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COUPLED SYSTEM OF NONLINEAR IMPULSIVE HYBRID DIFFERENTIAL EQUATIONS WITH LINEAR AND NONLINEAR PERTURBATIONS

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Abstract. In the present paper, we prove the existence and uniqueness of solutions to impulsive coupled system of nonlinear hybrid fractional differential equations involving Caputo fractional derivative of order $\alpha \in (0,1)$ with linear and nonlinear perturbations. We prove our main results by applying the nonlinear alternative of Leray-Schauder type and Banach's fixed-point theorem. As application, On example is included to show the applicability of our results.

Keywords: Coupled systems; Impulsive hybrid fractional differential equations; Fixed point theorems; linear perturbation.

AMS Subject Classification: 83-02, 99A00

1. Introduction

The subject of fractional calculus attracted much attentions and is rapidly growing area of research because of its numerous applications in engineering and scientific disciplines such as signal processing, nonlinear control theory, viscoelasticity, optimization theory, controlled thermonuclear fusion, chemistry, nonlinear biological systems, mechanics, electric networks, fluid dynamics, diffusion, oscillation, relaxation, turbulence, stochastic dynamical system, plasma physics, polymer physics, chemical physics, astrophysics, and economics. Therefore, it deserve an independent theory parallel to the theory of ordinary differential equations (DEs).(See[33]).
Nonlinear differential equations are crucial tools in the modeling of nonlinear real phenomena corresponding to a great variety of events, in relation with several fields of the physical sciences and technology. For instance, they appear in the study of the air motion or the

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fluids dynamics, electricity, electromagnetism, or the control of nonlinear processes, among others (see [21]). The resolution of nonlinear differential equations requires, in general, the development of different techniques in order to deduce the existence and other essential properties of the solutions. There are still many open problems related the solvability of nonlinear systems, apart form the fact that this is a field where advances are continuously taking place.

Perturbation techniques are useful in the nonlinear analysis for studying the dynamical systems represented by linear differential and integral equations. Evidently, some differential equations representing a certain dynamical system have no analytical solution, so the perturbation of such problems can be helpful. The perturbed differential equations are categorized into various types. An important type of these such perturbations is called a hybrid differential equation (i.e. quadratic perturbation of a linear differential equation). Another important class of differential equations is known as impulsive differential equations. This class plays the role of an effective mathematical tools for those evolution processes that are subject to abrupt changes in their states. There are many physical systems that exhibit impulsive behavior such as the action of a pendulum clock, mechanical systems subject to impacts, the maintenance of a species through periodic stocking or harvesting, the thrust impulse maneuver of a spacecraft, and the function of the heart, we refer to [12] for an introduction to the theory of impulsive differential equations. It is well known that in the evolution processes the impulsive phenomena can be found in many situations. For example, disturbances in cellular neural networks [9], operation of a damper subjected to the percussive effects [10], change of the valve shutter speed in its transition from open to closed state [15], fluctuations of pendulum systems in the case of external impulsive effects [8],percussive systems with vibrations [11], relaxational oscillations of the electromechanical systems [7], dynamic of system with automatic regulation [13], control of the satellite orbit, using the radial acceleration [13] and so on.

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown hybrid function with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [23]-[25].

Hybrid differential equations is a rich field of differential equations. It is quadratic perturbations of non linear differential equations. It has lately years been an object of increasing interest because of its vast applicability in several fields. For more details about hybrid differential equations, we refer to [19], [22], [17], [31], [20].

In [34], S. Melliani, A. El Allaoui and L. S. Chadli considered boundary value problem of nonlinear hybrid differential equations with linear and nonlinear perturbations:

$$\begin{cases} \frac{d}{dt}\Big(x(t)f(t,x(t)) - g(t,x(t))\Big) = h(t,x(t)), t \in I = [0,a], a > 0 \\ x(0)f(0,x(0)) + \alpha x(a)f(a,x(a)) = x(0)g(0,x(0)) + \alpha g(a,x(a)) + \beta, \end{cases} \tag{1}$$

where $f \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$ and $g, h \in C(I \times \mathbb{R}, \mathbb{R})$ are given functions and α, $\beta \in \mathbb{R}$ such that $\alpha \neq -1$.

The propose of this paper is to study the following coupled system of nonlinear impulsive hybrid differential equations with linear and nonlinear perturbations.

Motivated by the good effect of model (1), we consider the following problem of coupled impulsive hybrid fractional differential equations:

$$\begin{cases} D^{\alpha}\Big(u(t)f_1(t,u(t),v(t))-g_1(t,u(t),v(t))\Big)=h_1(t,u(t),v(t)), t\in J=[0,1], t\neq t_i,\\ i=1,2,\dots,n, 0<\alpha<1,\\ u(t_i^+)=u(t_i^-)+I_i(u(t_i^-)), t_i\in(0,1), i=1,2,\dots,n,\\ D^{\beta}\Big(v(t)f_2(t,u(t),v(t))-g_2(t,u(t),v(t))\Big)=h_2(t,u(t),v(t)), t\in J=[0,1], t\neq t_j,\\ j=1,2,\dots,m, 0<\beta<1,\\ v(t_j^+)=v(t_j^-)+I_j(v(t_j^-)), t_j\in(0,1), j=1,2,\dots,m,\\ u(0)f_1(0,u(0),v(0))-g_1(0,u(0),v(0))=\phi(u),\\ v(0)f_2(0,u(0),v(0))-g_2(0,u(0),v(0))=\psi(v), \end{cases} \tag{2}$$

where D^{α}, D^{β} denotes the Caputo fractional derivative of order α, β, respectively $f_i\in\mathcal{C}(J\times\mathbb{R}\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $g_i,h_i\in\mathcal{C}(J\times\mathbb{R}\times\mathbb{R},\mathbb{R}),(i=1,2)$ and $\phi,\psi:\mathcal{C}(J,\mathbb{R})\longrightarrow\mathbb{R}$ are continuous functions defined by $\phi(u)=\sum\limits_{i=1}^{n}\lambda_i u(\xi_i)$, $\psi(v)=\sum\limits_{j=1}^{n}\delta_j v(\eta_j)$, where ξ_i, $\eta_j\in(0,1)$ for $i=1,2,...,n$, $j=1,2,...,m$, and $I_k:\mathbb{R}\longrightarrow\mathbb{R}$ and $u(t_k^+)=\lim\limits_{\epsilon\to0^+}u(t_k+\varepsilon)$ and $u(t_k^-)=\lim\limits_{\epsilon\to0^-}u(t_k+\varepsilon)$ represent the right and left limits of $u(t)$ at $t=t_k$, $(k=i,j)$.

The HDEP (2) is a linear perturbation of second type of an initial value problem of first order nonlinear differential equations and has been discussed in Dhage and Jadhav [27] for existence theory for different aspects of the solutions. The details of different types of nonlinear perturbations of a differential equation appears in Dhage [?]. The specialty of the results of present paper lies in our constructive approach for the solutions to the HDEP (2) on J.
His paper is arranged as follows. In Section 2, we recall some tools related to the fractional calculus as well as some needed results. In Section 3, we present the main results. Section 4 is dedicated to a concrete application.The conclusions are drawn in Section 5.

2. Preliminaries

In this section, we recall some basic definitions and properties of the fractional calculus theory and preparation results.
Throughout this paper, let $J_0=[0,t_1]$, $J_1=(t_1,t_2]$, $\dots$, $J_{n-1}=(t_{n-1},t_n]$, $J_n=(t_n,1]$, $n\in\mathbb{N}, n>1$.
For $t_i\in(0,1)$ such that $t_1<t_2<\dots<t_n$ we define the following spaces:
$I'=I\setminus\{t_1,t_2,\dots t_n\}$,

$$X=\{u\in C([0,1],\mathbb{R}): u\in C(I') \text{ and left } u(t_i^+) \text{ and right limit } u(t_i^-)) \\ \text{exist and } u(t_i^-)=u(t_i), 1\leq i\leq n\}.$$

Then, clearly $(X,\|.\|)$ is the Banach space under the norm $\|u\|=\max\limits_{t\in[0,1]}|u(t)|$.
Similarly for $t_j\in(0,1)$ such that $t_1<t_2<\dots<t_m$ we define the following spaces:
$J'=J\setminus\{t_1,t_2,\dots,t_m\}$,

$$Y=\{v\in C([0,1],\mathbb{R}): v\in C(J') \text{ and left } v(t_j^+) \text{ and right limit } v(t_j^-)) \\ \text{exist and } v(t_j^-)=v(t_j), 1\leq j\leq n\}.$$

Then, clearly $(Y, \|.\|)$ is the Banach space under the norm $\|v\| = \max_{t\in[0,1]} |v(t)|$.
Consequently, the product $X \times Y$ is a Banach space under the norms $\|(u, v)\| = \|u\| + \|v\|$ and $\|(u, v)\| = max\{\|u\|, \|v\|\}$.

Definition 2.1. [18] *The fractional integral of the function $h \in L^1([a, b], \mathbb{R}^+)$ of order $\alpha \in \mathbb{R}^+$ is defined by*

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function.

Definition 2.2. [18] *For a function h given on the interval $[a, b]$, the Riemann-Liouville fractional-order derivative of h, is defined by*

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. [18] *For a function h given on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by*

$$({}^cD_{a^+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)} h^{(n)}(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Lemma 2.1. [18] *Let $n \in \mathbb{N}$ and $n-1 < \alpha < n$. If f is a continuous function, then we have*

$$I^\alpha \ {}^cD^\alpha f(t) = f(t) + a_0 + \alpha_1 t + a_2 t^2 + ... + a_{n-1} t^{n-1}.$$

3. Main results

In this section, we will prove the existence of a mild solution for (2) .
To do so, we will need the following assumptions:

(H_1) The function $u \longrightarrow uf_1(t, u, v)$ is increasing in $\mathbb{R}$ for every $t \in [0, 1]$.
(H_2) The function $v \longrightarrow vf_2(t, u, v)$ is increasing in $\mathbb{R}$ for every $t \in [0, 1]$.
(H_3) i) The functions f_i and g_i are continuous and bounded that is, there exist positive numbers $\nu_{f_i} > 0$ and $\mu_{g_i} > 0$ such that:
$|f_i(t, u, v)| \geq \nu_{f_i}$ and $|g_i(t, u, v)| \leq \mu_{g_i}$ for all $(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ $(i = 1, 2)$.
ii) There exist positive numbers $M_{f_i} > 0$ and $M_{g_i} > 0$ such that:

$$|f_i(t, u, v) - f_i(t, \bar{u}, \bar{v})| \leq M_{f_i}[\|u - \bar{u}\| + \|v - \bar{v}\|] \quad (i = 1, 2),$$

and

$$|g_i(t, u, v) - g_i(t, \bar{u}, \bar{v})| \leq M_{g_i}[\|u - \bar{u}\| + \|v - \bar{v}\|] \quad (i = 1, 2),$$

for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$.
(H_4) There exist positive numbers $M_{h_i} > 0$, such that:

$$|h_i(t, u, v) - h_i(t, \bar{u}, \bar{v})| \leq M_{h_i}[\|u - \bar{u}\| + \|v - \bar{v}\|] \quad (i = 1, 2),$$

for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}$ and $t \in [0, 1]$.
(H_5) There exist constants $A, B > 0$ such that:
$|I_i(u) - I_i(\bar{u})| \leq A|u - \bar{u}|, \qquad i = 1, 2, ..., n$ for all $u, \bar{u}, \in \mathbb{R}$,
$|I_j(v) - I_j(\bar{v})| \leq B|v - \bar{v}|, \qquad j = 1, 2, ..., m$ for all $v, \bar{v}, \in \mathbb{R}$.

(H_6) There exist constants K_ϕ, $K_\psi > 0$, such that:
$|\phi(u) - \phi(v)| \leq K_\phi \|u - v\|$, for all $u, v \in C([0,1], \mathbb{R})$,
$|\psi(u) - \psi(v)| \leq K_\psi \|u - v\|$, for all $u, v \in C([0,1], \mathbb{R})$.

(H_7) There exist constants $M_\phi, M_\psi > 0$, and $N_u, N_v > 0$, such that:
$|\phi(u)| \leq M_\phi \|u - v\|$, for all $u, v \in C([0,1], \mathbb{R})$.
$|\varphi(u)| \leq M_\psi \|u - v\|$, for all $u, v \in C([0,1], \mathbb{R})$.

$$|I_i(u)| \leq N_u |u|, \qquad i = 1, 2, ..., n,$$

$$|I_j(v)| \leq N_v |v|, \qquad j = 1, 2, ..., m.$$

(H_8) There exist constants $C, D > 0$ such that:
$|I_i(u_i)| \leq C, \quad i = 1, 2, \ldots, n$, for all $u \in \mathbb{R}$ and $|I_j(v_j)| \leq D, \quad j = 1, 2, \ldots, m$, $v \in \mathbb{R}$.

(H_9) There exist constants ρ, $\mu > 0$ such that
$|\phi(u)| \leq \rho, \quad \forall u \in X, \quad |\psi(v)| \leq \mu, \quad \forall v \in Y.$

(H_{10}) There exist constants ρ_0, $\delta_0 > 0$ and ρ_i, $\delta_i > (i = 1, 2)$ such that

$$|h_1(t, u, v)| \leq \rho_0 + \rho_1 \|u\| + \rho_2 \|v\|,$$

and

$$|h_2(t, u, v)| \leq \delta_0 + \delta_1 \|u\| + \delta_2 \|v\|,$$

for all $(u, v) \in X \times Y$.

For brevity, let us set

$$\begin{aligned} \pi_1 &= \frac{1}{\nu_{f_1}} \Big(M_{g_1} + K_\phi + M_{f_1} + nA + \frac{M_{h_1}}{\Gamma(\alpha + 1)} \Big), \\ \pi_2 &= \frac{1}{\nu_{f_2}} \Big(M_{g_2} + K_\psi + M_{f_2} + mB + \frac{M_{h_2}}{\Gamma(\beta + 1)} \Big). \end{aligned} \tag{3}$$

Lemma 3.1. *Let's assume that hypotheses (H_1) and (H_3) holds. Let $\alpha \in (0,1)$ and $h : J \longrightarrow \mathbb{R}$ be continuous. A function u is a solution to the fractional integral equation*

$$\begin{aligned} u(t) = \frac{1}{f_1(t, u(t), v(t))} \Big(& \ \phi(u) + g_1(t, u(t), v(t)) + \theta(t) \sum_{i=1}^{n} I_i(u(t_i^-)) f(t_i, u(t_i), v(t_i)) \\ & + \int_0^t \frac{(t-s)^{\alpha - 1}}{\Gamma(\alpha)} h_1(s) ds \Big), t \in [t_i, t_{i+1}]. \end{aligned} \tag{4}$$

Where

$$\theta(t) = \begin{cases} 0, & t \in [t_0, t_1[\\ 1, & t \in [t_i, t_{i+1}[. i = 1, 2 ... n, \end{cases} \tag{5}$$

if and only if u is a solution of the following impulsive problem:

$$\begin{cases} D^\alpha \Big(u(t) f_1(t, u(t), v(t)) - g_1(t, u(t), v(t)) \Big) = h_1(t), t \in J = [0,1], t \neq t_i, \\ i = 1, 2, \ldots, n, 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), t_i \in (0,1), i = 1, 2, \ldots, n, \\ u(0) f_1(0, u(0), v(0)) - g_1(0, u(0), v(0)) = \phi(u). \end{cases} \tag{6}$$

Proof. Assume that u satisfies (6). If $t \in [t_0, t_1[$, then

$$D^{\alpha}\Big(u(t)f_1(t, u(t), v(t)) - g_1(t, u(t), v(t))\Big) = h_1(t), \quad t \in [t_0, t_1[, \tag{7}$$

$$u(0)f_1(0, u(0), v(0)) - g_1(0, u(0), v(0)) = \phi(u). \tag{8}$$

Applying I^{α} on both sides of (7), we can obtain

$$\begin{aligned} u(t)f_1(t, u(t), v(t)) - g_1(t, u(t), v(t)) &= u(0)f_1(0, u(0), v(0)) - g_1(0, u(0), v(0)) \\ &\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds \\ &= \phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds. \end{aligned}$$

Then we get

$$u(t) = \frac{1}{f_1(t, u(t), v(t))}\Big(g_1(t, u(t), v(t)) + \phi(u) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds\Big).$$

If $t \in [t_1, t_2[$, then

$$D^{\alpha}\Big(u(t)f_1(t, u(t), v(t)) - g_1(t, u(t), v(t))\Big) = h_1(t), \quad t \in [t_1, t_2[, \tag{9}$$

$$u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)). \tag{10}$$

According to Lemma 3.3 and the continuity of $t \longrightarrow f_1(t, u(t), v(t))$, we have

$$\begin{aligned} u(t)f_1(t, u(t), v(t)) - g_1(t, u(t), v(t)) &= u(t_1^+)f_1(t_1, u(t_1), v(t_1)) - g_1(t_1, u(t_1), v(t_1)) \\ &\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds \\ &= (u(t_1^-) + I_1(u(t_1^-)))f_1(t_1, u(t_1), v(t_1)) \\ &\quad - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds. \end{aligned}$$

Since

$$\begin{aligned} u(t_1^-) = \frac{1}{f_1(t_1, u(t_1), v(t_1))}\Big(&g_1(t_1, u(t_1), v(t_1)) + \phi(u) \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds\Big), \end{aligned}$$

then we get

$$\begin{aligned}
u(t)f_1(t,u(t),v(t)) - g_1(t,u(t),v(t)) &= \Big[\frac{1}{f_1(t_1,u(t_1),v(t_1))}\Big(g_1(t_1,u(t_1),v(t_1)) + \phi(u) \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds\Big) + I_1(u(t_1^-))\Big] \\
&\quad \times f_1(t_1,u(t_1),v(t_1)) - \int_0^{t_1} \frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds \\
&= \phi(u) + I_1(u(t_1^-))f_1(t_1,u(t_1),v(t_1)) \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds.
\end{aligned}$$

so, one has

$$\begin{aligned}
u(t) = {}& \frac{1}{f_1(t,u(t),v(t))}\Big(\phi(u) + g_1(t,u(t),v(t)) + I_1(u(t_1^-))f_1(t_1,u(t_1),v(t_1)) \\
& + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds\Big).
\end{aligned}$$

If $t \in [t_2, t_3[$, we have

$$\begin{aligned}
u(t)f_1(t,u(t),v(t)) - g_1(t,u(t),v(t)) &= u(t_2^+)f_1(t_2,u(t_2),v(t_2)) - g_1(t_2,u(t_2),v(t_2)) \\
&\quad - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds \\
&= (u(t_2^-) + I_2(u(t_2^-)))f_1(t_2,u(t_2),v(t_2)) \\
&\quad - \int_0^{t_2} \frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds,
\end{aligned}$$

and

$$\begin{aligned}
u(t_2^-) = {}& \frac{1}{f_1(t_2,u(t_2),v(t_2))}\Big(\phi(u) + g_1(t_2,u(t_2),v(t_2)) + I_1(u(t_1^-))f_1(t_1,u(t_1),v(t_1)) \\
& + \int_0^{t_2} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s)ds\Big).
\end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
u(t)f_1(t,u(t),v(t)) - g_1(t,u(t),v(t)) &= \Big[\frac{1}{f_1(t_2,u(t_2),v(t_2))}\Big(\phi(u) + g_1(t_2,u(t_2),v(t_2))\\
&+ I_1(u(t_1^-))f_1(t_1,u(t_1),v(t_1))\\
&+ \int_0^{t_2}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\Big)\Big]f_1(t_2,u(t_2),v(t_2))\\
&+ I_2(u(t_2^-)))f_1(t_2,u(t_2),v(t_2))\\
&- \int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\\
&+ \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds,\\
&= \phi(u) + I_1(u(t_1^-))f_1(t_1,u(t_1),v(t_1))\\
&+ I_2(u(t_2^-))f_1(t_2,u(t_2),v(t_2))\\
&+ \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
u(t) = {}& \frac{1}{f_1(t,u(t),v(t))}\Big(g_1(t,u(t),v(t)) + \phi(u) + \sum_{i=1}^{2} I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i))\\
&+ \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\Big).
\end{aligned}
$$

By using the same method, for $t \in [t_i,t_{i+1}[(i = 3,4,\dots,n)$, one has

$$
\begin{aligned}
u(t) = {}& \frac{1}{f_1(t,u(t),v(t))}\Big(g_1(t,u(t),v(t)) + \phi(u) + \sum_{i=1}^{n} I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i))\\
&+ \int_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\Big).
\end{aligned}
$$

Conversely, assume that u satisfies (4). If $t \in [t_0,t_1[$, then we have

$$
u(t) = \tfrac{1}{f_1(t,u(t),v(t))}\Big(g_1(t,u(t),v(t)) + \phi(u) + \int_0^t \tfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\Big). \tag{11}
$$

Then, we multiplied by $f_1(t,u(t),v(t))$ and applying D^α on both sides of (11), we get equation (7).
Again, substituting $t = 0$ in (11), we obtain $u(0)f_1(0,u(0),v(0)) - g_1(0,u(0),v(0)) = \phi(u)$. Since $u \longrightarrow uf_1(t,u,v)$ is increasing in $\mathbb{R}$ for $t \in [t_0,t_1[$, the map $u \longrightarrow uf_1(t,u,v)$ is injective in $\mathbb{R}$. Then we get (8).
If $t \in [t_1,t_2[$, then we have

$$
\begin{aligned}
u(t) &= \frac{1}{f_1(t,u(t),v(t))}\Big(\phi(u) + g_1(t,u(t),v(t)) \\
&+ I_1(u(t_1^-))f_1(t_1,u(t_1),v(t_1)) + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s)ds\Big).
\end{aligned} \tag{12}
$$

Then, we multiplied by $f_1(t,u(t),v(t))$ and applying D^α on both sides of (13), we get equation (13) . Again by (H_3), substituting $t = t_1$ in (11) and taking the limit of (13),

then (13) minus (11) gives (14).
Similarly, for $t \in [t_i, t_{i+1}[(i = 2, 3, \dots, n)$, we get

$$D^\alpha\Big(u(t)f_1(t, u(t), v(t)) - g_1(t, u(t), v(t))\Big) = h_1(t), t \in [t_1, t_2[, \tag{13}$$

$$u(t_1^+) = u(t_1^-) + I_1(u(t_1^-)). \tag{14}$$

□

This completes the proof.

Lemma 3.2. *Lets h_1, h_2 are continuous, then $(u, v) \in X \times Y$ is a solution of (2) if and only if (u, v) is the solution of the integral equations:*

$$\begin{aligned} u(t) &= \frac{1}{f_1(t, u(t), v(t))}\Big(g_1(t, u(t), v(t)) + \phi(u) + \theta(t)\sum_{i=1}^{n} I_i(u(t_i^-))f_1(t_i, u(t_i), v(t_i)) \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s, u(s), v(s))ds\Big), t \in [t_i, t_{i+1}] \\ v(t) &= \frac{1}{f_2(t, u(t), v(t))}\Big(g_2(t, u(t), v(t)) + \psi(v) + \omega(t)\sum_{j=1}^{n} I_j(u(t_j^-))f_2(t_j, u(t_j), v(t_j)) \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_2(s, u(s), v(s))ds\Big), t \in [t_j, t_{j+1}], \end{aligned}$$

Where

$$\theta(t) = \begin{cases} 0, t \in [t_0, t_1[\\ 1, t \in [t_i, t_{i+1}[, i = 1, 2...n, \end{cases}$$

and

$$\omega(t) = \begin{cases} 0, t \in [t_0, t_1[\\ 1, t \in [t_j, t_{j+1}[, j = 1, 2...m. \end{cases}$$

First result. Now we are in a position to present our first result which deals with the existence and uniqueness of solution for problem (2). This result is based on Banach's fixed point theorem. To do so, we define the operator $\Theta : X \times Y \longrightarrow X \times Y$ by

$$\Theta(u, v)(t) = (\Theta_1(u, v)(t), \Theta_2(u, v)(t)), \tag{15}$$

where

$$\begin{aligned} \Theta_1(u, v)(t) &= \frac{1}{f_1(t, u(t), v(t))}\Big(g_1(t, u(t), v(t)) + \phi(u) + \theta(t)\sum_{i=1}^{n} I_i(u(t_i^-))f_1(t_i, u(t_i), v(t_i)) \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s, u(s), v(s))ds\Big), \end{aligned} \tag{16}$$

and

$$\begin{aligned} \Theta_2(u, v)(t) &= \frac{1}{f_2(t, u(t), v(t))}\Big(g_2(t, u(t), v(t)) + \psi(v) + \omega(t)\sum_{j=1}^{n} I_j(u(t_j^-))f_2(t_j, u(t_j), v(t_j)) \\ &+ \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_2(s, u(s), v(s))ds\Big). \end{aligned} \tag{17}$$

Theorem 3.1. *Supposedly that the condition* $(H_1)-(H_7)$ *holds and that* h_1, $h_2 : [0,1]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ *are continuous functions. In addition, there exist positive constants* λ_i, ζ_i, $i=1,2$ *such that*

$$|h_1(t,u,v)-h_1(t,\bar{u},\bar{v})| \leq \lambda_1|u-\bar{u}|+\zeta_1|v-\bar{v}|,$$

$$|h_2(t,u,v)-h_2(t,\bar{u},\bar{v})| \leq \lambda_2|u-\bar{u}|+\zeta_2|v-\bar{v}|.$$

If $\max(\pi_1,\pi_2)<1$, π_1 *and* π_2 *given by* (3), *then the impulsive coupled system (2) has a unique mild solution.*

Proof. Let us set $\sup\limits_{t\in J} h_1(t,0,0)=\kappa_1<\infty$, $\sup\limits_{t\in J}|h_2(t,0,0)|=\kappa_2<\infty$ and defne a closed ballas follows

$$\bar{B}=\{(u,v)\in X\times Y : \|(u,v)\|\leq r\},$$

where

$$r \geq \max\left\{\frac{\mu_{g_1}+\frac{\kappa_1}{\Gamma(\alpha+1)}}{\nu_{f_1}-(M_\phi+nN_u+\frac{1}{\Gamma(\alpha+1)}(\lambda_1+\lambda_2))}, \frac{\mu_{g_2}+\frac{\kappa_2}{\Gamma(\beta+1)}}{\nu_{f_2}-(M_\psi+nN_v+\frac{1}{\Gamma(\beta+1)}(\zeta_1+\zeta_2))}\right\}. \tag{18}$$

Then we show that $\Theta\bar{B}\subset\bar{B}$. For $(u,v)\in\bar{B}$, we obtain

$$\begin{aligned}
|\Theta_1(u,v)(t)| &\leq \frac{1}{|f_1(t,u(t),v(t))|}\Big|\Big(g_1(t,u(t),v(t))+\phi(u)\\
&+\theta(t)\sum_{i=1}^{n} I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i))\\
&+\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h_1(s,u(s),v(s))ds\Big)\Big|\\
&\leq \frac{1}{\nu_{f_1}}\Big(\mu_{g_1}+M_\phi\|u\|+nN_u\|u\|+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|h_1(s,u(s),v(s))\\
&-h_1(s,0,0)|+|h_1(s,0,0)|)ds\\
&\leq \frac{1}{\nu_{f_1}}\Big(\mu_{g_1}+M_\phi\|u\|+nN_u\|u\|+\frac{1}{\Gamma(\alpha+1)}\big((\lambda_1+\lambda_2)\|u\|+\kappa_1\big)\Big)\\
&\leq \frac{1}{\nu_{f_1}}\Big(\mu_{g_1}+(M_\phi+nN_u)r+\frac{1}{\Gamma(\alpha+1)}\big((\lambda_1+\lambda_2)r+\kappa_1\big)\Big).
\end{aligned}$$

Hence we get

$$\|\Theta_1(u,v)\| \leq \frac{1}{\nu_{f_1}}\Big(\mu_{g_1}+(M_\phi+nN_u)r+\frac{1}{\Gamma(\alpha+1)}\big((\lambda_1+\lambda_2)r+\kappa_1\big)\Big). \tag{19}$$

Working in a similar manner, one can find that

$$\|\Theta_2(u,v)(t)\| \leq \frac{1}{\nu_{f_2}}\Big(\mu_{g_2}+(M_\psi+nN_v)r+\frac{1}{\Gamma(\beta+1)}\big((\zeta_1+\zeta_2)r+\kappa_2\big)\Big). \tag{20}$$

From (19) and (20), it follows that $\|\Theta(u,v)\| \leq r$.
Next, for $(u,v),(\bar{u},\bar{v}) \in X \times Y$and for any $t \in [0,1]$, we have

$$\begin{aligned}|\Theta_1(u,v)(t)-\Theta_1(\bar{u},\bar{v})(t)| = \Big| & \frac{1}{f_1(t,u(t),v(t))}\Big(g_1(t,u(t),v(t))+\phi(u)+\theta(t)\sum_{i=1}^{n} I_i(u(t_i^-))f_1(t_i,u(t_i),v(t_i)) \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s,u(s),v(s))ds\Big) \\ & - \frac{1}{f_1(t,\bar{u}(t),\bar{v}(t))}\Big(g_1(t,\bar{u}(t),\bar{v}(t))+\phi(u)+\theta(t)\sum_{i=1}^{n} I_i(\bar{u}(t_i^-))f_1(t_i,\bar{u}(t_i),\bar{v}(t_i)) \\ & + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h_1(s,u(s),v(s))ds\Big)\Big| \\ & \leq \frac{1}{\nu_{f_1}}\Big(M_{g_1}(|u-\bar{u}|+|v-\bar{v}|)+K_\phi|u-\bar{u}|+M_{f_1}(|u-\bar{u}|+|v-\bar{v}|) \\ & + nA|u-\bar{u}| + \frac{M_{h_1}}{\Gamma(\alpha+1)}(|u-\bar{u}|+|v-\bar{v}|)\Big),\end{aligned}$$

which implies that

$$\begin{aligned}\|\Theta_1(u,v)-\Theta_1(\bar{u},\bar{v})\| &\leq \frac{1}{\nu_{f_1}}\Big(M_{g_1}+K_\phi+M_{f_1}+nA+\frac{M_{h_1}}{\Gamma(\alpha+1)}\Big)(\|u-\bar{u}\|+\|v-\bar{v}\|) \\ &= \pi_1(\|u-\bar{u}\|+\|v-\bar{v}\|). \quad (21)\end{aligned}$$

Similarly, we can show that

$$\|\Theta_2(u,v)-\Theta_2(\bar{u},\bar{v})\| \leq \pi_2(\|u-\bar{u}\|+\|v-\bar{v}\|). \quad (22)$$

From (21) and (22), we deduce that

$$\|\Theta(u,v)-\Theta(\bar{u},\bar{v})\| \leq \max(\pi_1,\pi_2)(\|u-\bar{u}\|+\|v-\bar{v}\|).$$

□

In view of this condition $\max(\pi_1,\pi_2) < 1$, it follows that Θ is a contraction. So Banach's fixed point theorem applies and hence the operator Θ has a unique fixed point. This, in turn, implies that the problem (2) has a unique solution on J. This completes the proof.

Second result. In our second result, we discuss the existence of solutions for the problem (2) by means of Leray-Schauder alternative.
For brevity, let us set

$$\Lambda_1 = \frac{1}{\nu_{f_1}\Gamma(\alpha+1)}, \qquad \Lambda_2 = \frac{1}{\nu_{f_2}\Gamma(\beta+1)}. \quad (23)$$

$$\Lambda_0 = \min\{1-(\Lambda_1\rho_1+\Lambda_2\delta_1), 1-(\Lambda_1\rho_2+\Lambda_2\delta_2)\}. \quad (24)$$

Lemma 3.3. *(Leray-Schauder alternative see* [16]*). Let $\Pi : G \longrightarrow G$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let $\mathcal{P}(\Pi) = \{u \in G : u = \lambda\Pi u \quad for \quad some \quad 0 < \lambda < 1\}$. Then either the set $\mathcal{P}(\Pi)$ is unbounded or Π has at least one fixed point.*

Theorem 3.2. *Let's assume that conditions $(H_1)-(H_3)$ and $(H_8)-(H_{10})$ holds. Furthermore, it is assumed that $\Lambda_1\rho_1+\Lambda_2\delta_1 < 1$ and $\Lambda_1\rho_2+\Lambda_2\delta_2 < 1$, where μ_1 and μ_2 are given by* (23)*. Then the boundary value problem* (2) *has at least one solution.*

Proof. We will show that the operator $\Pi : X \times Y \longrightarrow X \times Y$ satisfes all the assumptions of Lemma 3.3.
Step 1: We will prove that the operator Π is completely continuous.
Clearly, it follows by the continuity of functions f_1, f_2, g_1, g_2, h_1, and h_2 that the operator Π is continuous.
Let $S \subset X \times Y$ be bounded. Then we can find positive constants Ω_1 and Ω_2 such that $|h_1(t,u,v)| \leq \Omega_1$ and $|h_2(t,u,v)| \leq \Omega_2$, for all $(u,v) \in S$.
Thus, for any $u, v \in S$, we can get

$$\begin{aligned}|\Pi_1(u,v)(t)| &\leq \frac{1}{\nu_{f_1}}\Big(\mu_{g_1} + \rho + \sum_{i=1}^{n} C + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\Omega_1 ds\Big) \\ &\leq \frac{1}{\nu_{f_1}}\big(\mu_{g_1} + \rho + nC + \frac{\Omega_1}{\Gamma(\alpha+1)}\big),\end{aligned}$$

which yields

$$\|\Pi_1(u,v)\| \quad \leq \quad \frac{1}{\nu_{f_1}}\big(\mu_{g_1} + \rho + nC + \frac{\Omega_1}{\Gamma(\alpha+1)}\big). \tag{25}$$

In a similar manner, we can show that

$$\|\Pi_2(u,v)\| \quad \leq \quad \frac{1}{\nu_{f_2}}\big(\mu_{g_2} + \mu + mD + \frac{\Omega_2}{\Gamma(\beta+1)}\big). \tag{26}$$

From the inequalities (25) and (26), we deduce that the operator Π is uniformly bounded.
Setep 2: Now we show that the operator Π is equicontinuous.

We take $\tau_1, \tau_2 \in J$ with $\tau_1 < \tau_2$ and obtain

$$
\begin{aligned}
&|\Pi_1(u(\tau_2), v(\tau_2)) - \Pi_1(u(\tau_1), v(\tau_1))| \\
&\leq \Big| \frac{1}{f_1(\tau_2, u(\tau_2), v(\tau_2))} \Big(\phi(u) + g_1(\tau_2, u(\tau_2), v(\tau_2)) + \theta(\tau_2) \sum_{i=1}^{n} I_i(u(t_i^-)) f_1(t_i, u(t_i), v(t_i)) \\
&+ \Omega_1 \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big) \\
&- \frac{1}{f_1(\tau_1, u(\tau_1), v(\tau_1))} \Big(\phi(u) + g_1(\tau_1, u(\tau_1), v(\tau_1)) + \theta(\tau_1) \sum_{i=1}^{n} I_i(u(t_i^-)) f_1(t_i, u(t_i), v(t_i)) \\
&+ \Omega_1 \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big) \Big| \\
&\leq \frac{1}{\nu_{f_1}} \Big(\Big| (g_1(\tau_2, u(\tau_2), v(\tau_2)) - g_1(\tau_1, u(\tau_1), v(\tau_1)) \\
&+ (\theta(\tau_2) - \theta(\tau_1)) \sum_{i=1}^{n} I_i(u(t_i^-)) f_1(t_i, u(t_i), v(t_i)) \Big| + \Omega_1 \Big| \int_0^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&- \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big| \Big) \\
&\leq \frac{1}{\nu_{f_1}} \Big(\Big| (g_1(\tau_2, u(\tau_2), v(\tau_2)) - g_1(\tau_1, u(\tau_1), v(\tau_1)) \\
&+ (\theta(\tau_2) - \theta(\tau_1)) \sum_{i=1}^{n} I_i(u(t_i^-)) f_1(t_i, u(t_i), v(t_i)) \Big| + \Omega_1 \Big| \int_0^{\tau_1} \frac{(\tau_1 - s)^{\alpha-1} - (\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \\
&- \int_{\tau_2}^{\tau_2} \frac{(\tau_2 - s)^{\alpha-1}}{\Gamma(\alpha)} ds \Big| \Big),
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Pi_2(u(\tau_2), v(\tau_2)) - \Pi_2(u(\tau_1), v(\tau_1))| \\
&\leq \frac{1}{\nu_{f_2}} \Big(\Big| (g_2(\tau_2, u(\tau_2), v(\tau_2)) - g_2(\tau_1, u(\tau_1), v(\tau_1)) \\
&+ (\omega(\tau_2) - \omega(\tau_1)) \sum_{j=1}^{n} I_j(v(t_j^-)) f_2(t_j, u(t_j), v(t_j)) \Big| \\
&+ \Omega_2 \Big| \int_0^{\tau_1} \frac{(\tau_1 - s)^{\beta-1} - (\tau_2 - s)^{\alpha-1}}{\Gamma(\beta)} ds - \int_{\tau_2}^{\tau_2} \frac{(\tau_2 - s)^{\beta-1}}{\Gamma(\beta)} ds \Big| \Big).
\end{aligned}
$$

Which tend to 0 independently of (u, v). This implies that the operator $\Phi(u, v)$ is equicontinuous. Thus, by the above fndings, the operator $\Pi(u, v)$ is completely continuous.
Setep 3: In this step, it will be established that the set $P = \{(u, v) \in X \times Y/(u, v) = \lambda\Phi(u, v), 0 < \lambda < 1\}$ is bounded.
Let $(u, v) \in \mathcal{P}$. Then we have $(u, v) = \lambda\Pi(u, v)$. Thus, for any $t \in [0, 1]$, we can write

$$u(t) = \lambda\Pi_1(u, v)(t),$$

$$v(t) = \lambda\Pi_2(u, v)(t),$$

Then, we obtain

$$\begin{aligned}\|u\| &\leq \frac{1}{\nu_{f_1}}\Big(\mu_{g_1} + \rho + nC + \frac{1}{\Gamma(\alpha+1)}(\rho_0 + \rho_1\|u\| + \rho_2\|v\|)\Big)\\ &\leq \frac{1}{\nu_{f_1}}(\mu_{g_1} + \rho + nC) + \Lambda_1(\rho_0 + \rho_1\|u\| + \rho_2\|v\|),\end{aligned}$$

which impllies that,

$$\begin{aligned}\|v\| &\leq \frac{1}{\nu_{f_2}}\Big(\mu_{g_2} + \mu + mD + \frac{1}{\Gamma(\beta+1)}(\delta_0 + \delta_1\|u\| + \delta_2\|v\|)\Big)\\ &\leq \frac{1}{\nu_{f_2}}(\mu_{g_2} + \mu + mD) + \Lambda_2(\delta_0 + \delta_1\|u\| + \delta_2\|v\|).\end{aligned}$$

In consequence, we have

$$\begin{aligned}\|u\| + \|(v)\| \leq{}& \frac{1}{\nu_{f_1}}\big(\mu_{g_1} + \rho + nC\big) + \frac{1}{\nu_{f_2}}\big(\mu_{g_2} + \mu + mD\big) + \Lambda_1\rho_0 + \Lambda_2\delta_0\\ &+ (\Lambda_1\rho_1 + \Lambda_2\delta_1)\|u\| + (\Lambda_1\rho_2 + \Lambda_2\delta_2)\|v\|,\end{aligned}$$

which, in view of (24), can be expressed as

$$\|(u,v)\| \leq \frac{\frac{1}{\nu_{f_1}}\big(\mu_{g_1} + +\rho + nC\big) + \frac{1}{\nu_{f_2}}\big(\mu_{g_2} + \mu + mD\big) + \Lambda_1\rho_0 + \Lambda_2\delta_0}{\Lambda_0}.$$

This shows that the set is bounded. Hence, all the conditions of Lemma 3.3 are satisfed and consequently the operator Π has at least one fxed point, which corresponds to a solution of system (2). This completes the proof. $\square$

4. Example

We will consider the following impulsive coupled hybrid system :

$$\begin{cases} D^{\frac{1}{2}}\Big(u(t)f_1(t,u(t),v(t)) - g_1(t,u(t),v(t))\Big) = h_1(t,u(t),v(t)), t \in [0,1]\setminus\{t_1\},\\ u(t_1^+) = u(t_1^-) + (-2u(t_1^-)), t_1 \neq 0,1\\ D^{\frac{1}{2}}\Big(v(t)f_2(t,u(t),v(t)) - g_2(t,u(t),v(t))\Big) = h_2(t,u(t),v(t)), t \in [0,1]\setminus\{t_1\},\\ v(t_1^+) = v(t_1^-) + (-2v(t_1^-)), t_1 \neq 0,1\\ u(0)f_1(0,u(0),v(0)) - g_1(0,u(0),v(0)) = \sum_{i=1}^{n} c_i u(t_i),\\ v(0)f_2(0,u(0),v(0)) - g_2(0,u(0),v(0)) = \sum_{j=1}^{m} d_j u(t_j)). \end{cases} \tag{27}$$

Here,

$$\begin{aligned} f_1(t,u,v) &= \frac{\arctan t}{3}|u| + \frac{t^2}{3e^t} + \cos(|v(t)|),\\ f_2(t,u,v) &= \frac{t}{5\pi e^t}|v| + \frac{1}{10}\sin(|u(t)|) + \frac{\cos t}{2\pi t^3},\\ g_1(t,u,v) &= \frac{1}{7} + \frac{1}{9}u(t) + \frac{1}{10}v(t),\\ g_2(t,u,v) &= \frac{1}{7} + \frac{1}{9}v(t) + \frac{1}{10}u(t),\\ h_1(t,u,v) &= \frac{1}{4t^2}(u + \sqrt{v^2+1}),\\ h_2(t,u,v) &= \frac{3\cos \pi t|u|}{5} + \frac{1}{9}|v|, \end{aligned}$$

Note that

$$|g_1(t,u_1,v_1) - g_1(t,u_2,v_2)| \leq \frac{1}{9} \mid u_2 - u_1 \mid + \frac{1}{10} \mid v_2 - v_1 \mid,$$
$$|g_2(t,u_1,v_1) - g_2(t,u_2,v_2)| \leq \frac{1}{9} \mid u_2 - u_1 \mid + \frac{1}{10} \mid v_2 - v_1 \mid,$$
$$\forall t \in [0.1], u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

and

$$|h_1(t,u_1,v_1) - h_1(t,u_2,v_2)| \leq \frac{1}{4} \mid u_2 - u_1 \mid + \frac{1}{4} \mid v_2 - v_1 \mid$$
$$|h_2(t,u_1,v_1) - h_2(t,u_2,v_2)| \leq \frac{3}{5} \mid u_2 - u_1 \mid + \frac{1}{9} \mid v_2 - v_1 \mid,$$
$$\forall t \in [0.1], u_1, u_2, v_1, v_2 \in \mathbb{R}.$$

$$\pi_1 = \frac{1}{\nu_{f_1}}\Big(M_{g_1} + K_\phi + M_{f_1} + nA + \frac{M_{h_1}}{\Gamma(\alpha+1)}\Big) = 0.12345678,$$
$$\pi_2 = \frac{1}{\nu_{f_2}}\Big(M_{g_2} + K_\psi + M_{f_2} + mB + \frac{M_{h_2}}{\Gamma(\beta+1)}\Big) = 0.23456789.$$

Thus all the assumptions in Theorem 3.2 are satisfied, our results can be applied to problem (27) .

5. Conclusion

The aim of this paper is to discuss the existence of solutions for impulsive coupled system of nonlinear hybrid fractional differential equations with linear and nonlinear perturbations. Our results improve and generalize some known results. In future, the established theory can be applied for more general problems of IFDEs with linear and nonlinear perturbations. Also the mentioned fixed-point theorems may be applied for existence theory of other type of IFDEs involving various kinds of fractional derivatives like Hilfer's Hadamard, etc.

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References

[1] Erdelyi, A., (1956), Asymptotic expansions, Dover publications, New York.
[2] Anderson, J. D., Campbell, J. K., Ekelund, J. E., Ellis, J. and Jordan, J. F., (2008), Physical Review Letters, 100, 091102.
[3] Mould, R. A., (1994), Basic Relativity, Springer-Verlag Newyork Inc.
[4] Thompson, K. W., (1987), Time dependent boundary conditions for hyperbolic systems, J. Comp. Phys, 68, pp. 1-24.
[5] Hixon, R. and Turkel, E., (2000), Compact implicit MacCormack-type schemes with high accuracy, J. Comp. Phys, 158, pp. 51-70.
[6] Fan, E. and Jian, Z., (2002), Applications of the Jacobi elliptic function method to special-type nonlinear equations, Phys. Lett. A, 305 (6), pp. 383-392.
[7] Popov, E., (1964), The dynamics of automatic control systems, Moskow. Gostehizdat.
[8] Babitskii, V. I., Krupenin,V. A., (1985), Vibration in strongly nonlinear systems, Moskow: Nauka.
[9] Chua, L., Yang, L., (1988), Cellular neural networks: applications, IEEE International Symposium on Circuits and Systems, 35:1273-90.
[10] Chernousko, F., Akulenko, L., Sokolov B., (1980), Control of oscillations. Moskow: Nauka.
[11] Andronov, A., Witt, A., Haykin, S.,(1981), Oscilation theory, Moskow: Nauka.

[12] Lakshmikanthan, V., Bainov, D. D., Simeonov, P. S., (1989), Theory of impulsive differential equations, Singapore: World Scientific.

[13] Zavalishchin, S., Sesekin, A., (1991), Impulsive processes: models and applications, Moskow: Nauka.

[14] Podlubny, I., (1999), Fractional Differential Equations, Academic Press, San Diego.

[15] Bainov, D., Dimitrova, M., Dishliev, A., (2000), Oscillation of the bounded solutions of impulsive differential-difference equations of second order, Appl Math Comput, 114(1): 61-8.

[16] Granas, A., Dugundji, J., (2003), Fixed Point Teory, Springer, New York, NY, USA.

[17] Dhage, B. C., Salunkhe, S. N., Agarwal, R. P., Zhang, W., (2005), A functional differential equation in Banach algebras, Mathematical Inequalities and Applications, (8), 89-99.

[18] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., (2006), Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, 204, Elsevier Science B.V., Amsterdam.

[19] Dhage, B. C., (2009), Nonlinear quadratic first order functional integro-differential equations with periodic boundary conditions, Dynamic Systems and Applications, 18(2), 303-322.

[20] Zhou, Y., Jiao, F., Li, J., (2009), Existence and uniqueness for fractional neutral differential equations with infinite delay, Nonlinear Anal. TMA, (71), 3249-3256.

[21] Dhage, B. C., (2010), Quadratic perturbations of periodic boundary value problems of second order ordinary differential equations. Differ. Equ. Appl, (2), 465-486.

[22] Dhage, B. C., Lakshmikantham,V., (2010), Basic results on hybrid differential equations, Nonlinear Analysis: Hybrid Systems, (4), 414-424.

[23] Zhao, Y., Sun, S., Han,Z., Li, Q., (2011), Theory of fractional hybrid differential equations, Comput. Math. Appl, (62), 1312-1324.

[24] Zhang, L., Wang, G., (2011), Existence of solutions for nonlinear fractional differential equations with impulses and anti-periodic boundary conditions, Electronic Journal of Qualitative Theory of Differential Equations, (7), 1-11.

[25] Dhage, B. C., (2012), Basic results in the theory of hybrid differential equations with mixed perturbations of second type, Funct. Differ. Equ, 19, 1-20.

[26] Sun, S., Zhao, Y., Han, Z., Li, Y., (2012), The existence of solutions for boundary value problem of fractional hybrid differentialequations, Commun. Nonlinear Sci. Numer. Simul, (17), 4961-4967.

[27] Dhage, B. C, Jadhav, N. S., (2013), Basic results in the theory of hybrid differential equations with linear perturbations of second type, Tamkang Journal of Mathematics, (44), 171-186.

[28] Ahmad, B., Ntouyas, S. K., (2014), An existence theorem for fractional hybrid differential inclusions of Hadamard type with Dirichlet boundary conditions, Abstr. Appl. Anal, Article ID 705809.

[29] Ahmad, B., Ntouyas, S. K., Alsaedi, A., (2014), Existence results for a system of coupled hybrid fractional differential equations, Sci. World. J, Article ID 426438.

[30] Shah, K., Khalil, H., Khan, R. A., (2015), Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations, Chaos Solitons Fractals, (77), 240-246.

[31] Hilal, K., Kajouni, A., (2015), Boundary value problems for hybrid differential equations, Mathematical Theory and Modeling, 2224-5804.

[32] Dumitru, B., Vahid, H., Shahram, R., Al Qurashi, M. M., (2016), On two fractional differential inclusions, SpringerPlus. 5(1): 882.

[33] Iqbal, M., Shah, K., Khan, R. A., (2019), On using coupled fixed-point theorems for mild solutions to coupled system of multipoint boundary value problems of nonlinear fractional hybrid pantograph differential, Math Meth Appl Sci. 1-12.

[34] Melliani, S., El Allaoui, A. , Chadli, L. S., (2020), Boundary Value Problem of Nonlinear Hybrid Differential Equations with Linear and Nonlinear Perturbations, International Journal of Differential Equations, Article ID 9850924.

[35] Hannabou, M., Hilal, K., (2020), Existence Results for a System of Coupled Hybrid Differential Equations with Fractional Order, Hindawi Publishing Corporation International Journal of Differential Equations, Article ID 3038427.

[36] Bouaouid, M., Hannabou, M., Hilal, K., (2020), Nonlocal Conformable Fractional Differential Equations with a Measure of Noncompactness in Banach Spaces, Hindawi Publishing Corporation Journal of Mathematics, Article ID 5615080.

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