x^{\star}.$$

Thus, T is a reverse left $\star$-centralizer over $M_r(R)$. $\qquad\square$

Every left $\star$-centralizer is always a Jordan left $\star$-centralizer. The following example shows that every Jordan left $\star$-centralizer is not a left $\star$-centralizer.

Example 4.2. *We define $T : M_2(R) \to M_2(R)$ by–*

$$T \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x + y & z + t \\ 0 & 0 \end{bmatrix}.$$

Also, we define $\star : M_2(R) \to M_2(R)$ *by–*

$$\star \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} x & z \\ y & t \end{bmatrix}.$$

.

Then it is easy to see that T becomes a Jordan left $\star$-centralizer.
But $T(e_{11}e_{12}) = e_{11} \neq 0 = T(e_{11})e_{12}^{\star}$. Thus, T is not a left $\star$-centralizer.

Theorem 4.2. *If $T : M_r(R) \to M_r(R)$ is a reverse $\star$-centralizer, then there exists an $\alpha \in Z(R)$ such that $T(X) = \alpha x^{\star}$, for all $x \in M_r(R)$.*

Proof. Since $T(xy) = T(y)x^{\star}$ for all $x, y \in M_r(R)$. Putting $y = 1$,

$$T(x) = T(1)x^{\star} \text{ for all } x \in M_r(R). \tag{74}$$

Now, $T(xy) = y^{\star}T(x)$ for all $x, y \in M_r(R)$. Taking $y = x$ and $x = 1$, we have

$$T(x) = x^{\star}T(1) \text{ for all } x \in M_r(R). \tag{75}$$

Since $\star$ is a bijective map, $T(1) \in Z(M_r(R)) = Z(R)$. Therefore, $T(x) = \alpha x^{\star}$ for all $x \in M_r(R)$ where $\alpha = T(1)$.
$\square$

Theorem 4.3. *If $T : M_r(R) \to M_r(R)$ is a $\star$-centralizer, then $T = 0$.*

Proof. Since $T(xy) = T(x)y^{\star} = x^{\star}T(y)$ for all $x, y \in M_r(R)$. Putting $y = 1$ and $x = 1$, respectively, $T(x) = x^{\star}T(1) = T(1)x^{\star}$ for all $x \in M_r(R)$.

$$T(xy) = T(x)y^{\star} \implies T(1)(xy)^{\star} = T(1)x^{\star}y^{\star} \implies T(1)[(xy)^{\star} - x^{\star}y^{\star}] = 0$$
$$\implies \text{Either } T(1) = 0 \text{ or } (xy)^{\star} = x^{\star}y^{\star} \implies xy = yx.$$

Since $xy = yx$ does not hold true in $M_r(R)$, $T(1) = 0$. Thus, $T = 0$. $\square$

Theorem 4.4. *Let $m \geq 1, n \geq 1$ and $m, n \in \mathbb{Z}$, R be a ring with $n(m+n)^3$-torsion free. If $T : M_r(R) \to M_r(R)$ be an additive mapping such that there exists a reverse $\star$-centralizer T_0 satisfying*
$(m + n)T(x^2) = mx^{\star}T(x) + nT_0(x)x^{\star}$, *for all $x \in M_r(R)$, then T becomes reverse $\star$-centralizer. It also gives us $T = T_0$.*

Proof. Let $S(x) = (T(x))^{\star}$ and $S_0(x) = (T_0(x))^{\star}$. Then $S_0(x) = \beta x$, for some $\beta \in Z(R)$, by Theorem 4.2. Now,

$$(m + n)S(x^2) = (m + n)(T(x^2))^{\star} = [(m + n)T(x^2)]^{\star} = [mx^{\star}T(x) + nT_0(x)x^{\star}]^{\star}$$
$$= m(T(x))^{\star}x + nx(T_0(x))^{\star} = mS(x)x + nxS_0(x). \tag{76}$$

Therefore, S satisfies the condition of Theorem 2.2. Hence, S is a two-sided centralizer, and T becomes the reverse $\star$-centralizer. Since $S = S_0$, we have $T = T_0$. $\square$

Theorem 4.5. *Let $m \geq 1, n \geq 1$ and $m, n \in \mathbb{Z}$, R be an $n(m + n)^3$-torsion free ring. If $T : M_r(R) \to M_r(R)$ be an additive mapping such that there exists a $\star$-centralizer T_0 satisfying*
$(m+n)T(x^2) = mx^{\star}T(x) + nT_0(x)x^{\star}$, *for all $x \in M_r(R)$, then $T = 0$.*

Proof. By Theorem 4.3, $T_0 = 0$, T_0 is a reverse $\star$-centralizer. Since T satisfies the conditions in Theorem 4.4, we have $T = T_0 = 0$. $\square$

Letting $S(x) = (T(x))^{\star}$ and using Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5, we can prove Theorem 4.6, Theorem 4.7, Theorem 4.8, Theorem 4.9 and Theorem 4.10.

Theorem 4.6. *Let R be a 2-torsion free ring. If $T : M_r(R) \to M_r(R)$ is an additive map satisfying $2T(x^2) = T(x)x^\star + x^\star T(x)$, then T is a reverse $\star$-centralizer and $T(x) = \alpha x^\star$, where $\alpha \in Z(R)$.*

Example 4.3. *Let $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in M_2(\mathbb{Z}_2)$ and $X^\star = \begin{bmatrix} x & z \\ y & t \end{bmatrix}$.*
  Let us define $T : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ by

$$T(X) = \begin{bmatrix} x + y + z + t & 0 \\ 0 & x + y + z + t \end{bmatrix}.$$

  Then T satisfies $2T(X^2) = T(X)X^\star + X^\star T(X)$, for all $X \in M_2(\mathbb{Z}_2)$.
  Now, for $X = e_{11}$ and $Y = e_{12}$, $T(XY) = e_{11} + e_{12} \neq e_{11} = T(Y)X^\star$. Therefore, T is not a reverse $\star$-centralizer. It shows that the torsion condition on R is necessary for Theorem 4.6.

  If T satisfies the condition in Theorem 4.6, then T need not be a $\star$-centralizer.

Example 4.4. *Let us define $T(x) = x^\star$ for all $x \in M_2(R)$. Define $\star$ as the transpose of matrix x. Then T satisfies $2T(x^2) = T(x)x^\star + x^\star T(x)$, for all $x \in M_2(R)$. But $T(e_{11}e_{12}) \neq T(e_{11})e_{12}^\star$, hence T is not a $\star$-centralizer.*

Theorem 4.7. *Let R be a 2-torsion free ring. If $T : M_r(R) \to M_r(R)$ is an additive map satisfying $2T(x^2) = T_0(x)x^\star + x^\star T(x)$ where $T_0 : M_r(R) \to M_r(R)$ is an additive map satisfying $2T_0(x^2) = T_0(x)x^\star + x^\star T_0(x)$, then T is a reverse $\star$-centralizer and $T(x) = \alpha x^\star$, where $\alpha \in Z(R)$.*

Theorem 4.8. *Let R be a ring. If T is an additive map on $M_r(R)$ with $T(xyx) = x^\star T(y)x^\star$, for all $x, y \in M_r(R)$, then T becomes reverse $\star$-centralizer. In particular, $T(x) = \alpha x^\star$, where $\alpha \in Z(R)$.*

Theorem 4.9. *Let R be a ring. If T is an additive map on $M_r(R)$ with $2T(xyx) = T(x)y^\star x^\star + x^\star y^\star T(x)$ for all $x, y \in M_r(R)$, then T is a reverse $\star$-centralizer. In particular, $T(x) = \alpha x^\star$ where $\alpha \in Z(R)$.*

Theorem 4.10. *Let R be a ring with 2-torsion free. If T is an additive map on $M_r(R)$ with $3T(xyx) = T(x)y^\star x^\star + x^\star T(y)x^\star + x^\star y^\star T(x)$ for all $x, y \in M_r(R)$, then T is a reverse $\star$-centralizer. In particular, $T(x) = \alpha x^\star$, where $\alpha \in Z(R)$.*

Example 4.5. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}_2)$ and $A^\star = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$.*
  Let $T : M_2(\mathbb{Z}_2) \to M_2(\mathbb{Z}_2)$ be defined by

$$T(A) = \begin{bmatrix} 0 & c \\ b & 0 \end{bmatrix}.$$

  Then T satisfies $3T(ABC) = T(A)B^\star A^\star + A^\star T(B)A^\star + A^\star B^\star T(A)$, for all A, $B \in M_2(\mathbb{Z}_2)$.
  Now, for $A = e_{11}$ and $B = e_{12}$, $T(AB) = e_{21} \neq 0 = B^\star T(A)$. Therefore, T is not a reverse $\star$-centralizer. It shows that the torsion condition on R is necessary for the Theorem 4.10.

  Example 4.4 shows that even T satisfies the conditions in Theorems 4.7, 4.8, 4.9 and 4.10, but need not be a $\star$-centralizer. Theorem 2.2 motivates us to post a conjecture as follows.

Conjecture 4.1. *Let $m \geq 1$, $n \geq 1$ and m, $n \in \mathbb{Z}$, R be a ring with some suitable torsion restrictions. If $T : M_r(R) \to M_r(R)$ be an additive mapping such that*

$$(m + n)T(x^2) = mT(x)x + nxT(x), \text{ for all } x \in M_r(R), \tag{77}$$

then T become a two-sided centralizer.

5. Conclusions

In this paper, we discussed centralizers over matrix rings. Then we proved that the Jordan left centralizer over the matrix ring becomes the left centralizer, which is generally not true. Also, we showed that every two-sided centralizer over the matrix ring is of some particular form. Later, it showed that the map becomes a two-sided centralizer for the matrix ring when it satisfies some functional equation. We also prove every Jordan left $\star$-centralizer is a reverse left $\star$-centralizer over matrix ring with involution. Finally, we have established that every reverse $\star$-centralizer over $\star$-matrix ring has some particular form, and a map satisfying some equations over $\star$-matrix ring becomes reverse $\star$-centralizer.

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Arindam Ghosh is working as a Lecturer in the Department of Mathematics at Government Polytechnic Kishanganj, Thakurganj, Kishanganj, Bihar, India. He completed his M. Sc. at Indian Institute of Technology Madras, Chennai, Tamilnadu, and Ph. D. in Mathematics at Indian Institute of Technology Patna.

Om Prakash is a Professor at the Department of Mathematics, Indian Institute of Technology Patna. He completed M. Sc. in Mathematics from Patna University in 1999, and Ph. D. in Mathematics from Banasthali University, Rajasthan, India in 2010. His main research interest includes Rings and Modules, Algebraic Coding Theory, Algebraic Graph Theory and Algebraic Number Theory.

Sushma Singh is working as an Assistant Professor in the Department of Mathematics at Kalinga Institute of Industrial Technology, Bhubaneswar, Odisha, India. She completed her her M. Sc. at Banaras Hindu University, Varanasi, Uttar Pradesh and Ph. D. in Mathematics at Indian Institute of Technology Patna, India.