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HAAR WAVELETS FOR THE NUMERICAL STUDY OF A SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM WITH DISCONTINUITIES

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Abstract. In this article, we presented a numerical approach based on non-uniform Haar wavelets to approximate the solution of a second-order singularly perturbed problems with discontinuous data. The solutions to such problems have strong interior layer due to the discontinuity. Accordingly, we have utilized a special type of piecewise uniform mesh called a Shishkin mesh to resolve the layer behaviour of the solution. As the Haar functions are discontinous, the approximate solution is obtained with the integration approach. The second-order derivative is approximated by the linear combinations of Haar functions and then integrated to obtain the numerical approximations. The convergence analysis of the numerical method proposed is carried out, showing that the proposed method is of order two. The adaptability of the proposed method is established by numerical results on two test problems. Even at lower resolution levels, the proposed method provides high accuracy. In any programming language, the proposed method can be easily implemented and is computationally efficient.

Keywords: Haar wavelets, Singularly perturbed problems, Collocation method.

AMS Subject Classification: 65L11, 65L20

1. Introduction

Singularly perturbed differential equations (SPDEs) occurs frequently in diverse areas of applied mathematics, including quantum mechanics, fluid dynamics, transport phenomena in biology, meterology oceanography, water pollution problems [1] etc. These problems are well known for exhibiting boundary layers in their solutions. As a result of the irregular nature of the solution in some neighbourhood of the domain, layers are formed, which makes using traditional numerical techniques difficult. In this case, one must choose non-traditional numerical methods. Researches in SPDEs [2, 3, 4] has progressed significantly over the years, but there is still a lot of platform for more pertinent research to be conducted. In this article, we have considered a reaction-diffusion problem with discontinuities. Numerous methods exist in the literature for numerically solving SPDEs with discontinuities. Miller et al. [5] proposed a Schwarz method for SPDEs based

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on a Shishkin mesh. In [6, 7] Chandru and Shanti proposed a boundary value technique considering a hybrid difference scheme for SPDE of reaction-diffusion type with nonsmooth data. A first-order uniformly convergent scheme on a Shishkin-type mesh for SPDEs was developed by Farrell et al. [8]. A pseudo spectral technique was proposed by Nevenka et al. [9] for similar problems. A finite volume difference scheme for two-dimensional reaction diffusion problems with discontinuities was presented by Brayanov [10]. Roos and Zarin [11] developed a Galerkin finite element method on a Bakhvalov-Shishkin type mesh for SPDEs (1)-(2). Recently, Yang [12] has introduced a rational spectral collocation method for the SPDE (1)-(2) with nonsmooth data.

In this paper, we considered the following singularly perturbed reaction-diffusion problem with discontinuities:

$$-\epsilon^2 y''(x) + a(x)y(x) = g(x),\ x \in \Omega^- \cup \Omega^+, \tag{1}$$

with the boundary conditions

$$y(0) = \beta_1,\ y(1) = \beta_2, \tag{2}$$

and the the reaction coefficient $a(x)$ and the source term $g(x)$ have a discontinuity at a point $d \in \Omega = (0,1)$. where $0 < \epsilon << 1$ is the singular perturbation parameter, and β_1 and β_2 are given constants. $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $\Omega = (0,1)$ and $\bar{\Omega} = [0,1]$. This type of problem was first considered by Farrell et al. [13] with the assumptions that the functions $a(x)$ and $g(x)$ are sufficiently smooth on $\bar{\Omega}\backslash\{d\}$ and $a(x) \geq \alpha^2 > 0$. Further, $g(x)$ and its derivatives are considered to have a jump discontinuity at $x = d$. Thus $u \notin C^2(\Omega)$, but $y \in C^1(\Omega)$. The jump at a point d of a function ψ is denoted by $[\psi](d) = \psi(d+) - \psi(d-)$, where $\psi(d^{\pm}) = \lim_{x\to d\pm 0}\psi(x)$. These assumptions ensure the existence of a unique smooth solution $y \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ of problem (1)-(2). For more details, readers are referred to [6, 8]. In this article, we have developed an efficient numerical method based on non-uniform Haar wavelets to provide an accurate solution of the above problem. A detailed description of wavelet-based numerical methods can be found at [14].

During the past two decades, a number of studies based on Haar wavelets have been conducted. Haar wavelets are an orthonormal system with compact support which is used for numerical approximations. A wide range of problems, including boundary value problems [15], fractional-order problems [16, 17], multiresolution analysis [18], and two-dimensional problems [19], can be solved using Haar wavelets. A Haar wavelet approach was used by Oruc et al. [20, 21, 22, 23] to solve the regularized long wave, KdV, modified burgers, and coupled nonlinear Schrodinger$-KdV$ equations. The Haar wavelet approach was used by Lepik [24, 25, 26] to solve some well-known differential equations. Haar wavelets have been used to solve a two-dimensional time fractional reaction-sub diffusion equation [27]. It is shown in [28] that Haar wavelets are used to analyze vibrations of nanobeams. Since Haar wavelets are piecewise constant, orthogonal, they produce an invertible sparse matrix when solving SPBVP (1)-(2). In order to understand Haar wavelets in more detail, one may refer to [29]. This method transforms the SPDE (1)-(2) into a system of linear equations.

The idea of this paper is to solve a second order SPBVP with discontinuities using non-uniform Haar wavelets. A description of non-uniform Haar wavelets and their integrals is provided in Section 2. The domain discretization and collocation points are provided in Section 3. A description of the solution methodology can be found in Section 4. The proposed method's convergence analysis is studied in Section 5. The numerical simulations are discussed in Section 6, and the paper is concluded in Section 7.

2. Haar Wavelets

We follow [32] for defining the Haar wavelets and its properties. A wavelet-based method has become one of the most effective tool to solve differential equations in modern times. The applicability and proficiency of this method have made it a very useful tool for solving differential equations, especially when dealing with nonlinear terms, discontinuities, and boundary conditions of various kinds. As Haar functions have compact support, they are conceptually simple, fast, and memory efficient. The Haar basis is localized, i.e., except for a few entries, the vector is zero. Localization improves with increasing resolution. Localization plays a significant role in the speed of the method. Haar functions take care of the discontinuities present in differential equations since they are discontinuous. When wavelet number is increased, the wavelet coefficients decrease rapidly and are practically zero at higher levels. In this way, we can constrain the wavelet series to a smaller number of terms.

This paper explores non-uniform Haar wavelet's applications to solve a class of second order singularly perturbed problems with discontinuous data. Haar wavelets can be described by two parameters: a dilation parameter j, which receives values corresponding to $0, 1, 2, ..., J$, and a translation parameter l, which receives values corresponding to $0, 1, 2, ..., m-1$, where $m = 2^j$. In order to calculate the number of each wavelet, the index $i = l + m + 1$ is used. l has a minimum value of 0 and m has a minimum value of 1, resulting in $i = 2$. The maximum value is $i = 2M$, with $M = 2^J$ (J indiactes the maximum level of resolution). We denote $h_1(x)$ as the first wavelet and is stated as

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0, 1], \\ 0 & \text{elsewhere}, \end{cases} \tag{3}$$

A wavelet with $i = 2, 3, ..., 2M$ is defined as follows:

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\nu_1(i), \nu_2(i)), \\ -c_i & \text{for } x \in [\nu_2(i), \nu_3(i)), \\ 0 & \text{elsewhere}, \end{cases} \tag{4}$$

Using the notations: $\nu_1(i) = x(2l\mu),\ \nu_2(i) = x((2l+1)\mu), \nu_3(i) = x(2l+2)\mu),\ \mu = \dfrac{M}{m}$.
Here $x(.)$ denote the grid point and the generation of grid points has been discussed in the next section.
The coefficients

$$c_i = \frac{\nu_2(i) - \nu_1(i)}{\nu_3(i) - \nu_2(i)}$$

are calculated from the requirement that $\int_0^1 h_i(x)dx = 0,\ i \geq 2$.
Pseudocode to obtain the first eight Haar wavelets:

Initialize $J = 2$
$M = 2^J$
$N = 2M$
$stepsize = \frac{1}{N}$
set $x(1) = 0$
for $r = 2 \rightarrow N + 1$ **do**
    $x(r) = x(r-1) + stepsize$
    $xs(r-1) = \dfrac{x(r-1) + x(r)}{2}$
    $h(1, r-1) = 1$

end for
Set $\nu_1(1) = 0$, $\nu_2(1) = 1$, $\nu_3(1) = 1$
for $j = 0 \to J$ **do**
    $m = 2^j$
    for $L1 = 1 \to m$ **do**
        $l = L1 - 1$
        $i = L1 + m$
        set $\nu_1(i) = \dfrac{l}{m}$
        set $\nu_2(i) = \dfrac{l + 0.5}{m}$
        set $\nu_3(i) = \dfrac{l + 1}{m}$
        for $r = 1 \to N$ **do**
            if $xs(r) < \nu_1(i)$ **then**
                $h(i, r) = 0$
            else if $xs(r) < \nu_2(i)$ **then**
                $h(i, r) = 1$
            else if $xs(r) < \nu_3(i)$ **then**
                $h(i, r) = -1$
            else
            end if
        end for
    end for
end for
Plot each row of h in interval $[0, 1]$.

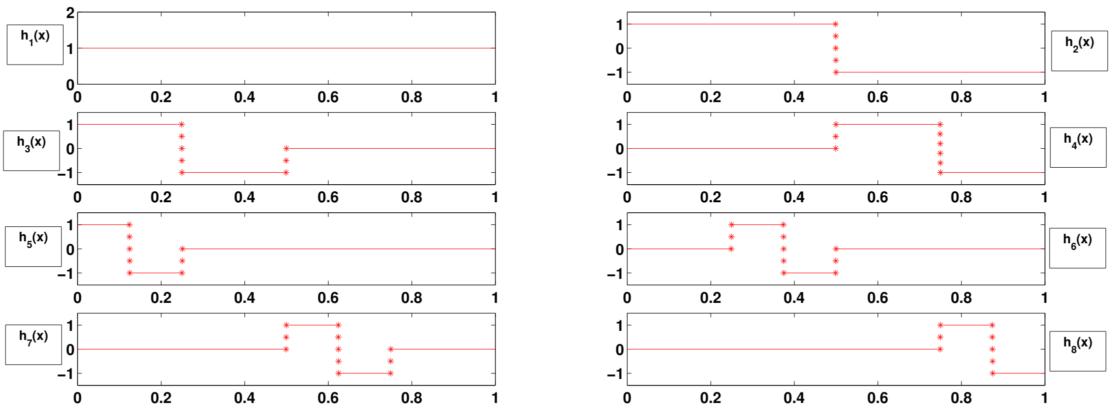

FIGURE 1. First eight Haar wavelets when $J = 2$.

The integrals of Haar wavelets are required to find the solution to (1)-(2). Using the following notations:

$$Q_{i,1}(x) = \int_0^x h_i(t)dt;$$

$$Q_{i,s+1}(x) = \int_0^x Q_{i,s}(t)dt;\ i = 1, 2, 3, ..., 2M;\ s = 1, 2, ...$$

Based on Eq.(3), we can obtain

$$Q_{1,1}(x) = x, \quad Q_{1,2}(x) = \frac{x^2}{2!}.$$

Based Eq.(4), the integrals for $i > 1$ can be calculated analytically, resulting in

$$Q_{i,1}(x) = \begin{cases} x - \nu_1(i) & \text{for } x \in [\nu_1(i), \nu_2(i)), \\ c_i(\nu_3(i) - x) & \text{for } x \in [\nu_2(i), \nu_3(i)), \\ 0 & \text{elsewhere,} \end{cases}$$

$$Q_{i,2}(x) = \begin{cases} 0 & \text{for } x < \nu_1(i), \\ \frac{1}{2}(x - \nu_1(i))^2 & \text{for } x \in [\nu_1(i), \nu_2(i)), \\ L - \frac{1}{2}c_i(\eta_3(i) - x)^2 & \text{for } x \in [\nu_2(i), \nu_3(i)), \\ L & \text{for } x \geq \nu_3(i), \end{cases}$$

where

$$L = \frac{1}{2}(\nu_2(i) - \nu_1(i))(\nu_3(i) - \nu_1(i)).$$

Remark: We can express any square integrable function $\phi(x)$ on the interval $(0, 1)$ as an infinite sum of Haar wavelets in the form:[30]

$$\phi(x) = \sum_{i=1}^{\infty} \lambda_i h_i(x). \tag{5}$$

If the function $\phi(x)$ is piecewise constant or is approximated by piecewise constant functions on appropriate subintervals, then the above series has a finite number of terms.

3. Mesh Construction

We considered a wavelet based scheme on a Shishkin-type mesh for problem (1)-(2). It is observed that there are abrupt changes in the solution of the proposed problem in the interior region and at the boundary points. Hence, we have considered a non-uniform mesh that requires a larger number of collocation points in the boundary layer region. To construct the mesh with $2M$ subintervals, we divide the domain $\bar{\Omega}$ into six-subdomains as follows

$$\begin{aligned} &\Omega_1 = [0, \tau_1], && \Omega_2 = (\tau_1, d - \tau_1], && \Omega_3 = (d - \tau_1, d], \\ &\Omega_4 = (d, d + \tau_2], && \Omega_5 = (d + \tau_2, 1 - \tau_2], && \Omega_6 = (1 - \tau_2, 1], \end{aligned}$$

where τ_1 and τ_2 are transition parameters given by

$$\tau_1 = \min\left\{\frac{d}{4}, \frac{3\epsilon}{2\alpha}\ln(2M)\right\},$$

$$\tau_2 = \min\left\{\frac{1 - d}{4}, \frac{3\epsilon}{2\alpha}\ln(2M)\right\}.$$

Then we divide each of the subintervals $[0, \tau_1), [d - \tau_1, d), [d, d + \tau_2), [1 - \tau_2, 1]$ into $\frac{M}{4}$ equidistant subintervals and each of $[\tau_1, d - \tau_1), [d + \tau_2, 1 - \tau_2)$ is subdivided into $\frac{M}{2}$

subintervals. Hence, the grid points $\{x(k)\}_{k=0}^{2M}$ of a Shishkin-type mesh can be calculated as follows

$$x(k) = \begin{cases} \frac{4}{M}\tau_1 k, & k = 0, 1, \ldots, \frac{M}{4}, \\ \tau_1 + \frac{2}{M}(d - 2\tau_1)(k - \frac{M}{4}), & k = \frac{M}{4} + 1, \ldots, \frac{3M}{4}, \\ d - \tau_1 + \frac{4}{M}(\tau_1)(k - \frac{3M}{4}), & k = \frac{3M}{4} + 1, \ldots, M, \\ d + \frac{4}{M}(\tau_2)(k - M), & k = M + 1, \ldots, \frac{5M}{4}, \\ d + \tau_2 + \frac{2}{M}(1 - d - 2\tau_2)(k - \frac{5M}{4}), & k = \frac{5M}{4} + 1, \ldots, \frac{7M}{4}, \\ 1 - \tau_2 + \frac{4}{M}(\tau_2)(k - \frac{7M}{4}), & k = \frac{7M}{4} + 1, \ldots, 2M. \end{cases}$$

Finally, on each sub interval, we take the collocation points as the midpoints on that interval, i.e.,

$$x_c(k) = \frac{x(k-1) + x(k)}{2}, k = 1, 2, ..., 2M. \tag{6}$$

4. Numerical Method

The method can be described by considering the SPBVP given in (1)-(2) and the collocation points in (6). Assume y and Y are the exact and approximate Haar solutions to the equations (1) and (2). Accordingly, we can approximate the second derivative of (1)-(2) in the following manner:

$$Y''(x) = \sum_{i=1}^{2M} \lambda_i h_i(x), \tag{7}$$

where the λ_i's are wavelet coefficients. Integration of equation (7) from 0 to x gives

$$Y'(x) = Y'(0) + \sum_{i=1}^{2M} \lambda_i Q_{i,1}(x). \tag{8}$$

The equation (8) is integrated from 0 to 1 in order to obtain

$$Y(1) - Y(0) = Y'(0) + \sum_{i=1}^{2M} \lambda_i Q_{i,2}(1),$$

Based on the boundary conditions in (2), we can write it as follows:

$$Y'(0) = \beta_2 - \beta_1 - \sum_{i=1}^{2M} \lambda_i Q_{i,2}(1). \tag{9}$$

Integration of equation (8) from 0 to x gives

$$Y(x) = \beta_1 + Y'(0)x + \sum_{i=1}^{2M} \lambda_i Q_{i,2}(x). \tag{10}$$

Inserting ,$Y''(x)$ and $Y(x)$ in equation (1) and as a result, we can calculate it at the collocation points stated in (6) to get the system

$$\begin{aligned} &\sum_{i=1}^{2M} \lambda_i \left[-\epsilon^2 h_i(x_c(k)) + a(x_c(k))(Q_{i,2}(x_c(k)) - x_c(k)Q_{i,2}(1))\right] \\ &= g(x_c(k)) - a(x_c(k))[(\beta_2 - \beta_1)x_c(k) + \beta_1], \end{aligned} \tag{11}$$

for $k = 1, 2, \ldots, 2M$.
This system (11) is solved to get the Haar coefficients. We use the equations (9) and (10) to get the approximate Haar wavelet solution.

5. Error Analysis

Now, we demonstrate the convergent nature of the proposed method on the basis of following error analysis. We follow the steps given by the present authors in [31] for proving the convergence analysis. The true solution to (1)-(2) can be written as follows:

$$y(x) = \beta_1 + (\beta_2 - \beta_1)x + \sum_{i=1}^{\infty} \lambda_i \Big[Q_{i,2}(x) - xQ_{i,2}(1)\Big],$$

while the Haar solution at the Jth resolution level given in (10) can be written as

$$Y(x) = \beta_1 + (\beta_2 - \beta_1)x + \sum_{i=1}^{2M} \lambda_i \Big[Q_{i,2}(x) - xQ_{i,2}(1)\Big].$$

The error can be determined as follows:

$$E = |y(x) - Y_J(x)| = \Bigg| \sum_{i=2M+1}^{\infty} \lambda_i \Big[Q_{i,2}(x) - xQ_{i,2}(1)\Big]\Bigg|.$$

and the squared error norm is obtained through the inner product, that is,

$$\begin{aligned}
\|E\|^2 &= \Big\langle \sum_{i=2M+1}^{\infty} \lambda_i \Big[Q_{i,2}(x) - xQ_{i,2}(1)\Big], \sum_{r=2M+1}^{\infty} \lambda_r \Big[Q_{r,2}(x) - xQ_{r,2}(1)\Big] \Big\rangle \\
&= \int_{-\infty}^{\infty} \Bigg(\sum_{i=2M+1}^{\infty} \lambda_i \Big[Q_{i,2}(x) - xQ_{i,2}(1)\Big]\Bigg) \Bigg(\sum_{r=2M+1}^{\infty} \lambda_r \Big[Q_{r,2}(x) - xQ_{r,2}(1)\Big]\Bigg) dx \\
&= \sum_{i=2M+1}^{\infty} \sum_{r=2M+1}^{\infty} \lambda_i \lambda_r \int_0^1 \Big(Q_{i,2}(x) - xQ_{i,2}(1)\Big)\Big(Q_{r,2}(x) - xQ_{r,2}(1)\Big) dx \\
&\leq \sum_{i=2M+1}^{\infty} \sum_{r=2M+1}^{\infty} \lambda_i \lambda_r K_{i,r},
\end{aligned}$$

where

$$K_{i,r} = \sup_{i,r} \int_0^1 \Big(Q_{i,2}(x) - xQ_{i,2}(1)\Big)\Big(Q_{r,2}(x) - xQ_{r,2}(1)\Big) dx.$$

On the other hand, the Haar coefficients λ_r take the form [32]:

$$\lambda_r = \int_0^1 2^{\frac{j}{2}} y(x) h(2^j x - l) dx,$$

where

$$h(2^j x - l) = \begin{cases} 1 & \text{for } x \in [\nu_1(i), \nu_2(i)), \\ -c_i & \text{for } x \in [\nu_2(i), \nu_3(i)), \\ 0 & \text{elsewhere}, \end{cases} \tag{12}$$

for $l = 0, 1, ..., m-1,\ j = 0, 1, ..., J$, and

$$\nu_1(i) = x(2l\mu),\ \nu_2(i) = x[(2l+1)\mu],\ \nu_3(i) = x[2(l+1)\mu],\ \mu = \frac{M}{m}.$$

Therefore, it is

$$\lambda_r = 2^{\frac{j}{2}}\left[\int_{\nu_1(r)}^{\nu_2(r)} u(x)dx - c_r\int_{\nu_2(r)}^{\nu_3(r)} u(x)dx\right],$$

The Mean Value Theorem for integrals allows us to write this equation as:

$$\begin{aligned}\lambda_r &= 2^{\frac{j}{2}}[\nu_2(r) - \nu_1(r)]u(\nu_a) - c_r[\nu_3(r) - \nu_2(r)]u(\nu_b)\\ &= 2^{\frac{j}{2}}[\nu_2(r) - \nu_1(r)]u(\nu_a) - 2^{\frac{j}{2}}[\nu_2(r) - \nu_1(r)]u(\nu_b),\end{aligned}$$

where ν_a, ν_b are the intermediate values such that $\nu_a \in (\nu_1(r), \nu_2(r))$ and $\nu_b \in (\nu_2(r), \nu_3(r))$.

Using Lagrange's Mean Value Theorem, we have

$$\lambda_r = 2^{\frac{j}{2}}[\nu_2(r) - \nu_1(r)]u'(\nu)(\nu_a - \nu_b),$$

where $\nu \in (\nu_a, \nu_b)$.
Now, we have

$$\begin{aligned}\nu_1(r) &= x\left(\frac{2lM}{m}\right) \le \frac{2lM}{m},\\ \nu_2(r) &= x\left(\frac{(2l+1)M}{m}\right) \le \frac{(2l+1)M}{m},\\ \nu_2(r) - \nu_1(r) &\le \frac{M}{m} = M2^{-j},\\ \nu_3(r) &= x\left(\frac{(2l+2)M}{m}\right) \le \frac{(2l+2)M}{m},\\ \nu_3(r) - \nu_1(r) &\le \frac{2M}{m} = 2M2^{-j},\\ \nu_a - \nu_b &\le \nu_3(r) - \nu_1(r) \le 2M2^{-j}.\end{aligned}$$

Hence, $\lambda_r \le 2^{\frac{-3j}{2}}M^2D$ where D is an upper bound of $u'(\nu)$, that is, $u'(\nu) \le D$. Thus, we can write

$$\lambda_r \le 2^{\frac{-3j}{2}}R,$$

where $R = M^2 D$ is a constant.
Let denote $K_i = \sup_r(K_{i,r})$. Then, we have

$$\begin{aligned}
\|E\|^2 &\leq \sum_{i=2M+1}^{\infty} \lambda_i K_i \sum_{r=2M+1}^{\infty} \lambda_r \\
&\leq \sum_{i=2M+1}^{\infty} \lambda_i K_i \sum_{r=2M+1}^{\infty} 2^{\frac{-3j}{2}} R \\
&\leq \sum_{i=2M+1}^{\infty} \lambda_i K_i \sum_{\substack{j=2(J+1) \\ j \bmod 2=0}}^{\infty} \sum_{r=2^{\frac{j}{2}}+1}^{2^{\frac{j}{2}+1}} 2^{\frac{-3j}{2}} R \\
&\leq \sum_{i=2M+1}^{\infty} \lambda_i K_i R \sum_{\substack{j=2(J+1) \\ j \bmod 2=0}}^{\infty} 2^{-j} \\
&\leq \sum_{i=2M+1}^{\infty} \lambda_i K_i R \frac{2^{-2J-2}}{1-\frac{1}{4}}.
\end{aligned}$$

Let denote $K = \sup_i(K_i)$. Then, we have that

$$\begin{aligned}
\|E\|^2 &\leq KR^2 \left(\frac{2^{-2J-2}}{1-\frac{1}{4}}\right)\left(\frac{2^{-2J-2}}{1-\frac{1}{4}}\right) \\
&\leq \frac{KR^2}{9} 2^{-4J},
\end{aligned}$$

from which we get

$$\begin{aligned}
\|E\| &\leq \frac{\sqrt{K}R}{3} 2^{-2J} \\
&\leq \mathcal{O}\left(\frac{1}{2^{2J}}\right).
\end{aligned}$$

The above ineqaulity implies that as $J \to \infty$, $\|E\| \to 0$. Therefore, convergence is assured by the method proposed. It is clear that the error of approximation at the maximal level of resolution J is

$$\|E_J\|_2 \approx \left(\frac{1}{2^J}\right)^2.$$

From this it is clear that $\log_2\left(\frac{\|E_J\|_2}{\|E_{J+1}\|_2}\right) \approx 2$. Hence the order of the proposed method is 2.

6. Numerical Results

We have applied the proposed method to two test problems in order to demonstrate its applicability and accuracy. We have compared the exact solutions and the approximate Haar solutions by calculating the maximum absolute errors for different resolution levels and different values of ϵ to check the efficiency of the proposed method. The maximum absolute error is computed as

$$E_{\varepsilon}^{2M} = \max_{1\leq k\leq 2M} |Y_{\epsilon}^{2M}(x_c(k)) - y_{\epsilon}(x_c(k))|.$$

where $y_\epsilon(x)$ is the exact solution of the proposed problem (1)-(2) and $Y_\epsilon^{2M}(x)$ is its approximated Haar wavelet solution taking $2M$ collocation points. Tables and graphs are presented as numerical results. As shown in the tables (1)-(2), even at lower resolution level, the proposed method produces better results.

Example 1: [33]Consider the following reaction-diffusion problem with discontinuities:

$$-\epsilon^2 y''(x) + a(x)y(x) = g(x),\ x \in \Omega^- \cup \Omega^+,$$
$$y(0) = 0,\ y(1) = 0,$$

where the function

$$a(x) = \begin{cases} 1 & \text{if } x < 0.5, \\ 2 - x & \text{if } x > 0.5. \end{cases}$$

and $g(x)$ is chosen such that the exact solution is

$$y(x) = \begin{cases} \dfrac{\exp\left(\dfrac{-x}{\epsilon}\right) + \exp\left(\dfrac{2x-1}{2\epsilon}\right)}{\exp\left(\dfrac{-1}{2\epsilon}\right) + 1} - 1 & \text{if } x < 0.5, \\[2ex] \dfrac{-\exp\left(\dfrac{x-1}{\epsilon}\right) - \exp\left(\dfrac{1-2x}{2\epsilon}\right)}{\exp\left(\dfrac{-1}{2\epsilon}\right) + 1} + 1 & \text{if } x > 0.5. \end{cases}$$

For various values of ϵ and resolution levels, Table 1 presents the maximum absolute errors. A plot of Haar solution is shown in Figure 2.

Table 1. Maximum absolute errors and CPU time (in seconds) for Example 1. for different values of ϵ and different resolution levels

$\epsilon \setminus 2M$	16	32	64	128	256	512	1024
1	4.67E-06	1.20E-06	3.03E-07	7.59E-08	1.90E-08	4.74E-09	1.19E-09
Time	0.095 s	0.105 s	0.110 s	0.171 s	0.312 s	0.948 s	4.102 s
10^{-1}	3.41E-03	8.89E-04	2.26E-04	5.67E-05	1.42E-05	3.55E-06	8.87E-07
Time	0.102 s	0.108 s	0.124 s	0.157 s	0.310 s	0.991 s	4.279 s
10^{-2}	1.16E-02	8.72E-03	4.19E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.102 s	0.104 s	0.122 s	0.153 s	0.302 s	1.008 s	4.322 s
10^{-3}	1.04E-02	8.63E-03	4.18E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.103 s	0.105 s	0.113 s	0.149 s	0.309 s	1.005 s	4.123 s
10^{-4}	1.02E-02	8.62E-03	4.18E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.100 s	0.111 s	0.113 s	0.158 s	0.319 s	0.980 s	4.173 s
10^{-5}	1.02E-02	8.62E-03	4.18E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.095 s	0.101 s	0.118 s	0.156 s	0.310 s	0.978 s	4.188 s
10^{-6}	1.02E-02	8.62E-03	4.18E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.096 s	0.102 s	0.112 s	0.151 s	0.323 s	0.979 s	4.149 s
10^{-7}	1.02E-02	8.62E-03	4.18E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.097 s	0.104 s	0.124 s	0.154 s	0.324 s	0.979 s	4.192 s
10^{-8}	1.02E-02	8.62E-03	4.18E-03	1.56E-03	5.30E-04	1.70E-04	5.28E-05
Time	0.096 s	0.105 s	0.116 s	0.151 s	0.319 s	0.990 s	4.249 s

Example 2:[34] Consider the following reaction diffusion problem with discontinuous source term

$$-\varepsilon^2 y''(x) + y(x) = g(x), \quad x \in \Omega^- \cup \Omega^+,$$
$$y(0) = y(1) = g(0),$$

where

$$g(x) = \begin{cases} -0.5x, & 0 \le x \le 0.5, \\ 0.5, & 0.5 \le x \le 1. \end{cases}$$

The exact solution of this example is

$$y(x) = \begin{cases} 0.25(\gamma+\xi)(e^{-(0.5-x)/\varepsilon} - e^{-(0.5+x)/\varepsilon}) - 0.5x, & 0 \le x \le 0.5, \\ 0.25(\gamma-\xi)e^{-1/(2\varepsilon)}(e^{-(x-1)/\varepsilon} - e^{-(1-x)/\varepsilon}) + 0.5(1 - e^{-(1-x)/\varepsilon}), & 0.5 < x \le 1, \end{cases}$$

where the constants γ and ξ are

$$\gamma = \frac{\varepsilon - e^{-1/(2\varepsilon)}}{1 + e^{-1/\varepsilon}},$$

$$\xi = \frac{1.5 - e^{-1/(2\varepsilon)}}{1 - e^{-1/\varepsilon}}.$$

For various values of ϵ and resolution levels, Table 2 presents the maximum absolute errors. A plot of Haar solution is shown in Figure 3.

TABLE 2. Maximum absolute errors and CPU time (in seconds) for Example 2. for different values of ϵ and different resolution levels

$\epsilon \setminus 2M$	16	32	64	128	256	512	1024
1	3.82E-05	9.82E-06	2.49E-06	6.27E-07	1.57E-07	3.94E-08	9.85E-09
Time	0.102 s	0.105 s	0.117 s	0.152 s	0.334 s	0.997 s	4.110 s
10^{-1}	1.57E-03	4.13E-04	1.05E-04	2.64E-05	6.60E-06	1.65E-06	4.13E-07
Time	0.104 s	0.112 s	0.137 s	0.174 s	0.332 s	1.051 s	4.065 s
10^{-2}	5.80E-03	4.31E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.109 s	0.111 s	0.130 s	0.173 s	0.346 s	1.174 s	4.546 s
10^{-3}	5.12E-03	4.27E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.104 s	0.116 s	0.125 s	0.161 s	0.371 s	1.100 s	4.420 s
10^{-4}	5.12E-03	4.26E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.106 s	0.108 s	0.127 s	0.165 s	0.346 s	1.109 s	4.507 s
10^{-5}	5.12E-03	4.26E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.106 s	0.111 s	0.128 s	0.171 s	0.331 s	1.099 s	4.183 s
10^{-6}	5.12E-03	4.26E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.103 s	0.120 s	0.118 s	0.176 s	0.326 s	1.074 s	4.152 s
10^{-7}	5.12E-03	4.26E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.103 s	0.118 s	0.123 s	0.205 s	0.331 s	1.009 s	4.529 s
10^{-8}	5.12E-03	4.26E-03	2.02E-03	7.52E-04	2.53E-04	8.13E-05	2.52E-05
Time	0.110 s	0.118 s	0.122 s	0.193 s	0.357 s	1.154 s	4.662 s

TABLE 3. Comparision of Maximum absolute errors for Example 2 for $2M = 32$ and for different values of ϵ

$\epsilon = 10^{-k}$	Method in [34]	Present method
k=2	2.7131E–2	4.31E-03
k=3	2.5663E–2	4.27E-03
k=4	2.5058E–2	4.26E-03
k=5	2.4983E–2	4.26E-03
k=6	2.4975E–2	4.26E-03
k=7	2.4974E–2	4.26E-03
k=8	2.4974E–2	4.26E-03
k=9	2.4974E–2	4.26E-03

TABLE 4. Comparision of Maximum absolute errors for Example 2 for $2M = 64$ and for different values of ϵ

$\epsilon = 10^{-k}$	Method in [34]	Present method
k=2	3.4894E–3	2.02E-03
k=3	7.3009E–3	2.02E-03
k=4	7.2974E–3	2.02E-03
k=5	7.2970E–3	2.02E-03
k=6	7.2970E–3	2.02E-03
k=7	7.2970E–3	2.02E-03
k=8	7.2970E–3	2.02E-03
k=9	7.2970E–3	2.02E-03

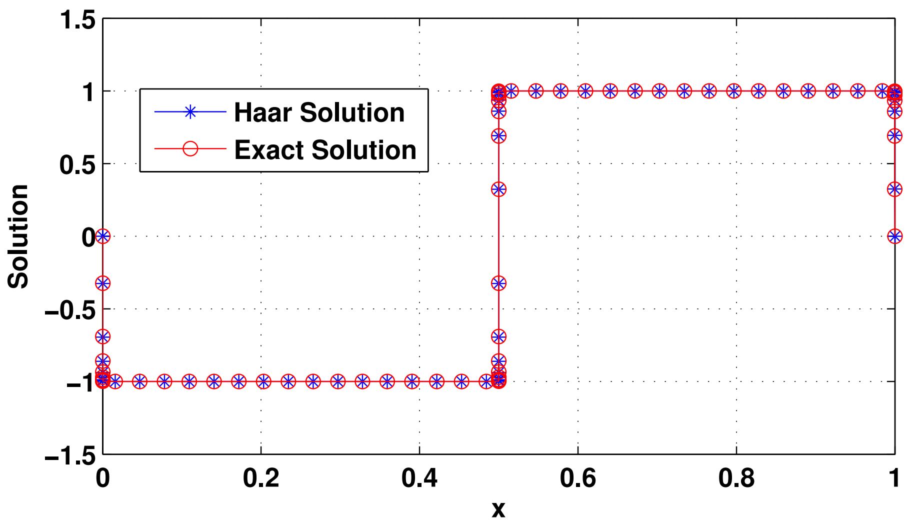

FIGURE 2. Haar solution of Example 1 for $\epsilon = 10^{-5}$, $2M = 64$.

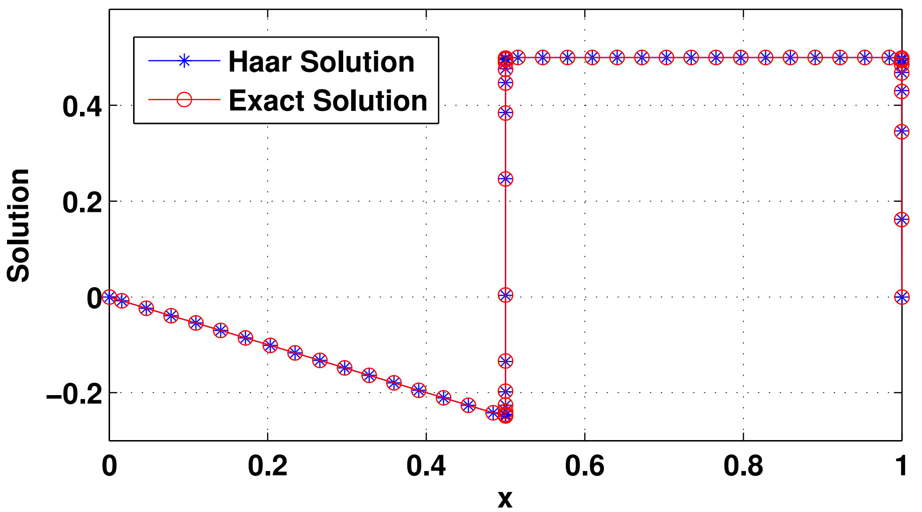

FIGURE 3. Haar solution of Example 2 for $\epsilon = 10^{-5}$, $2M = 64$.

7. Conclusion

In this paper, a numerical method was developed to solve a class of second-order singularly perturbed reaction-diffusion problems with discontinuities by utilizing Haar wavelets. A convergence analysis is conducted on the method. Upon examining the convergence analysis, the proposed method appears to be of order 2. It can be seen from the numerical results, the proposed method has superior accuracy even at lower wavelet resolutions. The method available in [34] is not able to give good results on less number of mesh points. The advantage of the proposed method is that it captures the layer behaviour of the solution even at less number of mesh points. It is easy to implement the proposed method, and it becomes more accurate as the wavelet resolution level J increases. Moreover, a wavelet based method is also quite simpler than other conventional numerical methods in terms of computational complexity.

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