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ON THE k-MERSENNE AND k-MERSENNE-LUCAS OCTONIONS

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Abstract. This paper aims to introduce the k-Mersenne and k-Mersenne-Lucas octonions. We investigate the algebraic properties of these octonions in closed form and give some well-known identities like Catalan identity, d'Ocagne identity, Simson identity, etc. Moreover, we present various generating functions and partial sum formulae for these octonions. Lastly, we study the combined identities and matrix representation for these octonions.

Keywords: k-Mersenne Octonions, Binet Formula, Catalan's Identity, Finite sum, Generating Function.

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1. Introduction

In 1843, W.R.Hamilton extended the concept of set of complex numbers to the set of quaternions denoted as $\mathbb{H}$. For $a, b, c, d \in \mathbb{R}$, a quaternion $q \in \mathbb{H}$ is of the form $q = a + bi + cj + dl$, where $i^2 = j^2 = l^2 = ijl = -1$.
Inspired by Hamilton's work, J.T. Graves defined the concept of the octonions in 1843. Later, in 1845, A. Cayley also defined the octonions. The set of octonions is usually denoted by $\mathbb{O}$. With a natural basis $\{e_0 = 1, e_1 = i, e_2 = j, e_3 = l, e_4 = e, e_5 = ie, e_6 = je, e_7 = le\}$, $\mathbb{O}$ forms an 8-dimensional non-associative division algebra over $\mathbb{R}$.

If $a \in \mathbb{O}$ then it takes the form,

$$a = \sum_{r=0}^{7} a_r e_r, \quad \text{where } a_r \in \mathbb{R}. \tag{1}$$

And, the conjugate of the octonion a is given as

$$\overline{a} = a_0 - \sum_{r=1}^{7} a_r e_r. \tag{2}$$

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The norm of an octonion a is given as

$$N(a) = \sqrt{a\bar{a}} = \sqrt{\bar{a}a} = \sqrt{\sum_{r=0}^{7} a_r^2}. \tag{3}$$

The octonion basis $\{e_0 = 1, e_1 = i, e_2 = j, e_3 = l, e_4 = e, e_5 = ie, e_6 = je, e_7 = le\}$ follows a special multiplication rule[1] given in the Table 1.

Table 1. The multiplication table for the basis of $\mathbb{O}$

.	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

For more reading on the quaternions and octonions, reader referred to [2, 3].

In recent years, recursive sequences are of great interest among the researchers. Study of recursive sequences in division algebra was firstly presented by Horadam[4] where they introduced Fibonacci and Lucas quaternions. Later, researchers extended this study to other number sequences like Pell, Pell-Lucas, Jacobsthal, k-Jacobsthal, etc. (for example, see[5, 6, 7, 8, 9]). Octonions with Fibonacci and Lucas components were introduced by Akkus and Keçilioglu[10] and they studied their properties like Binet formula, generating function and some well-known identities. Recently A.D. Godse[11] studied the hyperbolic Octonions involving k-Fibonacci & k-Lucas sequences and Özkan et.al. [12] studied the hyperbolic Octonions with k-Jacobsthal & k-Jacobsthal Lucas sequences. Some recent work on octonions with a number sequence like Pell, Pell-Lucas, Jacobsthal, Mersenne, Horadam etc. can be seen in[13, 8, 14, 15, 16].

Motivated essentially by recent works on octonions with the components from a recursive sequence, here we are considering the generalized recursive sequences so-called the k-Mersenne sequence and the k-Mersenne-Lucas sequence, a generalization of the Mersenne sequence. Many papers are dedicated to Mersenne sequence and their generalizations (see, for example [17, 18, 19, 20]). Daşdemir and Göksal [21] have defined Mersenne quaternions and obtained Binet's formula and generating function of them.
The Mersenne sequence $\{M_n\}_{n\geq 0}$ is defined [22] by

$$M_0 = 0, \quad M_1 = 1, \quad M_{n+1} = 3M_n - 2M_{n-1}, \quad n \geq 1,$$

and the k-Mersenne sequence $\{M_{k,n}\}_{n\geq 0}$ is defined [23] recursively by

$$M_{k,0} = 0, \quad M_{k,1} = 1, \quad M_{k,n+1} = 3kM_{k,n} - 2M_{k,n-1}, \quad n \geq 1. \tag{4}$$

The Mersenne-Lucas sequence $\{m_n\}_{n\geq 0}$ is defined [24] by

$$m_0 = 2, \quad m_1 = 3, \quad m_{n+1} = 3m_n - 2m_{n-1}, \quad n \geq 1,$$

and the k-Mersenne-Lucas sequence $\{m_{k,n}\}_{n\geq 0}$ is defined [19] by

$$m_{k,0} = 2, \quad m_{k,1} = 3k, \quad m_{k,n+1} = 3km_{k,n} - 2m_{k,n-1}, \quad n \geq 1. \tag{5}$$

The Binet formulae of the k-Mersenne and k-Mersenne-Lucas sequences are given, respectively, by

$$M_{k,n} = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}, \quad \text{and} \quad m_{k,n} = \lambda_1^n + \lambda_2^n, \tag{6}$$

where $\lambda_1 = \dfrac{3k + \sqrt{9k^2 - 8}}{2}$ and $\lambda_2 = \dfrac{3k - \sqrt{9k^2 - 8}}{2}$ are the roots of the characteristic equation $\lambda^2 - 3k\lambda + 2 = 0$ associated with the above recurrence relations.
Note that λ_1 and λ_2 satisfy the following properties:

$$\lambda_1 + \lambda_2 = 3k, \quad \lambda_1\lambda_2 = 2, \quad \lambda_1 - \lambda_2 = \sqrt{9k^2 - 8} \tag{7}$$

and also

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1^2}{2}, \quad \frac{\lambda_2}{\lambda_1} = \frac{\lambda_2^2}{2}.$$

2. Octonions with k-Mersenne/k-Mersenne-Lucas Numbers

In this section first we define the k-Mersenne octonions, obtain their algebraic properties in closed form and present some well-known identities. Then we define and investigate the k-Mersenne-Lucas octonions. Let $M_{k,n}$ and $m_{k,n}$ be the nth k-Mersenne and k-Mersenne-Lucas numbers, respectively, throughout the paper.

Definition 2.1. *For $n \geq 0$, any n^{th} k-Mersenne octonion $M\mathbb{O}_{k,n}$ is defined as*

$$M\mathbb{O}_{k,n} = \sum_{r=0}^{7} M_{k,n+r} e_r. \tag{8}$$

Using the Definition 2.1 and equation (4), after some basic calculations, we get the recurrence relation for the k-Mersenne octonions as follows:

$$M\mathbb{O}_{k,n+1} = 3kM\mathbb{O}_{k,n} - 2M\mathbb{O}_{k,n-1}, \quad n \geq 1, \tag{9}$$

where $M\mathbb{O}_{k,0} = \sum_{r=0}^{7} M_{k,r} e_r$ and $M\mathbb{O}_{k,1} = \sum_{r=0}^{7} M_{k,r+1} e_r$.
By (2), the conjugate of the k-Mersenne octonion is defined as

$$\overline{M\mathbb{O}}_{k,n} = M_{k,0} - \sum_{r=1}^{7} M_{k,n+r} e_r. \tag{10}$$

For $k = 1$ expression (9) gives the recursive formula for the Mersenne octonion i.e.

$$M\mathbb{O}_{n+1} = 3M\mathbb{O}_n - 2M\mathbb{O}_{n-1}, \quad n \geq 1,$$

where $M\mathbb{O}_0 = \sum_{r=0}^{7} M_r e_r$ and $M\mathbb{O}_1 = \sum_{r=0}^{7} M_{r+1} e_r$.

Theorem 2.1. *For any $n \in \mathbb{N} \cup \{0\}$, the norm of the n^{th} k-Mersenne octonion $M\mathbb{O}_{k,n}$ is*

$$N(M\mathbb{O}_{k,n}) = \sqrt{\frac{\lambda_1^{2n}(1 + \lambda_1^2 + ... + \lambda_1^{14}) + \lambda_2^{2n}(1 + \lambda_2^2 + ... + \lambda_2^{14}) - 255.2^{n+1}}{9k^2 - 8}}.$$

Proof. From (3), we have

$$\begin{aligned} N^2(M\mathbb{O}_{k,n}) &= \sum_{r=0}^{7} M_{k,n+r}^2 \\ &= \sum_{r=0}^{7} \left(\frac{\lambda_1^{n+r} - \lambda_2^{n+r}}{\lambda_1 - \lambda_2}\right)^2 \\ &= \frac{\lambda_1^{2n}(1 + \lambda_1^2 + ... + \lambda_1^{14}) + \lambda_2^{2n}(1 + \lambda_2^2 + ... + \lambda_2^{14}) - 255.2^{n+1}}{9k^2 - 8}. \end{aligned}$$

□

Theorem 2.2. *The closed form formula of the k-Mersenne octonions is given as*

$$M\mathbb{O}_{k,n} = \frac{\alpha\lambda_1^n - \beta\lambda_2^n}{\sqrt{9k^2 - 8}}, \tag{11}$$

where $\alpha = \sum_{r=0}^{7} \lambda_1^r e_r$ and $\beta = \sum_{r=0}^{7} \lambda_2^r e_r$.

Proof. By using the Binet formula of k-Mersenne numbers sequences (6) in the Definition 2.1, we get

$$\begin{aligned} M\mathbb{O}_{k,n} &= \sum_{r=0}^{7} \left(\frac{\lambda_1^{n+r} - \lambda_2^{n+r}}{\lambda_1 - \lambda_2}\right) e_r \\ &= \frac{1}{\lambda_1 - \lambda_2} \left(\lambda_1^n \sum_{r=0}^{7} \lambda_1^r e_r - \lambda_2^n \sum_{r=0}^{7} \lambda_2^r e_r\right) \\ &= \frac{\alpha\lambda_1^n - \beta\lambda_2^n}{\sqrt{9k^2 - 8}}, \end{aligned}$$

where $\alpha = \sum_{r=0}^{7} \lambda_1^r e_r$ and $\beta = \sum_{r=0}^{7} \lambda_2^r e_r$. □

With the help of this closed form formula, we give several identities of k-Mersenne octonions given in the following theorems. Throughout the paper, we use $\alpha = \sum_{r=0}^{7} \lambda_1^r e_r$ and $\beta = \sum_{r=0}^{7} \lambda_2^r e_r$. Note that $\mathbb{O}$ is a non-commutative algebra and hence $\alpha\beta \neq \beta\alpha$.

Theorem 2.3. *[Catalan's Identity] For $n, r \in \mathbb{N}$ such that $n \geq r$, we have*

(1) $M\mathbb{O}_{k,n+r} M\mathbb{O}_{k,n-r} - M\mathbb{O}_{k,n}^2 = \dfrac{2^{n-r}[\alpha\beta(2^r - \lambda_1^{2r}) + \beta\alpha(2^r - \lambda_2^{2r})]}{9k^2 - 8}$,

(2) $M\mathbb{O}_{k,n-r} M\mathbb{O}_{k,n+r} - M\mathbb{O}_{k,n}^2 = \dfrac{2^{n-r}[\alpha\beta(2^r - \lambda_2^{2r}) + \beta\alpha(2^r - \lambda_1^{2r})]}{9k^2 - 8}$.

Proof (1). Using (11) in the LHS, we write

$$\begin{aligned} M\mathbb{O}_{k,n+r} M\mathbb{O}_{k,n-r} - M\mathbb{O}_{k,n}^2 &= \left(\frac{\alpha\lambda_1^{n+r} - \beta\lambda_2^{n+r}}{\sqrt{9k^2 - 8}}\right)\left(\frac{\alpha\lambda_1^{n-r} - \beta\lambda_2^{n-r}}{\sqrt{9k^2 - 8}}\right) - \left(\frac{\alpha\lambda_1^n - \beta\lambda_2^n}{\sqrt{9k^2 - 8}}\right)^2 \\ &= \frac{\alpha\beta\lambda_1^n\lambda_2^n + \beta\alpha\lambda_1^n\lambda_2^n - \alpha\beta\lambda_1^{n+r}\lambda_2^{n-r} - \beta\alpha\lambda_1^{n-r}\lambda_2^{n+r}}{9k^2 - 8} \\ &= \frac{2^n[\alpha\beta(1 - \lambda_1^r\lambda_2^{-r}) + \beta\alpha(1 - \lambda_1^{-r}\lambda_2^r)]}{9k^2 - 8} \\ &= \frac{2^{n-r}[\alpha\beta(2^r - \lambda_1^{2r}) + \beta\alpha(2^r - \lambda_2^{2r})]}{9k^2 - 8}. \end{aligned}$$

The argument for identity (2) is similar to (1). □

Theorem 2.4. *For $n \geq 1$, we have*

(1) $M\mathbb{O}_{k,n+1}M\mathbb{O}_{k,n-1} - M\mathbb{O}_{k,n}^2 = \dfrac{2^{n-1}[\alpha\beta(2-\lambda_1^2)+\beta\alpha(2-\lambda_2^2)]}{9k^2-8}$,

(2) $M\mathbb{O}_{k,n-1}M\mathbb{O}_{k,n+1} - M\mathbb{O}_{k,n}^2 = \dfrac{2^{n-1}[\alpha\beta(2-\lambda_2^2)+\beta\alpha(2-\lambda_1^2)]}{9k^2-8}$.

Proof. For $r = 1$ in the Catalan's identity given in the Theorem 2.3, we get the above results. □

Theorem 2.5 (d'Ocagne's Identity). *Let n and r be any nonnegative integers, then the d'Ocagne's identity for k-Mersenne octonions is given by*

$$M\mathbb{O}_{k,r}M\mathbb{O}_{k,n+1} - M\mathbb{O}_{k,r+1}M\mathbb{O}_{k,n} = \frac{\alpha\beta\lambda_1^r\lambda_2^n - \beta\alpha\lambda_1^n\lambda_2^r}{\sqrt{9k^2-8}}.$$

Proof. From Binet formula (11), we have

$$\begin{aligned} M\mathbb{O}_{k,r}M\mathbb{O}_{k,n+1} - M\mathbb{O}_{k,r+1}M\mathbb{O}_{k,n} &= \left(\frac{\alpha\lambda_1^r - \beta\lambda_2^r}{\sqrt{9k^2-8}}\right)\left(\frac{\alpha\lambda_1^{n+1} - \beta\lambda_2^{n+1}}{\sqrt{9k^2-8}}\right) \\ &- \left(\frac{\alpha\lambda_1^{r+1} - \beta\lambda_2^{r+1}}{\sqrt{9k^2-8}}\right)\left(\frac{\alpha\lambda_1^{n} - \beta\lambda_2^{n}}{\sqrt{9k^2-8}}\right) \\ &= \frac{\alpha\beta\lambda_1^{r+1}\lambda_2^n + \beta\alpha\lambda_1^n\lambda_2^{r+1} - \alpha\beta\lambda_1^r\lambda_2^{n+1} - \beta\alpha\lambda_1^{n+1}\lambda_2^r}{9k^2-8} \\ &= \frac{\alpha\beta\lambda_1^r\lambda_2^n(\lambda_1-\lambda_2) - \beta\alpha\lambda_1^n\lambda_2^r(\lambda_1-\lambda_2)}{9k^2-8} \\ &= \frac{\alpha\beta\lambda_1^r\lambda_2^n - \beta\alpha\lambda_1^n\lambda_2^r}{\sqrt{9k^2-8}}. \end{aligned}$$

As required. □

Theorem 2.6 (Vajda's Identity). *Let n, r & s be any non-negative integers then we have*

$$M\mathbb{O}_{k,n+r}M\mathbb{O}_{k,n+s} - M\mathbb{O}_{k,n}M\mathbb{O}_{k,n+r+s} = \frac{2^n M_{k,r}[\beta\alpha\lambda_1^s - \alpha\beta\lambda_2^s]}{\sqrt{9k^2-8}}.$$

Proof. By using the Binet formula for the k-Mersenne octonions, we have
$M\mathbb{O}_{k,n+r}M\mathbb{O}_{k,n+s} - M\mathbb{O}_{k,n}M\mathbb{O}_{k,n+r+s}$

$$\begin{aligned} &= \left(\frac{\alpha\lambda_1^{n+r} - \beta\lambda_2^{n+r}}{\sqrt{9k^2-8}}\right)\left(\frac{\alpha\lambda_1^{n+s} - \beta\lambda_2^{n+s}}{\sqrt{9k^2-8}}\right) \\ &\quad - \left(\frac{\alpha\lambda_1^{n} - \beta\lambda_2^{n}}{\sqrt{9k^2-8}}\right)\left(\frac{\alpha\lambda_1^{n+r+s} - \beta\lambda_2^{n+r+s}}{\sqrt{9k^2-8}}\right) \\ &= \frac{\alpha\beta\lambda_1^n\lambda_2^{n+r+s} + \beta\alpha\lambda_1^{n+r+s}\lambda_2^n - \alpha\beta\lambda_1^{n+r}\lambda_2^{n+s} - \beta\alpha\lambda_1^{n+s}\lambda_2^{n+r}}{9k^2-8} \\ &= \frac{(\lambda_1\lambda_2)^n[\alpha\beta\lambda_2^s(\lambda_2^r - \lambda_1^r) + \beta\alpha\lambda_1^s(\lambda_1^r - \lambda_2^r)]}{9k^2-8} \\ &= \frac{2^n M_{k,r}[\beta\alpha\lambda_1^s - \alpha\beta\lambda_2^s]}{\sqrt{9k^2-8}}. \end{aligned}$$

As required. □

Theorem 2.7. *The ordinary and exponential generating function for the k-Mersenne octonions are given, respectively, as*

(1) $\sum_{n=0}^{\infty} M\mathbb{O}_{k,n}x^n = \dfrac{M\mathbb{O}_{k,0} + x\left(M\mathbb{O}_{k,1} - 3kM\mathbb{O}_{k,0}\right)}{1 - 3x + 2x^2}$,

(2) $\sum_{n=0}^{\infty} \dfrac{M\mathbb{O}_{k,n}x^n}{n!} = \dfrac{\alpha e^{\lambda_1 x} - \beta e^{\lambda_2 x}}{\sqrt{9k^2-8}}$.

Proof (1). Let $gM\mathbb{O}(x)$ be the ordinary generating function for the k-Mersenne octonion $\{M\mathbb{O}_{k,n}\}_{n\geq 0}$, i.e.

$$gM\mathbb{O}(x) = \sum_{n=0}^{\infty} M\mathbb{O}_{k,n}x^n.$$

Now using the closed form formula (11), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} M\mathbb{O}_{k,n}x^n &= \sum_{n=0}^{\infty}\left(\frac{\alpha\lambda_1^n - \beta\lambda_2^n}{\sqrt{9k^2-8}}\right)x^n \\
&= \frac{1}{\sqrt{9k^2-8}}\left[\alpha\sum_{n=0}^{\infty}(\lambda_1 x)^n - \beta\sum_{n=0}^{\infty}(\lambda_2 x)^n\right] \\
&= \frac{1}{\sqrt{9k^2-8}}\left[\alpha\left(\frac{1}{1-\lambda_1 x}\right) - \beta\left(\frac{1}{1-\lambda_2 x}\right)\right] \\
&= \frac{1}{\sqrt{9k^2-8}}\left[\frac{(\alpha-\beta) + x\left(\beta\lambda_1 - \alpha\lambda_2\right)}{1-3kx+2x^2}\right] \\
&= \frac{M\mathbb{O}_{k,0} + x\left(M\mathbb{O}_{k,1} - 3kM\mathbb{O}_{k,0}\right)}{1-3kx+2x^2}.
\end{aligned}$$

Proof of (*2*) is same as of (*1*), so we omit it. □

Theorem 2.8. *For $k \neq 1$, the finite sum formula for k-Mersenne octonions is given by,*

$$\sum_{s=0}^{n} M\mathbb{O}_{k,s} = \frac{2M\mathbb{O}_{k,n} - M\mathbb{O}_{k,n+1} + M\mathbb{O}_{k,1} + M\mathbb{O}_{k,0}(1-3k)}{3(1-k)}.$$

Proof. Using the Binet formula in LHS, we write

$$\begin{aligned}
\sum_{s=0}^{n} M\mathbb{O}_{k,s} &= \sum_{s=0}^{n}\left(\frac{\alpha\lambda_1^s - \beta\lambda_2^s}{\sqrt{9k^2-8}}\right) \\
&= \frac{1}{\sqrt{9k^2-8}}\left[\alpha\sum_{s=0}^{n}\lambda_1^s - \beta\sum_{s=0}^{n}\lambda_2^s\right] \\
&= \frac{1}{\sqrt{9k^2-8}}\left[\alpha\left(\frac{\lambda_1^{n+1}-1}{\lambda_1-1}\right) - \beta\left(\frac{\lambda_2^{n+1}-1}{\lambda_2-1}\right)\right] \\
&= \frac{\lambda_1\lambda_2(\alpha\lambda_1^n - \beta\lambda_2^n) - (\alpha\lambda_1^{n+1} - \beta\lambda_2^{n+1}) + (\beta\lambda_1 - \alpha\lambda_2) + (\alpha-\beta)}{\sqrt{9k^2-8}(\lambda_1\lambda_2 - (\lambda_1+\lambda_2) + 1)} \\
&= \frac{2M\mathbb{O}_{k,n} - M\mathbb{O}_{k,n+1} + M\mathbb{O}_{k,1} + M\mathbb{O}_{k,0}(1-3k)}{3(1-k)}.
\end{aligned}$$

As required. □

2.1. **k-Mersenne-Lucas Octonions.** Now, we define the k-Mersenne-Lucas Octonions and present their properties like the Binet formula, generating function, combinatorial identities, etc.

Definition 2.2. *For $n \geq 0$, the n^{th} k-Mersenne-Lucas octonion $m\mathbb{O}_{k,n}$ is defined as*

$$m\mathbb{O}_{k,n} = \sum_{r=0}^{7} m_{k,n+r} e_r, \tag{12}$$

where $m_{k,n}$ is the n^{th} k-Mersenne-Lucas number.

Using expression (5) in the Definition 2.2 and after some basic calculations gives the recurrence relation for the k-Mersenne-Lucas octonions as follows:

$$m\mathbb{O}_{k,n+1} = 3km\mathbb{O}_{k,n} - 2m\mathbb{O}_{k,n-1}, \quad n \geq 1, \tag{13}$$

where $m\mathbb{O}_{k,0} = \sum_{r=0}^{7} m_{k,r} e_r$ and $m\mathbb{O}_{k,1} = \sum_{r=0}^{7} m_{k,r+1} e_r$.
Note that for $k = 1$, we have the definition of the Mersenne-Lucas octonion given recursively by

$$m\mathbb{O}_{n+1} = 3m\mathbb{O}_n - 2m\mathbb{O}_{n-1}, \quad n \geq 1, \tag{14}$$

where $m\mathbb{O}_0 = \sum_{r=0}^{7} m_r e_r$ and $m\mathbb{O}_1 = \sum_{r=0}^{7} m_{r+1} e_r$.

The conjugate of the k-Mersenne-Lucas octonion $m\mathbb{O}_{k,n}$ can be written as

$$\overline{m\mathbb{O}}_{k,n} = m_{k,0} - \sum_{r=1}^{7} m_{k,n+r} e_r. \tag{15}$$

Theorem 2.9. *For $n \geq 0$, the norm of the n^{th} k-Mersenne-Lucas octonion $m\mathbb{O}_{k,n}$ is*

$$N(m\mathbb{O}_{k,n}) = \sqrt{\lambda_1^{2n}(1 + \lambda_1^2 + ... + \lambda_1^{14}) + \lambda_2^{2n}(1 + \lambda_2^2 + ... + \lambda_2^{14}) + 255.2^{n+1}}. \tag{16}$$

Proof. By the definition of norm, we have

$$\begin{aligned} N^2(m\mathbb{O}_{k,n}) &= \sum_{r=0}^{7} m_{k,n+r}^2 \\ &= \sum_{r=0}^{7} \left(\lambda_1^{n+r} + \lambda_2^{n+r}\right)^2 \\ &= \lambda_1^{2n}(1 + \lambda_1^2 + ... + \lambda_1^{14}) + \lambda_2^{2n}(1 + \lambda_2^2 + ... + \lambda_2^{14}) + 255.2^{n+1}. \end{aligned}$$

As required. □

Theorem 2.10. *The closed form formula of the k-Mersenne-Lucas octonions is given as*

$$m\mathbb{O}_{k,n} = \alpha\lambda_1^n + \beta\lambda_2^n, \tag{17}$$

where $\alpha = \sum_{r=0}^{7} \lambda_1^r e_r$ and $\beta = \sum_{r=0}^{7} \lambda_2^r e_r$.

Proof. By using the Binet formula for k-Mersenne-Lucas in the Definition 2.2, we get

$$\begin{aligned} m\mathbb{O}_{k,n} &= \sum_{r=0}^{7} \left(\lambda_1^{n+r} + \lambda_2^{n+r}\right) e_r \\ &= \left(\lambda_1^n \sum_{r=0}^{7} \lambda_1^r e_r + \lambda_2^n \sum_{r=0}^{7} \lambda_2^r e_r\right) \\ &= \alpha\lambda_1^n + \beta\lambda_2^n, \end{aligned}$$

where $\alpha = \sum_{r=0}^{7} \lambda_1^r e_r$ and $\beta = \sum_{r=0}^{7} \lambda_2^r e_r$. □

Theorem 2.11 (Catalan's Identity)**.** *For $n, r \in \mathbb{N}$ such that $n \geq r$, we have*

(1) $m\mathbb{O}_{k,n+r}m\mathbb{O}_{k,n-r} - m\mathbb{O}_{k,n}^2 = 2^{n-r}[\alpha\beta(\lambda_1^{2r} - 2^r) + \beta\alpha(\lambda_2^{2r} - 2^r)]$,
(2) $m\mathbb{O}_{k,n-r}m\mathbb{O}_{k,n+r} - m\mathbb{O}_{k,n}^2 = 2^{n-r}[\alpha\beta(\lambda_2^{2r} - 2^r) + \beta\alpha(\lambda_1^{2r} - 2^r)]$.

Proof. Using the Binet formula for k-Mersenne-Lucas octonions, we write

$$\begin{aligned} m\mathbb{O}_{k,n+r}m\mathbb{O}_{k,n-r} - m\mathbb{O}_{k,n}^2 &= \left(\alpha\lambda_1^{n+r} + \beta\lambda_2^{n+r}\right)\left(\alpha\lambda_1^{n-r} + \beta\lambda_2^{n-r}\right) - (\alpha\lambda_1^n + \beta\lambda_2^n)^2 \\ &= \alpha\beta\lambda_1^{n+r}\lambda_2^{n-r} + \beta\alpha\lambda_1^{n-r}\lambda_2^{n+r} - \alpha\beta\lambda_1^n\lambda_2^n - \beta\alpha\lambda_1^n\lambda_2^n \\ &= 2^n[\alpha\beta(\lambda_1^r\lambda_2^{-r} - 1) + \beta\alpha(\lambda_1^{-r}\lambda_2^r - 1)] \\ &= 2^{n-r}[\alpha\beta(\lambda_1^{2r} - 2^r) + \beta\alpha(\lambda_2^{2r} - 2^r)]. \end{aligned}$$

By a similar argument, (2) can be proved so we omit it. □

Theorem 2.12. *For $n \geq 1$, the Cassini's identity is given as*
(1) $m\mathbb{O}_{k,n+1}m\mathbb{O}_{k,n-1} - m\mathbb{O}_{k,n}^2 = 2^{n-1}[\alpha\beta(\lambda_1^2 - 2) + \beta\alpha(\lambda_2^2 - 2)]$,
(2) $m\mathbb{O}_{k,n-1}m\mathbb{O}_{k,n+1} - m\mathbb{O}_{k,n}^2 = 2^{n-1}[\alpha\beta(\lambda_2^2 - 2) + \beta\alpha(\lambda_1^2 - 2)]$.

Proof. The results can be established by substituting $r = 1$ in the Catalan's identity given in the Theorem 2.11. □

Theorem 2.13. *Let n, r be any nonnegative integers, then d'Ocagne's identity for k-Mersenne-Lucas octonions is given by*

$$m\mathbb{O}_{k,r}m\mathbb{O}_{k,n+1} - m\mathbb{O}_{k,r+1}m\mathbb{O}_{k,n} = (\sqrt{9k^2 - 8})(\beta\alpha\lambda_1^n\lambda_2^r - \alpha\beta\lambda_1^r\lambda_2^n).$$

Proof. By the Binet formula (17), we have

$$\begin{aligned} m\mathbb{O}_{k,r}m\mathbb{O}_{k,n+1} - m\mathbb{O}_{k,r+1}m\mathbb{O}_{k,n} &= (\alpha\lambda_1^r + \beta\lambda_2^r)\left(\alpha\lambda_1^{n+1} + \beta\lambda_2^{n+1}\right) \\ &\quad - \left(\alpha\lambda_1^{r+1} + \beta\lambda_2^{r+1}\right)(\alpha\lambda_1^n + \beta\lambda_2^n) \\ &= \alpha\beta\lambda_1^r\lambda_2^{n+1} + \beta\alpha\lambda_1^{n+1}\lambda_2^r - \alpha\beta\lambda_1^{r+1}\lambda_2^n - \beta\alpha\lambda_1^n\lambda_2^{r+1} \\ &= \alpha\beta\lambda_1^r\lambda_2^n(\lambda_2 - \lambda_1) + \beta\alpha\lambda_1^n\lambda_2^r(\lambda_1 - \lambda_2) \\ &= (\sqrt{9k^2 - 8})(\beta\alpha\lambda_1^n\lambda_2^r - \alpha\beta\lambda_1^r\lambda_2^n). \end{aligned}$$

As required. □

Theorem 2.14. *Let n, r and s be any non-negative integers then the Vajda identity is given as*

$$m\mathbb{O}_{k,n+r}m\mathbb{O}_{k,n+s} - m\mathbb{O}_{k,n}m\mathbb{O}_{k,n+r+s} = 2^n M_{k,r}(\sqrt{9k^2 - 8})(\alpha\beta\lambda_2^s - \beta\alpha\lambda_1^s).$$

Proof. By the Binet formula (17), we have

$$\begin{aligned} m\mathbb{O}_{k,n+r}m\mathbb{O}_{k,n+s} - m\mathbb{O}_{k,n}m\mathbb{O}_{k,n+r+s} &= \left(\alpha\lambda_1^{n+r} + \beta\lambda_2^{n+r}\right)\left(\alpha\lambda_1^{n+s} + \beta\lambda_2^{n+s}\right) \\ &\quad - (\alpha\lambda_1^n + \beta\lambda_2^n)\left(\alpha\lambda_1^{n+r+s} + \beta\lambda_2^{n+r+s}\right) \\ &= \alpha\beta\lambda_1^{n+r}\lambda_2^{n+s} + \beta\alpha\lambda_1^{n+s}\lambda_2^{n+r} - \alpha\beta\lambda_1^n\lambda_2^{n+r+s} \\ &\quad - \beta\alpha\lambda_1^{n+r+s}\lambda_2^n \\ &= (\lambda_1\lambda_2)^n[\alpha\beta\lambda_2^s(\lambda_1^r - \lambda_2^r) + \beta\alpha\lambda_1^s(\lambda_2^r - \lambda_1^r)] \\ &= 2^n M_{k,r}(\sqrt{9k^2 - 8})(\alpha\beta\lambda_2^s - \beta\alpha\lambda_1^s). \end{aligned}$$

As required. □

Theorem 2.15. *The ordinary and exponential generating function for the k-Mersenne-Lucas octonions are given, respectively, as*

(1) $\sum_{n=0}^{\infty} m\mathbb{O}_{k,n}x^n = \dfrac{m\mathbb{O}_{k,0} + x\,(m\mathbb{O}_{k,1} - 3km\mathbb{O}_{k,0})}{1 - 3kx + 2x^2}$,

(2) $\sum_{n=0}^{\infty} \dfrac{m\mathbb{O}_{k,n}x^n}{n!} = \alpha e^{\lambda_1 x} + \beta e^{\lambda_2 x}$.

Proof. Let $gm\mathbb{O}(x) = \sum_{n=0}^{\infty} m\mathbb{O}_{k,n}x^n$ be the ordinary generating function for the k-Mersenne-Lucas octonion $m\mathbb{O}_{k,n}$. Now using the closed form formula (17), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} m\mathbb{O}_{k,n}x^n &= \sum_{n=0}^{\infty} \left(\alpha\lambda_1^n + \beta\lambda_2^n\right) x^n \\
&= \alpha\sum_{n=0}^{\infty}(\lambda_1 x)^n + \beta\sum_{n=0}^{\infty}(\lambda_2 x)^n \\
&= \alpha\left(\frac{1}{1-\lambda_1 x}\right) + \beta\left(\frac{1}{1-\lambda_2 x}\right) \\
&= \frac{(\alpha+\beta) - x\left(\beta\lambda_1 + \alpha\lambda_2\right)}{1-3kx+2x^2} \\
&= \frac{m\mathbb{O}_{k,0} + x\left(m\mathbb{O}_{k,1} - 3km\mathbb{O}_{k,0}\right)}{1-3kx+2x^2}.
\end{aligned}$$

As required.
Proof of (*2*) is same as of (*1*), so we omit it. □

Theorem 2.16. *For $k \neq 1$, the finite sum formula for k-Mersenne-Lucas octonions is given by*

$$\sum_{s=0}^{n} m\mathbb{O}_{k,s} = \frac{2m\mathbb{O}_{k,n} - m\mathbb{O}_{k,n+1} + m\mathbb{O}_{k,1} + m\mathbb{O}_{k,0}(1-3k)}{3(1-k)}.$$

Proof. Using the Binet formula for k-Mersenne-Lucas octonions, we write

$$\begin{aligned}
\sum_{s=0}^{n} m\mathbb{O}_{k,s} &= \sum_{s=0}^{n}\left(\alpha\lambda_1^s + \beta\lambda_2^s\right) = \alpha\sum_{s=0}^{n}\lambda_1^s + \beta\sum_{s=0}^{n}\lambda_2^s \\
&= \alpha\left(\frac{\lambda_1^{n+1}-1}{\lambda_1 - 1}\right) + \beta\left(\frac{\lambda_2^{n+1}-1}{\lambda_2-1}\right) \\
&= \frac{\lambda_1\lambda_2(\alpha\lambda_1^n + \beta\lambda_2^n) - (\alpha\lambda_1^{n+1} + \beta\lambda_2^{n+1}) - (\alpha\lambda_2 + \beta\lambda_1) + (\alpha+\beta)}{\lambda_1\lambda_2 - (\lambda_1+\lambda_2) + 1} \\
&= \frac{2m\mathbb{O}_{k,n} - m\mathbb{O}_{k,n+1} + m\mathbb{O}_{k,1} + m\mathbb{O}_{k,0}(1-3k)}{3(1-k)}.
\end{aligned}$$

As required. □

For $k = 1$, some of the properties of the k-Mersenne and k-Mersenne-Lucas octonions are presented in [16].

3. Combined identity and Matrix representation

Now we give combined identities for the k-Mersenne and k-Mersenne-Lucas octonions then we present their matrix representation. Further, we give closed formula for these octonions viz determinant of tridiagonal matrices.

By following a similar argument to Theorem 2.8, for $k \neq \pm 1$ the finite sum of even and odd indexed k-Mersenne/k-Mersenne-Lucas octonions are, respectively

$$\sum_{s=0}^{n} \mathbb{O}_{k,2s} = \frac{4\mathbb{O}_{k,2n} - \mathbb{O}_{k,2(n+1)} + 3k\mathbb{O}_{k,1} - (9k^2-3)\mathbb{O}_{k,0}}{9(1-k^2)}$$

and

$$\sum_{s=0}^{n}\mathbb{O}_{k,2s+1}=\frac{4\mathbb{O}_{k,2n+1}-\mathbb{O}_{k,2n+3}+3\mathbb{O}_{k,1}-6k\mathbb{O}_{k,0}}{9(1-k^2)},$$

where either $\mathbb{O}_{k,n}=M\mathbb{O}_{k,n}$ or $m\mathbb{O}_{k,n}$.

Theorem 3.1. *The ordinary generating function for even and odd indexed k-Mersenne/k-Mersenne Lucas octonions are given by*

$$\sum_{n=0}^{\infty}\mathbb{O}_{k,2n}x^n=\frac{\mathbb{O}_{k,0}+x\left(3k\mathbb{O}_{k,1}+(2-9k^2)\mathbb{O}_{k,0}\right)}{1-5x+4x^2}$$

and

$$\sum_{n=0}^{\infty}\mathbb{O}_{k,2n+1}x^n=\frac{3k\mathbb{O}_{k,0}+\mathbb{O}_{k,1}+2x\left(\mathbb{O}_{k,1}-3k\mathbb{O}_{k,0}\right)}{1-5x+4x^2},$$

where either $\mathbb{O}_{k,n}=M\mathbb{O}_{k,n}$ *or* $m\mathbb{O}_{k,n}$.

Proof. The argument is very similar to Theorem 2.7. □

For any sequence $\{a_n\}$, let $\xi(x)=\sum_{n=0}^{\infty}\frac{a_nx^n}{n!}$ be the exponential generating functions then clearly it satisfy

$$\xi(x)+\xi(-x)=2\sum_{n=0}^{\infty}\frac{a_{2n}x^{2n}}{2n!}\quad\text{and}\quad\xi(x)-\xi(-x)=2x\sum_{n=0}^{\infty}\frac{a_{2n+1}x^{2n}}{(2n+1)!}.$$

Simplifying and replacing x with $\sqrt{x}$ gives the generating function for even and odd indexed sequence a_n as

$$\xi_{a_{2n}}(x)=\frac{\xi(\sqrt{x})+\xi(-\sqrt{x})}{2}=\sum_{n=0}^{\infty}\frac{a_{2n}x^n}{2n!}$$

and

$$\xi_{a_{2n+1}}(x)=\frac{\xi(\sqrt{x})-\xi(-\sqrt{x})}{2\sqrt{x}}=\sum_{n=0}^{\infty}\frac{a_{2n+1}x^n}{(2n+1)!}.$$

For k-Mersenne octonions, using Theorem 2.2 the exponential generating function $\xi(x)$ satisfy

$$\xi(x)\pm\xi(-x)=\frac{1}{\sqrt{9k^2-8}}\Big(\alpha(e^{\lambda_1x}\pm e^{-\lambda_1x})-\beta(e^{\lambda_2x}\pm e^{-\lambda_2x})\Big)$$

Using the trigonometric identity $\sinh(rx)=(e^{rx}-e^{-rx})/2$ and $\cosh(rx)=(e^{rx}+e^{-rx})/2$, we have

$$\xi(x)+\xi(-x)=\frac{2}{\sqrt{9k^2-8}}(\alpha\cosh\lambda_1x-\beta\cosh\lambda_2x)$$

and

$$\xi(x)-\xi(-x)=\frac{2}{\sqrt{9k^2-8}}(\alpha\sinh\lambda_1x-\beta\sinh\lambda_2x).$$

which gives

$$\xi_{M\mathbb{O}_{k,2n}}(x)=\frac{1}{\sqrt{9k^2-8}}(\alpha\cosh\lambda_1\sqrt{x}-\beta\cosh\lambda_2\sqrt{x})$$

$$\text{and}\quad\xi_{M\mathbb{O}_{k,2n+1}}(x)=\frac{1}{\sqrt{x}\sqrt{9k^2-8}}(\alpha\sinh\lambda_1\sqrt{x}-\beta\sinh\lambda_2\sqrt{x}).$$

Similarly, the exponential generating function for even and odd indexed k-Mersenne-Lucas octonions are, respectively

$$\begin{aligned} \xi_{m\mathbb{O}_{k,2n}}(x) &= (\alpha \cosh \lambda_1 \sqrt{x} + \beta \cosh \lambda_2 \sqrt{x}) \\ \text{and} \qquad \xi_{m\mathbb{O}_{k,2n+1}}(x) &= \frac{1}{\sqrt{x}}(\alpha \sinh \lambda_1 \sqrt{x} + \beta \sinh \lambda_2 \sqrt{x}). \end{aligned}$$

Theorem 3.2. *For positive integer n, let define the matrices $F^{(n)}, F^{(0)}$ and $G(k)$ as*

$$F^{(n)}(\mathbb{O}_{k,r}) = \begin{bmatrix} \mathbb{O}_{k,n+2} & \mathbb{O}_{k,n+1} \\ \mathbb{O}_{k,n+1} & \mathbb{O}_{k,n} \end{bmatrix}, \; F^{(0)}(\mathbb{O}_{k,r}) = \begin{bmatrix} \mathbb{O}_{k,2} & \mathbb{O}_{k,1} \\ \mathbb{O}_{k,1} & \mathbb{O}_{k,0} \end{bmatrix} \; \text{and } G(k) = \begin{bmatrix} 3k & -2 \\ 1 & 0 \end{bmatrix}.$$

Then we have

$$F^{(n)}(M\mathbb{O}_{k,r}) = (G(k))^n F^{(0)}(M\mathbb{O}_{k,r}) \quad \text{and} \quad F^{(n)}(m\mathbb{O}_{k,r}) = (G(k))^n F^{(0)}(m\mathbb{O}_{k,r}).$$

Proof. We prove the result using inductive hypothesis. For first identity (k-Mersenne octonion), for $n = 1$ the equality holds i.e. $G(k)F^{(0)}(M\mathbb{O}_{k,r}) = F^{(1)}(M\mathbb{O}_{k,r})$. Now we verify the fact for $n+1$ by assuming that the hypothesis is true for $n > 1$. We have

$$\begin{aligned} (G(k))^{n+1} F^{(0)}(M\mathbb{O}_{k,r}) &= G(k)(G(k))^n F^{(0)}(M\mathbb{O}_{k,r}) \\ &= G(k) F^{(n)}(M\mathbb{O}_{k,r}) \\ &= F^{(n+1)}(M\mathbb{O}_{k,r}). \end{aligned}$$

Thus the proof is completed and by a similar assertion the second identity (k-Mersenne-Lucas octonion) can be verified easily. □

From Theorem 3.2 we have $F^{(n)}(M\mathbb{O}_{k,r}) = (G(k))^n F^{(0)}(M\mathbb{O}_{k,r})$ and taking determinant on both sides give the Simson identity for Mersenne/Mersenne-Lucas octonions given by following identities:

$$\begin{aligned} M\mathbb{O}_{k,n+1} M\mathbb{O}_{k,n-1} - (M\mathbb{O}_{k,n})^2 &= 2^{n-1}(M\mathbb{O}_{k,2} M\mathbb{O}_{k,0} - (M\mathbb{O}_{k,1})^2), \\ M\mathbb{O}_{k,n-1} M\mathbb{O}_{k,n+1} - (M\mathbb{O}_{k,n})^2 &= 2^{n-1}(M\mathbb{O}_{k,0} M\mathbb{O}_{k,2} - (M\mathbb{O}_{k,1})^2), \\ m\mathbb{O}_{k,n+1} m\mathbb{O}_{k,n-1} - (m\mathbb{O}_{k,n})^2 &= 2^{n-1}(M\mathbb{O}_{k,2} m\mathbb{O}_{k,0} - (m\mathbb{O}_{k,1})^2), \\ m\mathbb{O}_{k,n-1} m\mathbb{O}_{k,n+1} - (m\mathbb{O}_{k,n})^2 &= 2^{n-1}(m\mathbb{O}_{k,0} m\mathbb{O}_{k,2} - (m\mathbb{O}_{k,1})^2). \end{aligned}$$

The nth term of Mersenne octonions ($M\mathbb{O}_{k,n}$) can also be expressed by the determinant of the tridiagonal matrix defined as follow:

$$T_{k,n} = \begin{bmatrix} M\mathbb{O}_{k,2} & M\mathbb{O}_{k,1} & & & & \\ 2 & 3k & 1 & & & \\ & 2 & 3k & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 2 & 3k & 1 \\ & & & & 2 & 3k \end{bmatrix}_{n \times n},$$

where determinant of $T_{k,n}$ satisfy

$$det(T_{k,n}) = M\mathbb{O}_{k,n+1}. \tag{18}$$

Similarly, if we replace the first row of the above tridiagonal matrix with $[m\mathbb{O}_{k,2}, m\mathbb{O}_{k,1}, 0, 0, \ldots, 0]$ then its determinant gives the $(n+1)$th term of the k-Mersenne Lucas octonions i.e. $det(T_{k,n}) = m\mathbb{O}_{k,n+1}$.

The above identity can be also verified with the hypothesis of mathematical induction on n.

4. Conclusion

In our present study, we have defined the octonions involving the k-Mersenne and k-Mersenne-Lucas sequences and obtained the closed form formulas of these octonions. Moreover, we have presented various results including norm, generating functions, Catalan's identity, Simson identity, d'Ocagne's identity, Vajda's identity, and the finite sum formula of these octonions. Lastly, we have studied the combined identities and matrix representation for these octonions.

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