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SOME NEW FIXED POINT RESULTS FOR SINGLE VALUED MAPPINGS IN b-METRIC SPACES

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Abstract. In recent investigations, the study of coincidence points and common fixed points is a new development in the domain of contractive type single valued theory. In this paper, we introduce the concepts of some new generalized contractive type mappings in b-metric spaces and discuss the existence and uniqueness of their fixed points. As some applications of this study, we obtain some coincidence point and common fixed point results for a pair of single valued mappings in b-metric spaces. Our results generalize, extend and unify several well known comparable results in the existing literature. Finally, we give some examples to justify the validity of our results.

Keywords: b-metric, weakly compatible mappings, coincidence point, fixed point.

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1. Introduction

It is well-known that Banach contraction principle [4] is one of the most important theorems in classical functional analysis. Its significance lies in its vast applicability in different branches of mathematics and applied sciences. In 1922, S. Banach [4] started a new field in mathematics, called fixed point theory. In fact, that was the starting point to start generalize his result by changing the contraction condition or by generalizing the underlying metric space; see [8, 13, 14, 15, 20]. In 1989, Bakhtin [3] introduced the concept of b-metric spaces, an extension of metric spaces and generalized the famous Banach contraction principle in metric spaces to b-metric spaces. After that, many research works were conducted on b-metric spaces under different contractive conditions. Starting from these considerations, the study of coincidence points and common fixed points of mappings satisfying a certain contractive type condition attracted many researchers, see for examples [12, 17, 18, 19]. Very recently, Hieu and Hung [10] proved some fixed point results for single valued and multi-valued weakly Picard operators in complete metric spaces. Inspired and motivated by the results in [10], we introduce the concepts of some new generalized contractive type mappings in b-metric spaces and prove some fixed point results for single valued mappings satisfying such contractive conditions. Moreover, we

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give some examples to illustrate our results. As some consequences, we obtain several coincidence point results in the framework of b-metric spaces.

2. Some Basic Concepts

In this section, we recall some basic known definitions, notations and results in b-metric spaces which will be used in the sequel.

Definition 2.1. [7] *Let X be a nonempty set and $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a b-metric on X if the following conditions hold:*

(i) $d(x, y) = 0$ *if and only if* $x = y$;
(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;
(iii) $d(x, y) \leq s\,(d(x, z) + d(z, y))$ *for all* $x, y, z \in X$.

The pair (X, d) is called a b-metric space.

It is to be noted that the class of b-metric spaces is effectively larger than that of the ordinary metric spaces. The following example illustrates the above fact.

Example 2.1. [16] *Let $X = \{-1, 0, 1\}$. Define $d : X \times X \to [0, \infty)$ by $d(x, y) = d(y, x)$ for all $x, y \in X$, $d(x, x) = 0$, $x \in X$ and $d(-1, 0) = 3$, $d(-1, 1) = d(0, 1) = 1$. Then (X, d) is a b-metric space, but not a metric space since the triangle inequality is not satisfied. Indeed, we have that*

$$d(-1, 1) + d(1, 0) = 1 + 1 = 2 < 3 = d(-1, 0).$$

It is easy to verify that $s = \frac{3}{2}$.

Example 2.2. [5] *Let $p \in (0, 1)$. Then the set $l^p(\mathbb{R}) := \{(x_n) \subseteq \mathbb{R} : \sum_{n=1}^{\infty} \mid x_n \mid^p < \infty\}$ endowed with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$ given by*

$$d((x_n), (y_n)) = \left(\sum_{n=1}^{\infty} \mid x_n - y_n \mid^p \right)^{\frac{1}{p}}$$

for all (x_n), $(y_n) \in l^p(\mathbb{R})$ is a b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 2.2. [6] *Let (X, d) be a b-metric space, $x \in X$ and (x_n) be a sequence in X. Then*

(i) (x_n) *converges to x if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$. *We denote this by* $\lim_{n\to\infty} x_n = x$ *or* $x_n \to x (n \to \infty)$.
(ii) (x_n) *is Cauchy if and only if* $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) *is complete if and only if every Cauchy sequence in X is convergent.*

Remark 2.1. [6] *In a b-metric space (X, d), the following assertions hold:*

(i) *A convergent sequence has a unique limit.*
(ii) *Each convergent sequence is Cauchy.*
(iii) *In general, a b-metric is not continuous.*

Definition 2.3. [11] *Let (X, d) be a b-metric space. A subset $A \subseteq X$ is said to be open if and only if for any $a \in A$, there exists $\epsilon > 0$ such that the open ball $B(a, \epsilon) \subseteq A$. The family of all open subsets of X will be denoted by τ.*

Theorem 2.1. [11] *τ defines a topology on (X, d).*

Theorem 2.2. [2] *Let (X, d) be a b-metric space and suppose that (x_n) and (y_n) converge to x, $y \in X$, respectively. Then, we have*

$$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Moreover, for each $z \in X$, we have

$$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$$

3. Main Results

In this section, we present some fixed point results for single valued mappings in b-metric spaces.

Theorem 3.1. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and f be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \text{ implies } d(fx,fy) \leq M_{sf}(x,y,\alpha)\, d(x,y) \tag{1}$$

for all x, $y \in X$, where $M_{sf}(x,y,\alpha) = \frac{\frac{1}{s}d(x,fy)+d(y,fx)+d(x,y)}{2d(x,fx)+d(y,fy)+\alpha}$. If there exists $x_0 \in X$ such that for $x_n = f^n x_0$ for all $n \in \mathbb{N}$ and $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{1}{s}$, then

(i) *f has at least one fixed point in X;*
(ii) *if u, $v \in X$ are two distinct fixed points, then $d(u, v) \geq \frac{\alpha s}{1+2s}$.*

Moreover, if $M_{sf}(x, y, \alpha) < 1$ for all x, $y \in X$, then f has a unique fixed point in X.

Proof. Suppose there exists $x_0 \in X$ such that for $x_n = f^n x_0$ for all $n \in \mathbb{N}$ and $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{1}{s}$. We note that

$$\frac{1}{2s}d(x_n,fx_n) = \frac{1}{2s}d(x_n,x_{n+1}) \leq d(x_n,x_{n+1}) \text{ for all } n \geq 0.$$

By using condition (1), we have

$$\begin{aligned}
d_{n+1} &= d(fx_n, fx_{n+1}) \\
&\leq M_{sf}(x_n,x_{n+1},\alpha)\, d(x_n,x_{n+1}) \\
&= \left[\frac{\frac{1}{s}d(x_n,fx_{n+1})+d(x_{n+1},fx_n)+d(x_n,x_{n+1})}{2d(x_n,fx_n)+d(x_{n+1},fx_{n+1})+\alpha}\right] d(x_n,x_{n+1}) \\
&= \left[\frac{\frac{1}{s}d(x_n,x_{n+2})+d(x_n,x_{n+1})}{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\alpha}\right] d(x_n,x_{n+1}) \\
&\leq \left[\frac{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})}{2d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\alpha}\right] d(x_n,x_{n+1}) \\
&= \left[\frac{2d_n+d_{n+1}}{2d_n+d_{n+1}+\alpha}\right] d_n \text{ for all } n \geq 0.
\end{aligned}$$

Let us put $\alpha_n = \frac{2d_n+d_{n+1}}{2d_n+d_{n+1}+\alpha}$ *for all* $n \geq 0$. Then, $0 \leq \alpha_n < 1$ with $s\alpha_0 < 1$ and $d_{n+1} \leq \alpha_n d_n$ for all $n \geq 0$. Now it is easy to compute that

$$d_n \leq d_{n-1} \text{ and } d_n \leq \alpha_{n-1}\alpha_{n-2}\cdots\alpha_0 d_0 \text{ for all } n \geq 1.$$

As the function $g(t)=\frac{t}{t+\alpha}$ is increasing on $[0,\infty)$, it follows that $\alpha_n \leq \alpha_{n-1}$ for all $n\geq 1$. Therefore, $0\leq d_n \leq \alpha_0^n d_0$. Taking limit as $n\to\infty$, we get $\lim\limits_{n\to\infty} d_n = 0$.

Since $s\alpha_0<1$, for m, $n\in\mathbb{N}$ with $m>n$, we have

$$\begin{aligned}
d(x_n,x_m) &\leq sd(x_n,x_{n+1})+s^2d(x_{n+1},x_{n+2})+\cdots\\
&\quad +s^{m-n-1}d(x_{m-2},x_{m-1})+s^{m-n-1}d(x_{m-1},x_m)\\
&\leq sd_n+s^2d_{n+1}+\cdots+s^{m-n-1}d_{m-2}+s^{m-n}d_{m-1}\\
&\leq [s\alpha_0^n+s^2\alpha_0^{n+1}+\cdots+s^{m-n-1}\alpha_0^{m-2}+s^{m-n}\alpha_0^{m-1}]\,d_0\\
&= s\alpha_0^n[1+s\alpha_0+s^2\alpha_0^2+\cdots+s^{m-n-2}\alpha_0^{m-n-2}+s^{m-n-1}\alpha_0^{m-n-1}]\,d_0\\
&\leq s\alpha_0^n[1+s\alpha_0+(s\alpha_0)^2+\cdots]\,d_0\\
&= \frac{s\alpha_0^n}{1-s\alpha_0}\,d_0\\
&\to 0 \ as\ n\to\infty.
\end{aligned}$$

Thus,

$$\lim_{n,\,m\to\infty} d(x_n,x_m)=0.$$

This gives that (x_n) is a Cauchy sequence in X. As (X,d) is complete, there exists $u\in X$ such that $\lim\limits_{n\to\infty} x_n = u$.

We now show that for any $n\geq 0$,

$$either\ \frac{1}{2s}d(x_n,fx_n)\leq d(x_n,u),\ or,\ \frac{1}{2s}d(fx_n,fx_{n+1})\leq d(fx_n,u). \tag{2}$$

If possible, suppose that for some $n\geq 0$, we have

$$d(x_n,u)<\frac{1}{2s}d(x_n,fx_n)\ and\ d(fx_n,u)<\frac{1}{2s}d(fx_n,fx_{n+1}).$$

Then,

$$\begin{aligned}
d_n=d(x_n,fx_n) &\leq s[d(x_n,u)+d(fx_n,u)]\\
&< \frac{1}{2}d(x_n,fx_n)+\frac{1}{2}d(fx_n,fx_{n+1})\\
&= \frac{1}{2}d_n+\frac{1}{2}d_{n+1}\\
&\leq d_n.
\end{aligned}$$

Thus we arrived at a contradiction. Thus, it follows from condition (2) that for every $n\geq 0$,

$$either\ d(x_{n+1},fu)\leq M_{sf}(x_n,u,\alpha)\,d(x_n,u),$$

$$or,\ d(x_{n+2},fu)\leq M_{sf}(x_{n+1},u,\alpha)\,d(x_{n+1},u).$$

This is equivalent with the fact that for every $n\geq 0$,

$$either\ d(x_{n+1},fu)\leq \left[\frac{\frac{1}{s}d(x_n,fu)+d(u,fx_n)+d(x_n,u)}{2d(x_n,fx_n)+d(u,fu)+\alpha}\right]d(x_n,u), \tag{3}$$

$$or,\ d(x_{n+2},fu)\leq \left[\frac{\frac{1}{s}d(x_{n+1},fu)+d(u,fx_{n+1})+d(x_{n+1},u)}{2d(x_{n+1},fx_{n+1})+d(u,fu)+\alpha}\right]d(x_{n+1},u) \tag{4}$$

holds true. Then at least one of the above inequalities holds for infinitely many natural numbers. Suppose (3) holds for infinitely many natural numbers and (n_k) is the corresponding increasing sequence of natural numbers. Thus, (x_{n_k}) is a subsequence of (x_n) and

$$\begin{aligned} d(x_{n_k+1}, fu) &\leq \left[\frac{\frac{1}{s}d(x_{n_k}, fu) + d(u, x_{n_k+1}) + d(x_{n_k}, u)}{2d(x_{n_k}, x_{n_k+1}) + d(u, fu) + \alpha}\right] d(x_{n_k}, u) \\ &\leq \left[\frac{d(x_{n_k}, u) + d(u, fu) + d(u, x_{n_k+1}) + d(x_{n_k}, u)}{2d_{n_k} + d(u, fu) + \alpha}\right] d(x_{n_k}, u). \end{aligned}$$

Passing to the limit as $k \to \infty$, we get $\lim\limits_{k\to\infty} x_{n_k+1} = fu$ and so $fu = u$. This proves that u is a fixed point of f. If (4) holds for infinitely many natural numbers, by an argument similar to that used above we can show that u is a fixed point of f.

Suppose that $v (\neq u)$ is another fixed point of f in X. Then, $0 = \frac{1}{2s}d(u, fu) \leq d(u, v)$ which implies that

$$\begin{aligned} d(u, v) &= d(fu, fv) \leq M_{sf}(u, v, \alpha)\, d(u, v) \\ &= \left[\frac{\frac{1}{s}d(u, fv) + d(v, fu) + d(u, v)}{2d(u, fu) + d(v, fv) + \alpha}\right] d(u, v) \\ &= \left[\frac{\frac{1}{s}d(u, v) + d(v, u) + d(u, v)}{2d(u, u) + d(v, v) + \alpha}\right] d(u, v) \\ &= \left[\frac{(1 + 2s)d(u, v)}{s\alpha}\right] d(u, v). \end{aligned}$$

This gives that $d(u, v) \geq \frac{s\alpha}{1+2s}$.

Finally, we suppose that $M_{sf}(x, y, \alpha) < 1$ for all $x, y \in X$. Let $w \in X$ be another fixed point of f. Then, $0 = \frac{1}{2s}d(u, fu) \leq d(u, w)$ which implies that

$$d(u, w) = d(fu, fw) \leq M_{sf}(u, w, \alpha)\, d(u, w).$$

This ensures that $d(u, w) = 0$ and hence $u = w$. This proves that f has a unique fixed point in X. □

Taking $s = 1$ in Theorem 3.1, we can obtain Theorem 2.1 of Hieu and Hung [10] as follows.

Theorem 3.2. *Let (X, d) be a complete metric space and f be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(x, fx) \leq d(x, y) \text{ implies } d(fx, fy) \leq M_f(x, y, \alpha)\, d(x, y)$$

for all $x, y \in X$, where $M_f(x, y, \alpha) = \frac{d(x,fy)+d(y,fx)+d(x,y)}{2d(x,fx)+d(y,fy)+\alpha}$. Then

(i) f has at least one fixed point in X;

(ii) if $u, v \in X$ are two distinct fixed points, then $d(u, v) \geq \frac{\alpha}{3}$.

Moreover, if $M_f(x, y, \alpha) < 1$ for all $x, y \in X$, then f has a unique fixed point in X.

Theorem 3.3. *Let (X, d) be a complete b-metric space with the coefficient $s \geq 1$ and f be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2s}d(x, fx) \leq d(x, y) \text{ implies } d(fx, fy) \leq \frac{1}{2s}M_{sf}(x, y, \alpha)\, [\frac{1}{s}d(x, fy) + d(y, fx)] \tag{5}$$

for all $x,\, y \in X$. *If there exists* $x_0 \in X$ *such that for* $x_n = f^n x_0$ *for all* $n \in \mathbb{N}$ *and* $d_n = d(x_n, x_{n+1})$ *for all* $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{2s}{1+s}$, *then*

(*i*) f *has at least one fixed point in* X;

(*ii*) *if* $u,\, v \in X$ *are two distinct fixed points, then* $d(u,v) \geq \frac{2s^3\alpha}{(1+s)(1+2s)}$.

Moreover, if $M_{sf}(x,y,\alpha) < \frac{2s^2}{1+s}$ *for all* $x,\, y \in X$, *then* f *has a unique fixed point in* X.

Proof. Suppose there exists $x_0 \in X$ such that for $x_n = f^n x_0$ for all $n \in \mathbb{N}$ and $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, we have $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{2s}{1+s}$. We note that

$$\frac{1}{2s} d(x_n, fx_n) = \frac{1}{2s} d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \text{ } for \text{ } all \text{ } n \geq 0.$$

By using condition (5), we have

$$\begin{aligned}
d_{n+1} &= d(fx_n, fx_{n+1}) \\
&\leq \frac{1}{2s} M_{sf}(x_n, x_{n+1}, \alpha)\, [\frac{1}{s} d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n)] \\
&\leq \frac{1}{2s} \left[\frac{\frac{1}{s} d(x_n, fx_{n+1}) + d(x_{n+1}, fx_n) + d(x_n, x_{n+1})}{2d(x_n, fx_n) + d(x_{n+1}, fx_{n+1}) + \alpha} \right] (d_n + d_{n+1}) \\
&= \frac{1}{2s} \left[\frac{\frac{1}{s} d(x_n, x_{n+2}) + d(x_n, x_{n+1})}{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \alpha} \right] (d_n + d_{n+1}) \\
&\leq \frac{1}{2s} \left[\frac{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \alpha} \right] (d_n + d_{n+1}) \\
&= \frac{1}{2s} \left[\frac{2d_n + d_{n+1}}{2d_n + d_{n+1} + \alpha} \right] (d_n + d_{n+1}) \text{ } for \text{ } all \text{ } n \geq 0. \qquad (6)
\end{aligned}$$

Let us put $c_n = \frac{1}{2} \frac{2d_n + d_{n+1}}{2d_n + d_{n+1} + \alpha}$ *for all* $n \geq 0$. Then, $0 \leq c_n < \frac{1}{2} \leq \frac{s}{2}$ which implies that $0 \leq \frac{c_n}{s - c_n} < 1$. Moreover, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{2s}{1+s} \Rightarrow c_0 < \frac{s}{1+s} \Rightarrow \frac{c_0}{s-c_0} < \frac{1}{s}$. Take $\alpha_n = \frac{c_n}{s-c_n}$ *for all* $n \geq 0$. Then, $0 \leq \alpha_n < 1$ with $\alpha_0 < \frac{1}{s}$.

We obtain from condition (6) that

$$d_{n+1} \leq \frac{c_n}{s} (d_n + d_{n+1}).$$

This gives that

$$d_{n+1} \leq \frac{c_n}{s - c_n} d_n = \alpha_n d_n \text{ } for \text{ } all \text{ } n \geq 0.$$

By the techniques that adapted in Theorem 3.1, we can show that $\lim_{n\to\infty} d_n = 0$ and (x_n) is a Cauchy sequence in X. As (X,d) is complete, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

We now show that for any $n \geq 0$,

$$either \text{ } \frac{1}{2s} d(x_n, fx_n) \leq d(x_n, u), \text{ } or, \text{ } \frac{1}{2s} d(fx_n, fx_{n+1}) \leq d(fx_n, u). \qquad (7)$$

If possible, suppose that for some $n \geq 0$, we have

$$d(x_n, u) < \frac{1}{2s} d(x_n, fx_n) \text{ } and \text{ } d(fx_n, u) < \frac{1}{2s} d(fx_n, fx_{n+1}).$$

Then,

$$\begin{aligned} d_n = d(x_n, fx_n) &\leq s[d(x_n, u) + d(fx_n, u)] \\ &< \frac{1}{2}d(x_n, fx_n) + \frac{1}{2}d(fx_n, fx_{n+1}) \\ &= \frac{1}{2}d_n + \frac{1}{2}d_{n+1} \\ &\leq d_n, \end{aligned}$$

which is a contradiction. Thus, it follows from condition (7) that for every $n \geq 0$,

$$\textit{either } d(x_{n+1}, fu) \leq \frac{1}{2s}\left[\frac{D_n + d(x_n, u)}{2d(x_n, fx_n) + d(u, fu) + \alpha}\right] D_n, \tag{8}$$

$$\textit{or, } d(x_{n+2}, fu) \leq \frac{1}{2s}\left[\frac{D_{n+1} + d(x_{n+1}, u)}{2d(x_{n+1}, fx_{n+1}) + d(u, fu) + \alpha}\right] D_{n+1} \tag{9}$$

holds true, where $D_n = \frac{1}{s}d(x_n, fu) + d(u, fx_n)$. Then at least one of the above inequalities holds for infinitely many natural numbers. Suppose (8) holds for infinitely many natural numbers and (n_k) is the corresponding increasing sequence of natural numbers. Thus, (x_{n_k}) is a subsequence of (x_n) and

$$d(x_{n_k+1}, fu) \leq \frac{1}{2s}\left[\frac{D_{n_k} + d(x_{n_k}, u)}{2d(x_{n_k}, x_{n_k+1}) + d(u, fu) + \alpha}\right] D_{n_k}.$$

We now compute that

$$\begin{aligned} d(u, fu) &\leq sd(u, x_{n_k+1}) + sd(x_{n_k+1}, fu) \\ &\leq sd(u, x_{n_k+1}) + \frac{1}{2}\left[\frac{D_{n_k} + d(x_{n_k}, u)}{2d_{n_k} + d(u, fu) + \alpha}\right] D_{n_k}. \end{aligned}$$

As $D_{n_k} \leq d(x_{n_k}, u) + d(u, fu) + d(u, x_{n_k+1})$, taking limit as $k \to \infty$, we get

$$d(u, fu) \leq \frac{1}{2}\left[\frac{d(u, fu}{d(u, fu) + \alpha}\right] d(u, fu). \tag{10}$$

If possible, suppose that $d(u, fu) \neq 0$. Then, it follows from (10) that $\frac{d(u,fu}{d(u,fu)+\alpha} \geq 2 > 1$, a contradiction. This ensures that $d(u, fu) = 0$ and so $fu = u$. This proves that u is a fixed point of f. If (9) holds for infinitely many natural numbers, by an argument similar to that used above we can show that u is a fixed point of f.

Suppose that $v(\neq u)$ is another fixed point of f in X. Then, $0 = \frac{1}{2s}d(u, fu) \leq d(u, v)$ which implies that

$$\begin{aligned} d(u, v) &= d(fu, fv) \leq \frac{1}{2s}M_{sf}(u, v, \alpha)\,[\frac{1}{s}d(u, fv) + d(v, fu)] \\ &= \frac{1}{2s}\left[\frac{\frac{1}{s}d(u, fv) + d(v, fu) + d(u, v)}{2d(u, fu) + d(v, fv) + \alpha}\right]\left(\frac{1}{s} + 1\right) d(u, v) \\ &= \frac{(1+s)}{2s^2}\left[\frac{\frac{1}{s}d(u, v) + d(v, u) + d(u, v)}{2d(u, u) + d(v, v) + \alpha}\right] d(u, v) \\ &= \frac{(1+s)}{2s^2}\left[\frac{(1+2s)d(u, v)}{s\alpha}\right] d(u, v). \end{aligned}$$

This gives that $d(u,v) \geq \frac{2s^3\alpha}{(1+s)(1+2s)}$.

Finally, we suppose that $M_{sf}(x,y,\alpha) < \frac{2s^2}{1+s}$ for all x, $y \in X$. Let $w \in X$ be another fixed point of f. Then, $0 = \frac{1}{2s}d(u,fu) \leq d(u,w)$ which implies that

$$\begin{aligned} d(u,w) = d(fu,fw) &\leq \frac{1}{2s}M_{sf}(u,w,\alpha)\,[\frac{1}{s}d(u,fw) + d(w,fu)] \\ &= \frac{(1+s)}{2s^2}M_{sf}(u,w,\alpha)d(u,w). \end{aligned}$$

This ensures that $d(u,w) = 0$ and hence $u = w$. This proves that f has a unique fixed point in X. □

Corollary 3.1. *Let (X,d) be a complete metric space and f be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(x,fx) \leq d(x,y) \text{ implies } d(fx,fy) \leq \frac{1}{2}M_f(x,y,\alpha)\,[d(x,fy) + d(y,fx)]$$

for all x, $y \in X$. Then

- *(i) f has at least one fixed point in X;*
- *(ii) if u, $v \in X$ are two distinct fixed points, then $d(u,v) \geq \frac{\alpha}{3}$.*

Moreover, if $M_f(x,y,\alpha) < 1$ for all x, $y \in X$, then f has a unique fixed point in X.

Proof. The proof follows from Theorem 3.3 by taking $s = 1$. □

Theorem 3.4. *Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$ and f be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \text{ implies } d(fx,fy) \leq \frac{1}{2s}M_{sf}(x,y,\alpha)\,[d(x,fx) + d(y,fy)] \quad (11)$$

for all x, $y \in X$. If there exists $x_0 \in X$ such that for $x_n = f^n x_0$ for all $n \in \mathbb{N}$ and $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{2s}{1+s}$, then f has a unique fixed point in X.

Proof. The proof of the first part, that is, the existence of a fixed point $u \in X$ of f is similar to Theorem 3.3. For uniqueness, let $w \in X$ be another fixed point of f. Then, $0 = \frac{1}{2s}d(u,fu) \leq d(u,w)$ which implies that

$$d(u,w) = d(fu,fw) \leq \frac{1}{2s}M_{sf}(u,w,\alpha)\,[d(u,fu) + d(w,fw)] = 0.$$

This gives that $d(u,w) = 0$ and hence $u = w$. This proves that f has a unique fixed point in X. □

Corollary 3.2. *Let (X,d) be a complete metric space and f be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2}d(x,fx) \leq d(x,y) \text{ implies } d(fx,fy) \leq \frac{1}{2}M_f(x,y,\alpha)\,[d(x,fx) + d(y,fy)]$$

for all x, $y \in X$. Then f has a unique fixed point in X.

Proof. The proof follows from Theorem 3.4 by taking $s = 1$. □

We note that if the condition $M_{sf}(x,y,\alpha) < 1$ for all x, $y \in X$ does not hold, then the uniqueness of the fixed point is not guaranteed in Theorem 3.1. The following example illustrates the above fact.

Example 3.1. *Let $X = \{a, b, c\}$ and $d : X \times X \to [0, \infty)$ be defined by*

$$d(a,a) = d(b,b) = d(c,c) = 0,\ d(a,b) = d(b,a) = 4,$$

$$d(a,c) = d(c,a) = \frac{3}{2},\ d(b,c) = d(c,b) = \frac{1}{2}.$$

Then (X, d) is a complete b-metric space with $s = 2$ but it is not a metric space since $d(a,b) = 4 > d(a,c) + d(c,b)$.

We define $f : X \to X$ by $fa = b,\ fb = b,\ fc = c$. For $\alpha = \frac{1}{4}$, we have

$$M_{sf}(b,c,\alpha) = \frac{\frac{1}{2}d(b,fc) + d(c,fb) + d(b,c)}{2d(b,fb) + d(c,fc) + \alpha} = 5,$$

$$M_{sf}(c,a,\alpha) = \frac{\frac{1}{2}d(c,fa) + d(a,fc) + d(c,a)}{2d(c,fc) + d(a,fa) + \alpha} = \frac{13}{17},$$

$$M_{sf}(c,b,\alpha) = \frac{\frac{1}{2}d(c,fb) + d(b,fc) + d(c,b)}{2d(c,fc) + d(b,fb) + \alpha} = 5,$$

$$M_{sf}(a,c,\alpha) = \frac{\frac{1}{2}d(a,fc) + d(c,fa) + d(a,c)}{2d(a,fa) + d(c,fc) + \alpha} = \frac{1}{3}.$$

We consider the following three cases:

Case-I: $0 = \frac{1}{2s}d(b,fb) \leq d(b,y)$ *for any $y \in X$ and*

$$0 = d(fb,fb) = M_{sf}(b,b,\alpha)\, d(b,b),$$

$$0 = d(fb,fa) < M_{sf}(b,a,\alpha)\, d(b,a),$$

$$\frac{1}{2} = d(fb,fc) < M_{sf}(b,c,\alpha)\, d(b,c) = \frac{5}{2}.$$

Thus, $\frac{1}{2s}d(b,fb) \leq d(b,y)$ implies that $d(fb,fy) \leq M_{sf}(b,y,\alpha)\, d(b,y)$ for all $y \in X$.

Case-II: $0 = \frac{1}{2s}d(c,fc) \leq d(c,y)$ *for any $y \in X$ and*

$$\frac{1}{2} = d(fc,fa) < M_{sf}(c,a,\alpha)\, d(c,a) = \frac{39}{34},$$

$$\frac{1}{2} = d(fc,fb) < M_{sf}(c,b,\alpha)\, d(c,b) = \frac{5}{2},$$

$$0 = d(fc,fc) = M_{sf}(c,c,\alpha)\, d(c,c).$$

Thus, $\frac{1}{2s}d(c,fc) \leq d(c,y)$ implies that $d(fc,fy) \leq M_{sf}(c,y,\alpha)\, d(c,y)$ for all $y \in X$.

Case-III: $1 = \frac{1}{2s}d(a,fa) < d(a,y)$ *for $y = b,\ c$ and*

$$0 = d(fa,fb) < M_{sf}(a,b,\alpha)\, d(a,b),$$

$$\frac{1}{2} = d(fa,fc) = M_{sf}(a,c,\alpha)\, d(a,c) = \frac{1}{2}.$$

Thus, $\frac{1}{2s}d(a,fa) < d(a,y)$ implies that $d(fa,fy) \leq M_{sf}(a,y,\alpha)\, d(a,y)$ for $y = b,\ c$. Consequently it follows that condition (1) of Theorem 3.1 *holds true for $\alpha = \frac{1}{4}$.*

Moreover, there exists $x_0 = b$ such that for $x_n = f^n x_0 = b$ for all $n \in \mathbb{N}$ and $d_n = d(x_n, x_{n+1}) = 0$ for all $n \geq 0$, $\frac{2d_0 + d_1}{2d_0 + d_1 + \alpha} = 0 < \frac{1}{s}$. Thus, f satisfies all the hypotheses of Theorem 3.1 *except $M_{sf}(x,y,\alpha) < 1$ for all $x,\ y \in X$. Then the existence of a fixed point* *of f follows from Theorem 3.1. It should be noticed that f has two distinct fixed points b, c and $d(b,c) = \frac{1}{2} > \frac{1}{10} = \frac{\alpha s}{1+2s}$.*

It is valuable to note that if the condition $M_{sf}(x,y,\alpha) < 1$ for all x, $y \in X$ holds, then the uniqueness of the fixed point is guaranteed in Theorem 3.1. The following example supports the above fact.

Example 3.2. *Let $X = \{a, b, c\}$ and $d : X \times X \to [0,\infty)$ be defined by*

$$d(a,a) = d(b,b) = d(c,c) = 0,\ d(a,b) = d(b,a) = \frac{1}{2},$$

$$d(a,c) = d(c,a) = 2,\ d(b,c) = d(c,b) = \frac{1}{2}.$$

Then (X,d) is a complete b-metric space with $s = 2$ but it is not a metric space since $d(a,c) = 2 > d(a,b) + d(b,c)$.

We define $f : X \to X$ by $fx = c$ for all $x \in X$. For $\alpha = 3$, we compute that

$$M_{sf}(a,a,\alpha) = \frac{\frac{1}{2}d(a,c) + d(a,c) + d(a,a)}{2d(a,c) + d(a,c) + \alpha} = \frac{1}{3},$$

$$M_{sf}(b,b,\alpha) = \frac{\frac{1}{2}d(b,c) + d(b,c) + d(b,b)}{2d(b,c) + d(b,c) + \alpha} = \frac{1}{6},$$

$$M_{sf}(c,c,\alpha) = \frac{\frac{1}{2}d(c,c) + d(c,c) + d(c,c)}{2d(c,c) + d(c,c) + \alpha} = 0,$$

$$M_{sf}(a,b,\alpha) = \frac{\frac{1}{2}d(a,c) + d(b,c) + d(a,b)}{2d(a,c) + d(b,c) + \alpha} = \frac{4}{15},$$

$$M_{sf}(b,a,\alpha) = \frac{\frac{1}{2}d(b,c) + d(a,c) + d(b,a)}{2d(b,c) + d(a,c) + \alpha} = \frac{11}{24},$$

$$M_{sf}(b,c,\alpha) = \frac{\frac{1}{2}d(b,c) + d(c,c) + d(b,c)}{2d(b,c) + d(c,c) + \alpha} = \frac{3}{16},$$

$$M_{sf}(c,b,\alpha) = \frac{\frac{1}{2}d(c,c) + d(b,c) + d(c,b)}{2d(c,c) + d(b,c) + \alpha} = \frac{2}{7},$$

$$M_{sf}(c,a,\alpha) = \frac{\frac{1}{2}d(c,c) + d(a,c) + d(c,a)}{2d(c,c) + d(a,c) + \alpha} = \frac{4}{5},$$

$$M_{sf}(a,c,\alpha) = \frac{\frac{1}{2}d(a,c) + d(c,c) + d(a,c)}{2d(a,c) + d(c,c) + \alpha} = \frac{3}{7}.$$

This shows that $M_{sf}(x,y,\alpha) < 1$ for all x, $y \in X$. Finally, $d(fx,fy) = 0$ for all x, $y \in X$ assures that

$$d(fx,fy) \le M_{sf}(x,y,\alpha)\, d(x,y)$$

for all x, $y \in X$. Also, there exists $x_0 = c$ such that for $x_n = f^n x_0 = c$ for all $n \in \mathbb{N}$ and $d_n = d(x_n,x_{n+1}) = 0$ for all $n \ge 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} = 0 < \frac{1}{s}$. Thus, f satisfies all the hypotheses of Theorem 3.1 for $\alpha = 3$. It should be noticed that f has a unique fixed point $c \in X$.

The following example illustrates our Theorem 3.3.

Example 3.3. *Let* $X = \{a,\, b,\, c\}$ *and* $d : X \times X \to [0, \infty)$ *be defined by*

$$d(a,a) = d(b,b) = d(c,c) = 0,\ d(a,b) = d(b,a) = 3,$$
$$d(a,c) = d(c,a) = \frac{3}{2},\ d(b,c) = d(c,b) = \frac{1}{12}.$$

Then (X, d) *is a complete b-metric space with* $s = \frac{36}{19}$ *but it is not a metric space since* $d(a, b) = 3 > d(a, c) + d(c, b)$.

We define $f : X \to X$ *by* $fa = b,\ fb = b,\ fc = c$. *For* $\alpha = \frac{1}{12}$, *we have*

$$M_{sf}(a,c,\alpha) = \frac{\frac{19}{36}d(a,fc) + d(c,fa) + d(a,c)}{2d(a,fa) + d(c,fc) + \alpha} = \frac{57}{146},$$
$$M_{sf}(c,a,\alpha) = \frac{\frac{19}{36}d(c,fa) + d(a,fc) + d(c,a)}{2d(c,fc) + d(a,fa) + \alpha} = \frac{1315}{1332},$$
$$M_{sf}(b,c,\alpha) = \frac{\frac{19}{36}d(b,fc) + d(c,fb) + d(b,c)}{2d(b,fb) + d(c,fc) + \alpha} = \frac{91}{36},$$
$$M_{sf}(c,b,\alpha) = \frac{\frac{19}{36}d(c,fb) + d(b,fc) + d(c,b)}{2d(c,fc) + d(b,fb) + \alpha} = \frac{91}{36}.$$

We consider the following three cases:

Case-I: $\frac{19}{24} = \frac{1}{2s}d(a, fa) < d(a, y)$ *for* $y = b,\ c$ *and*

$$0 = d(fa, fb) < \frac{1}{2s}M_{sf}(a,b,\alpha)\,[\frac{1}{s}d(a,fb) + d(b,fa)],$$
$$\frac{1}{12} = d(fa, fc) < \frac{1}{2s}M_{sf}(a,c,\alpha)\,[\frac{1}{s}d(a,fc) + d(c,fa)] = \frac{2527}{28032}.$$

Thus, $\frac{1}{2s}d(a, fa) < d(a, y)$ *implies that* $d(fa, fy) < \frac{1}{2s}M_{sf}(a, y, \alpha)\,[\frac{1}{s}d(a, fy) + d(y, fa)]$ *for* $y = b,\ c$.

Case-II: $0 = \frac{1}{2s}d(b, fb) \le d(b, y)$ *for any* $y \in X$ *and*

$$0 = d(fb, fa) < \frac{1}{2s}M_{sf}(b,a,\alpha)\,[\frac{1}{s}d(b,fa) + d(a,fb)],$$
$$0 = d(fb, fb) = \frac{1}{2s}M_{sf}(b,b,\alpha)\,[\frac{1}{s}d(b,fb) + d(b,fb)],$$
$$\frac{1}{12} = d(fb, fc) < \frac{1}{2s}M_{sf}(b,c,\alpha)\,[\frac{1}{s}d(b,fc) + d(c,fb)] = \frac{95095}{1119744}.$$

Thus, $\frac{1}{2s}d(b, fb) \le d(b, y)$ *implies that* $d(fb, fy) \le \frac{1}{2s}M_{sf}(b, y, \alpha)\,[\frac{1}{s}d(b, fy) + d(y, fb)]$ *for any* $y \in X$.

Case-III: $0 = \frac{1}{2s}d(c, fc) \le d(c, y)$ *for any* $y \in X$ *and*

$$\frac{1}{12} = d(fc, fa) < \frac{1}{2s}M_{sf}(c,a,\alpha)\,[\frac{1}{s}d(c,fa) + d(a,fc)] = \frac{16664995}{41430528},$$
$$\frac{1}{12} = d(fc, fb) < \frac{1}{2s}M_{sf}(c,b,\alpha)\,[\frac{1}{s}d(c,fb) + d(b,fc)] = \frac{95095}{1119744}.$$
$$0 = d(fc, fc) = \frac{1}{2s}M_{sf}(c,c,\alpha)\,[\frac{1}{s}d(c,fc) + d(c,fc)].$$

Thus, $\frac{1}{2s}d(c, fc) \le d(c, y)$ *implies that* $d(fc, fy) \le \frac{1}{2s}M_{sf}(c, y, \alpha)\,[\frac{1}{s}d(c, fy) + d(y, fc)]$ *for any* $y \in X$. *Consequently it follows that condition (5) of Theorem* 3.3 *holds true for*

$\alpha = \frac{1}{12}$.

Moreover, there exists $x_0 = c$ *such that for* $x_n = f^n x_0 = c$ *for all* $n \in \mathbb{N}$ *and* $d_n = d(x_n, x_{n+1}) = 0$ *for all* $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} = 0 < \frac{2s}{1+s}$. *Thus,* f *satisfies all the hypotheses of Theorem* 3.3 *except* $M_{sf}(x, y, \alpha) < \frac{2s^2}{1+s}$ *for all* x, $y \in X$. *In fact,* $M_{sf}(b, c, \alpha) = \frac{91}{36} > \frac{2s^2}{1+s}$. *Then the existence of a fixed point of* f *follows from Theorem* 3.3. *It should be noticed that* b, c *are fixed points of* f *in* X *and* $d(b, c) = \frac{1}{12} > \frac{7776}{95095} = \frac{2s^3\alpha}{(1+s)(1+2s)}$.

Remark 3.1. *It is worth mentioning that if we consider Example* 3.2, *then* f *satisfies all the hypotheses of Theorem* 3.3 *including* $M_{sf}(x, y, \alpha) < \frac{2s^2}{1+s}$ *for all* x, $y \in X$. *As a result, there exists a unique fixed point of* f *in* X.

The following example illustrates our Theorem 3.4.

Example 3.4. *Let* $X = \{a, b, c\}$ *and* $d : X \times X \to [0, \infty)$ *be defined by*

$$d(a, a) = d(b, b) = d(c, c) = 0,\ d(a, b) = d(b, a) = 3,$$
$$d(a, c) = d(c, a) = \frac{1}{2},\ d(b, c) = d(c, b) = \frac{3}{2}.$$

Then (X, d) *is a complete b-metric space with* $s = \frac{3}{2}$ *but it is not a metric space since* $d(a, b) = 3 > d(a, c) + d(c, b)$.

We define $f : X \to X$ *by* $fa = a$, $fb = c$, $fc = a$. *For* $\alpha = \frac{1}{140}$, *we have*

$$M_{sf}(b, c, \alpha) = \frac{\frac{2}{3}d(b, fc) + d(c, fb) + d(b, c)}{2d(b, fb) + d(c, fc) + \alpha} = \frac{490}{491},$$
$$M_{sf}(a, b, \alpha) = \frac{\frac{2}{3}d(a, fb) + d(b, fa) + d(a, b)}{2d(a, fa) + d(b, fb) + \alpha} = \frac{2660}{633},$$
$$M_{sf}(b, a, \alpha) = \frac{\frac{2}{3}d(b, fa) + d(a, fb) + d(b, a)}{2d(b, fb) + d(a, fa) + \alpha} = \frac{770}{421},$$
$$M_{sf}(c, b, \alpha) = \frac{\frac{2}{3}d(c, fb) + d(b, fc) + d(c, b)}{2d(c, fc) + d(b, fb) + \alpha} = \frac{70}{39}.$$

We consider the following three cases:

Case-I: $0 = \frac{1}{2s}d(a, fa) \leq d(a, y)$ *for any* $y \in X$ *and*

$$0 = d(fa, fa) = \frac{1}{2s}M_{sf}(a, a, \alpha)\,[d(a, fa) + d(a, fa)],$$
$$\frac{1}{2} = d(fa, fb) < \frac{1}{2s}M_{sf}(a, b, \alpha)\,[d(a, fa) + d(b, fb)] = \frac{1330}{633},$$
$$0 = d(fa, fc) < \frac{1}{2s}M_{sf}(a, c, \alpha)\,[d(a, fa) + d(c, fc)].$$

Thus, $\frac{1}{2s}d(a, fa) \leq d(a, y)$ *implies that* $d(fa, fy) \leq \frac{1}{2s}M_{sf}(a, y, \alpha)\,[d(a, fa) + d(y, fy)]$ *for all* $y \in X$.

Case-II: $\frac{1}{2} = \frac{1}{2s}d(b, fb) < d(b, y)$ *for* $y = a$, c *and*

$$\frac{1}{2} = d(fb, fa) < \frac{1}{2s}M_{sf}(b, a, \alpha)\,[d(b, fb) + d(a, fa)] = \frac{385}{421},$$
$$\frac{1}{2} = d(fb, fc) < \frac{1}{2s}M_{sf}(b, c, \alpha)\,[d(b, fb) + d(c, fc)] = \frac{980}{1473}.$$

Thus, $\frac{1}{2s}d(b,fb) < d(b,y)$ *implies that* $d(fb,fy) < \frac{1}{2s}M_{sf}(b,y,\alpha)\,[d(b,fb)+d(y,fy)]$ *for* $y = a,\ c$.

Case-III: $\frac{1}{6} = \frac{1}{2s}d(c,fc) < d(c,y)$ *for* $y = a,\ b$ *and*

$$0 = d(fc,fa) < \frac{1}{2s}M_{sf}(c,a,\alpha)\,[d(c,fc)+d(a,fa)],$$

$$\frac{1}{2} = d(fc,fb) < \frac{1}{2s}M_{sf}(c,b,\alpha)\,[d(c,fc)+d(b,fb)] = \frac{140}{117}.$$

Thus, $\frac{1}{2s}d(c,fc) < d(c,y)$ *implies that* $d(fc,fy) < \frac{1}{2s}M_{sf}(c,y,\alpha)\,[d(c,fc)+d(y,fy)]$ *for* $y = a,\ b$. *Consequently it follows that condition (11) of Theorem* 3.4 *holds true for* $\alpha = \frac{1}{140}$.

Moreover, there exists $x_0 = a$ *such that for* $x_n = f^n x_0 = a$ *for all* $n \in \mathbb{N}$ *and* $d_n = d(x_n, x_{n+1}) = 0$ *for all* $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} = 0 < \frac{2s}{1+s}$. *Thus,* f *satisfies all the hypotheses of Theorem* 3.4. *Then the existence of a unique fixed point of* f *follows from Theorem* 3.4. *It should be noticed that* a *is the unique fixed point of* f *in* X.

4. Some Coincidence Point Results

Definition 4.1. [1] *Let* f *and* g *be self mappings of a set* X. *If* $y = fx = gx$ *for some* x *in* X, *then* x *is called a coincidence point of* f *and* g *and* y *is called a point of coincidence of* f *and* g.

Definition 4.2. [12] *The mappings* $f, g : X \to X$ *are weakly compatible, if for every* $x \in X$, *the following holds:*

$$g(fx) = f(gx) \text{ whenever } fx = gx.$$

Proposition 4.1. [1] *Let* f *and* g *be weakly compatible self maps of a nonempty set* X. *If* f *and* g *have a unique point of coincidence* $y = fx = gx$, *then* y *is the unique common fixed point of* f *and* g.

We state the following lemma which is a key result in this section.

Lemma 4.1. [9] *Let* X *be a nonempty set and* $f : X \to X$ *a function. Then there exists a subset* $U \subseteq X$ *such that* $f(U) = f(X)$ *and* $f : U \to X$ *is one-to-one.*

As an application of Theorem 3.1, we obtain the following result.

Theorem 4.1. *Let* (X,d) *be a b-metric space with the coefficient* $s \geq 1$. *Let* f, g *be single valued mappings from* X *to itself,* $f(X) \subseteq g(X)$ *and* $g(X)$ *a complete subspace of* X. *Suppose there exists* $\alpha > 0$ *such that*

$$\frac{1}{2s}d(gx,fx) \leq d(gx,gy) \text{ implies } d(fx,fy) \leq M_{sgf}(x,y,\alpha)\,d(gx,gy) \tag{12}$$

for all $x,\ y \in X$, *where* $M_{sgf}(x,y,\alpha) = \frac{\frac{1}{s}d(gx,fy)+d(gy,fx)+d(gx,gy)}{2d(gx,fx)+d(gy,fy)+\alpha}$. *If there exists* $x_0 \in X$ *such that for* $gx_n = fx_{n-1}$ *for all* $n \in \mathbb{N}$ *and* $d_n = d(gx_n, gx_{n+1})$ *for all* $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{1}{s}$, *then* f *and* g *have a point of coincidence in* $g(X)$. *In addition, if* $M_{sgf}(x,y,\alpha) < 1$ *for all* $x,\ y \in X$, *then* f *and* g *have a unique point of coincidence in* $g(X)$. *Moreover, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point in* $g(X)$.

Proof. By Lemma 4.1, there exists $U \subseteq X$ such that $g(U) = g(X)$ and $g : U \to X$ is one-to-one. Define $h : g(U) \to g(U)$ by $h(gx) = fx$. This is possible as $f(X) \subseteq g(X)$.

Then h is well defined, as g is one-to-one on U.

For all gx, $gy \in g(U)$, we obtain from condition (12) that there exists $\alpha > 0$ such that

$$\frac{1}{2s}d(gx,h(gx)) \leq d(gx,gy) \text{ implies } d(h(gx),h(gy)) \leq M_{sh}(gx,gy,\alpha)\, d(gx,gy),$$

where $M_{sh}(gx,gy,\alpha) = \frac{\frac{1}{s}d(gx,h(gy))+d(gy,h(gx))+d(gx,gy)}{2d(gx,h(gx))+d(gy,h(gy))+\alpha}$. This proves that $h : g(U) \to g(U)$ satisfies condition (1) of Theorem 3.1. Moreover, taking $y_n = gx_n$ for all $n \geq 0$, we have $y_n = gx_n = fx_{n-1} = h(gx_{n-1}) = hy_{n-1}$ for all $n \geq 1$. By hypothesis, there exists $y_0 \in g(U)$ such that for $y_n = hy_{n-1}$ for all $n \in \mathbb{N}$ and $d_n = d(y_n, y_{n+1})$ for all $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{1}{s}$. Since $g(U) = g(X)$ is complete, by Theorem 3.1, there exists $gu_0 \in g(X)$ such that $h(gu_0) = gu_0 = u$, say. That is, $fu_0 = gu_0 = u$. Hence, f and g have a point of coincidence u in $g(X)$.

If $M_{sgf}(x,y,\alpha) < 1$ for all x, $y \in X$, then $M_{sh}(gx,gy,\alpha) < 1$ for all gx, $gy \in g(U)$ and hence by Theorem 3.1, there exists a unique $gu_0 \in g(X)$ such that $fu_0 = h(gu_0) = gu_0 = u$. This proves that f and g have a unique point of coincidence in $g(X)$.

If f and g are weakly compatible, then by Proposition 4.1 it follows that f and g have a unique common fixed point in $g(X)$. □

The following theorem is a consequence of Theorem 3.3 and Lemma 4.1.

Theorem 4.2. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Let f, g be single valued mappings from X to itself, $f(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2s}d(gx,fx) \leq d(gx,gy) \text{ implies } d(fx,fy) \leq \frac{1}{2s}M_{sgf}(x,y,\alpha)[\frac{1}{s}d(gx,fy) + d(gy,fx)]$$

for all x, $y \in X$. If there exists $x_0 \in X$ such that for $gx_n = fx_{n-1}$ for all $n \in \mathbb{N}$ and $d_n = d(gx_n, gx_{n+1})$ for all $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{2s}{1+s}$, then f and g have a point of coincidence in $g(X)$. In addition, if $M_{sgf}(x,y,\alpha) < \frac{2s^2}{1+s}$ for all x, $y \in X$, then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

The following theorem is a consequence of Theorem 3.4 and Lemma 4.1.

Theorem 4.3. *Let (X,d) be a b-metric space with the coefficient $s \geq 1$. Let f, g be single valued mappings from X to itself, $f(X) \subseteq g(X)$ and $g(X)$ a complete subspace of X. Suppose there exists $\alpha > 0$ such that*

$$\frac{1}{2s}d(gx,fx) \leq d(gx,gy) \text{ implies } d(fx,fy) \leq \frac{1}{2s}M_{sgf}(x,y,\alpha)\,[d(gx,fx) + d(gy,fy)]$$

for all x, $y \in X$. If there exists $x_0 \in X$ such that for $gx_n = fx_{n-1}$ for all $n \in \mathbb{N}$ and $d_n = d(gx_n, gx_{n+1})$ for all $n \geq 0$, $\frac{2d_0+d_1}{2d_0+d_1+\alpha} < \frac{2s}{1+s}$, then f and g have a unique point of coincidence in $g(X)$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Remark 4.1. *Taking $s = 1$ in Theorems 4.1, 4.2 and 4.3, we can obtain the corresponding coincidence point results in metric spaces.*

5. Conclusions

In this work, we first obtained some new fixed point results and then applied these results to establish some coincidence point and common fixed point results for a pair of self mappings in b-metric spaces. Our results generalized several well known results in the existing literature. Moreover, we have justified our results with the aid of competent examples.

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