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WHITTAKER'S EQUATION-BASED FORMULATION OF A NEW CLASS OF ANALYTIC FUNCTIONS COMBINED WITH GEOMETRIC ANALYSIS

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Abstract. A special function is a function with a particular use in mathematical physics or another branch of mathematics and is often named after an early scientist who researched its characteristics. A few noteworthy instances exist, such as the hypergeometric function and its distinct species. By using k-calculus, this sort of special function is made more generic. K-symbol calculus is utilized in this study to develop the k-convoluted operators associated with the k-Whittaker function (confluent hypergeometric function of the first kind). Through the use of this recently created operator, we propose a new geometric formula of normalized functions in the unit disk. Our strategy is to modify the theory of differential subordination, thus we geometrically investigate the most well-known characteristics of this new operator, including subordination features and coefficient bounds. We draw attention to some notable corollaries of our main conclusions as exceptional examples.

Keywords: univalent function, fractional calculus, the open unit disk, analytic function, subordination and superordination

AMS Subject Classification: 30C45

1. Introduction

An investigation of the many classes of operators done by function spaces is called operator theory. The features of the operators can lead to the formation of non-figurative organizations. In recent years, this principle has found considerable interest in presentations not merely in mathematics but also in other scientific fields, particularly physics. Currently, the operators in fractional calculus [1], fractal calculus, quantum calculus [2, 3] and k-symbol calculus [4] play a significant role in the development of applications in the fields of engineering, medicine, involving the dynamics of recent pandemics, economics,

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and computer sciences. Additional presentations of this theory emerged when certain classes of differential and integral operators convoluted with well-known analytic functions or special functions were protracted to the complex plane. The class of Whittaker functions [5] is generalized by using the concept of quantum calculus (q -deformed fractional calculus) by Kostant [6]. Recently, Schrader and Shapiro [7] established an integral transform using the theory of b -Whittaker functions which can be utilized to formulate certain hyperbolic hypergeometric integral evaluations. In a complex domain, the class of Whittaker functions is applied in the study of wave theory [8]. A novel method for the Whittaker function-based incomplete gamma function-based temporal discretization of the new Caputo-Hadamard fractional derivative is proposed in [9]. We extend our study in this direction by constructing the k-convoluted operators coupled to the k-Whittaker function. Utilizing this freshly created operator, we present a brand-new subclass of analytic functions for the open unit disk. Our strategy was motivated by the idea of differential subordination, and as a consequence, we geometrically studied the most well-known characteristics of this new operator, such as the subordination features and coefficient bounds. We highlight a few notable corollaries of our central conclusions as exemplary cases.

2. Methods

Whittaker (1903) developed an adapted description of the confluent hypergeometric equation, recognized as Whittaker's equation, which is a special instance of the resolution [5], in order to create the prescription encompassing the resolutions more symmetric. The arrangement of Whittaker's equation is as tails with the Whittaker's function Ω:

$$\frac{d^2\Omega(\zeta)}{d\zeta^2} + \left(-\frac{1}{4} + \frac{\vartheta}{\zeta} + \frac{1/4 - \upsilon^2}{\zeta^2}\right)\Omega(\zeta) = 0, \tag{1}$$

where the solutions are

$$\begin{aligned}\Omega_{\vartheta,\upsilon} &= \exp(-\zeta/2)\zeta^{\upsilon+1/2}\sum_{n=0}^{\infty}\left(\frac{(\upsilon-\vartheta+1/2)_n}{n!(2\upsilon+1)_n}\right)\zeta^n \\ &= \exp(-\zeta/2)\zeta^{\upsilon+1/2}\sum_{n=0}^{\infty}\left(\frac{(\upsilon-\vartheta+1/2)_n}{\Gamma(n+1)(2\upsilon+1)_n}\right)\zeta^n \\ &= \exp(-\zeta/2)\zeta^{\upsilon+1/2}\left(1+\frac{1/2+\upsilon-\vartheta}{1!(2\upsilon+1)}\zeta+\frac{(1/2+\upsilon-\vartheta)\,(3/2+\upsilon-\vartheta)}{2!(2\upsilon+1)(2\upsilon+2)}\zeta^2+...\right) \\ &= \exp(-\zeta/2)\zeta^{\upsilon+1/2}\left(1+\frac{1/2+\upsilon-\vartheta}{\Gamma(2)(2\upsilon+1)}\zeta+\frac{(1/2+\upsilon-\vartheta)\,(3/2+\upsilon-\vartheta)}{\Gamma(3)(2\upsilon+1)(2\upsilon+2)}\zeta^2+...\right)\end{aligned} \tag{2}$$

and

$$\Omega_{\vartheta,-\upsilon} = \exp(-\zeta/2)\zeta^{-\upsilon+1/2}\left(1+\frac{1/2-\upsilon-\vartheta}{\Gamma(2)(1-2\upsilon)}\zeta+\frac{(1/2-\upsilon-\vartheta)\,(3/2-\upsilon-\vartheta)}{\Gamma(3)(1-2\upsilon)(-2\upsilon+2)}\zeta^2+...\right),$$

where $(\omega)_n$ presents the Pochhammer symbol. Fig. 1 shows the 3D-plot of the function for different values of its parameters.

2.1. $K-$**symbol notion.** This notion is presented by Diaz and Osler in [10, 11], as tails:

Definition 2.1. *The resulting formula assumes the influence gamma function, occasionally known as the k-symbol gamma function:*

$$\Gamma_k(\zeta) = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{\zeta}{k}-1}}{(\zeta)_{n,k}}, \tag{3}$$

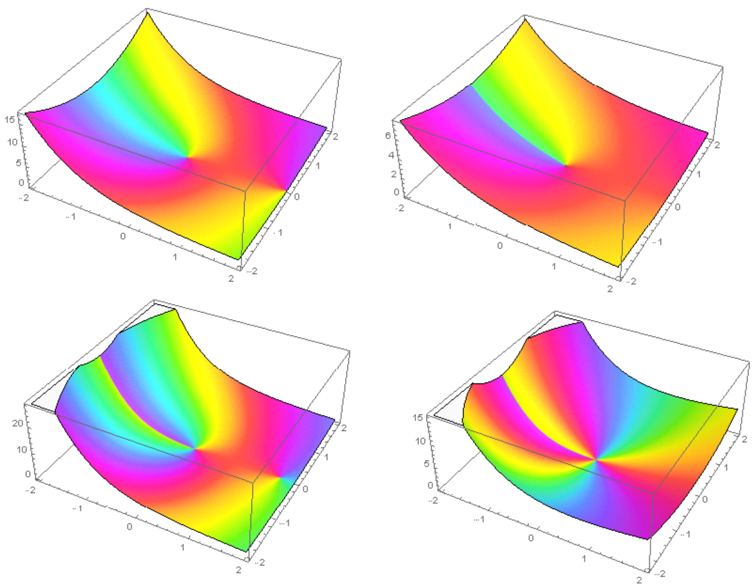

FIGURE 1. 3D-plot of Whittaker function for $(\vartheta, \upsilon) = \{(2,1/2),(1,1/4),(4,1),(2,2)\}$ respectively. Note that $\Omega_{2,1/2}(\zeta) = \zeta - \zeta^2 + \frac{3\zeta^3}{8} - \frac{\zeta^4}{12} + \frac{5\zeta^5}{384} - \frac{\zeta^6}{640} + O\left(\zeta^7\right)$

where

$$(\zeta)_{n,k} := \zeta(\zeta+k)(\zeta+2k)...(\zeta+(n-1)k)$$

and

$$(\zeta)_{n,k} = \frac{\Gamma_k(\zeta+nk)}{\Gamma_k(\zeta)}.$$

Note that $\Gamma_k(\zeta) \to \Gamma(\zeta)$ *when* $k \to 1$, *and*

$$\Gamma_k(\zeta+k) = \zeta\Gamma_k(\zeta), \quad \Gamma_k(k) = 1.$$

2.2. $K-$**symbol Whittaker's function.** Whittaker's function for $k-$symbols can be modified as given below, utilizing the $k-$symbol description:

$$\begin{aligned}\Omega^k_{\vartheta,\upsilon} &= \exp(-\zeta/2)\zeta^{\upsilon+1/2}\sum_{n=0}^{\infty}\left(\frac{(\upsilon-\vartheta+1/2)_{n,k}}{\Gamma_k(n+1)(2\upsilon+1)_{n,k}}\right)\zeta^n \qquad (4)\\ &= \exp(-\zeta/2)\zeta^{\upsilon+1/2}\left(1+\frac{1/2+\upsilon-\vartheta}{\Gamma_k(2)(2\upsilon+1)}\zeta+\frac{(1/2+\upsilon-\vartheta)(1/2+\upsilon-\vartheta+k)}{\Gamma_k(3)(2\upsilon+1)(2\upsilon+1+k)}\zeta^2+...\right)\end{aligned}$$

Obviously, for $k=1$, we have the traditional formula of the Whittaker's function (2). Generally, the coefficients of the $k-$symbol Whittaker's function are calculated in the following proposition:

Proposition 2.1. *The general formula of the $n-th$ coefficient of the $k-$symbol Whittaker's function is*

$$\begin{aligned}
\omega_{0,k} &= 1 \qquad (5)\\
\omega_{1,k} &= \frac{1/2+\upsilon-\vartheta}{\Gamma_k(2)(2\vartheta+1)}\\
\omega_{2,k} &= \frac{(1/2+\upsilon-\vartheta)\,(1/2+\upsilon-\vartheta+k)}{\Gamma_k(3)(2\upsilon+1)(2\upsilon+1+k)}\\
&\vdots\\
\omega_{n,k} &= \frac{(1/2+\upsilon-\vartheta)\,(1/2+\upsilon-\vartheta+k)\ldots(1/2+\upsilon-\vartheta+(n-1)k)}{\Gamma_k(n+1)(2\upsilon+1)(2\upsilon+1+k)\ldots(2\upsilon+1+(n-1)k)}.
\end{aligned}$$

Proof. The proof is occurred by using Eq.(4). □

2.3. **Convoluted operators.** In this part, we proceed to define the $k-$symbol convoluted operator by using the $k-$symbol Whittaker's function. The following definition can be found in [12].

Definition 2.2. *Suppose that Λ is the subspace of analytic functions in the open unit disk $\mathcal{O} := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ owning the power series*

$$\eta(\zeta) = \zeta + \sum_{n=2}^{\infty} \eta_n\,\zeta^n, \quad \zeta \in \mathcal{O}, \quad \eta(0) = \eta'(0) - 1 = 0.$$

The normalized functions $\eta, \psi \in \Lambda$ are convoluted ($\eta*\psi$) if they have the following product

$$(\eta*\psi)(\zeta) = \left(\zeta + \sum_{n=2}^{\infty} \eta_n\,\zeta^n\right) * \left(\zeta + \sum_{n=2}^{\infty} \psi_n\,\zeta^n\right) = \zeta + \sum_{n=2}^{\infty} \eta_n\,\psi_n\,\zeta^n.$$

Definition 2.3. *Define two classes of analytic functions in Λ: the starlike subclass $\mathcal{S}*$ and the convex subclass $\mathcal{C}$. Finally, the class $\mathcal{P} := \{\wp : \wp(\zeta) = 1 + \wp_1\zeta + \wp_2\zeta^2 + ..., \zeta \in \mathcal{O}\}$ is a special class of analytic functions in $\mathcal{O}$ with positive real part in $\mathcal{O}$ and $\wp(0) = 1$.*

The next concept can be located in [13] and [14]

Definition 2.4. *Two analytic functions χ_1, χ_2 in $\mathcal{O}$ are subordinated denoting by $\chi_1 \prec \chi_2$ or $\chi_1(\zeta) \prec \chi_2(\zeta), \zeta \in \mathcal{O}$ if for an analytic function $\alpha, |\alpha| \le |\zeta| < 1$ achieving the equation $\chi_1(\zeta) = \chi_2(\alpha(\zeta)), \zeta \in \mathcal{O}$.*

The $k-$symbol Whittaker's function can be normalized as follows:

$$\begin{aligned}
\Psi^k_{\vartheta,\upsilon}(\zeta) &:= \exp(\zeta/2)\zeta^{1-\upsilon-1/2}\Omega^k_{\vartheta,\upsilon}\\
&= \left(\zeta + \frac{1/2+\upsilon-\vartheta}{\Gamma_k(2)(2\upsilon+1)}\zeta^2 + \frac{(1/2+\upsilon-\vartheta)\,(1/2+\upsilon-\vartheta+k)}{\Gamma_k(3)(2\upsilon+1)(2\upsilon+1+k)}\zeta^3 + ...\right)\\
&= \zeta + \sum_{n=2}^{\infty}\left(\frac{(\upsilon-\vartheta+1/2)_{n-1,k}}{\Gamma_k(n)(2\upsilon+1)_{n-1,k}}\right)\zeta^n\\
&:= \zeta + \sum_{n=2}^{\infty}\psi_n(\vartheta,\upsilon;k)\zeta^n, \quad \psi_n(\vartheta,\upsilon;k) = \left(\frac{(\upsilon-\vartheta+1/2)_{n-1,k}}{\Gamma_k(n)(2\upsilon+1)_{n-1,k}}\right)
\end{aligned}$$

Thus, by using the definition of the convolution product, we formulate the following normalized $k-$symbol convoluted operator in $\mathcal{O}$:

$$[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta) = \left(\zeta + \sum_{n=2}^{\infty} \psi_n(\vartheta,\upsilon;k)\,\zeta^n\right) * \left(\zeta + \sum_{n=2}^{\infty} \eta_n\,\zeta^n\right) \tag{6}$$
$$= \zeta + \sum_{n=2}^{\infty} \psi_n(\vartheta,\upsilon;k)\eta_n\,\zeta^n.$$

When $k = 1$, we obtain the convoluted Whittaker's operator (see Fig.2)

$$[\Psi_{\vartheta,\upsilon} * \eta](\zeta) = \zeta + \sum_{n=2}^{\infty} \psi_n(\vartheta,\upsilon;1)\eta_n\,\zeta^n.$$

Since the Whittaker's function is a a modified types of the hypergeometric function,

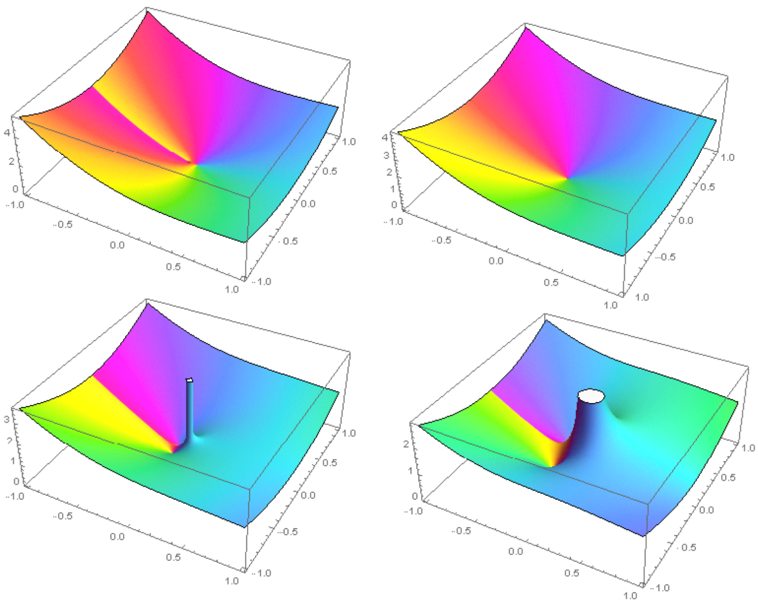

FIGURE 2. 3D-plot of convoluted Whittaker operator $[\Psi_{\vartheta,\upsilon} * \eta](\zeta)$ for $(\vartheta,\upsilon) = \{(2,1/4),(1,1/2),(2,3/4),(2,1)\}$ respectively, with $\eta(\zeta) = \zeta/(1-\zeta)$.

then the proposed operator $[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)$ involves different types of well-known convoluted operator, like Carlson and Shaffer convoluted operator [15], Salagean differential operator [16] and Ruscheweyh product [17], for the case $k = 1$; and it involves the $k-$symbol [18, 19]. Our aim in this illustration is to study the proposed operator $[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)$ geometry. We shall present a collection of results involving sufficient conditions for stralikness and convexity.

2.4. **New classes of analytic functions.** In this part, we propose new classes of analytic functions owing the structure of Whattaker equation Eq. (1). Therefore, from Eq. (1), we have the structure

$$\left(1+\frac{\zeta\Omega''(\zeta)}{\Omega'(\zeta)}\right)+\zeta^2\left(-\frac{1}{4}+\frac{\vartheta}{\zeta}+\frac{1/4-\upsilon^2}{\zeta^2}\right)\left(\frac{1}{\frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)}}\right)=1.$$

Consequently, we obtain

$$\left(1+\frac{\zeta\Omega''(\zeta)}{\Omega'(\zeta)}\right)-\left(\zeta-2\vartheta+\sqrt{4(\vartheta^2+\upsilon^2)-1}\right)\left(\zeta-2\vartheta-\sqrt{4(\vartheta^2+\upsilon^2)-1}\right)$$

$$\times\left(\frac{1}{\frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)}}\right)=1.$$

Dividing by $\frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)}$, we obtain

$$\frac{\left(1+\frac{\zeta\Omega''(\zeta)}{\Omega'(\zeta)}\right)}{\frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)}}-\left(\zeta-2\vartheta+\sqrt{4(\vartheta^2+\upsilon^2)-1}\right)\left(\zeta-2\vartheta-\sqrt{4(\vartheta^2+\upsilon^2)-1}\right)$$

$$\times\left(\frac{1}{\frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)}}\right)^2=\frac{1}{\frac{\zeta\Omega'(\zeta)}{\Omega(\zeta)}}.$$

We proceed to define the Whittaker's class of analytic function by using the subordination concept or its equivalent inequality. Then the analytic function $\eta\in\Lambda$ is called in the class Θ_β if and only if $\upsilon^2\neq 1/4$ and

$$\begin{aligned}\Theta_\beta &:= \left\{\eta : \left|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}-\frac{\sigma(\zeta)}{(\upsilon^2-1/4)}\left(\frac{1}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)^2\right|<\beta, \zeta\in\mathcal{O}, \beta\in(0,1)\right\} \quad (7)\\ &:= \left\{\eta : \left|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}-\rho(\zeta)\left(\frac{1}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)^2\right|<\beta, \zeta\in\mathcal{O}, \beta\in(0,1)\right\}\end{aligned}$$

where

$$\begin{aligned}\sigma(\zeta) &:= \left(\frac{\zeta^2}{4}-\vartheta\zeta-1/4+\upsilon^2\right)\\ &= \left(\zeta-2\vartheta+\sqrt{4(\vartheta^2+\upsilon^2)-1}\right)\left(\zeta-2\vartheta-\sqrt{4(\vartheta^2+\upsilon^2)-1}\right).\end{aligned}$$

And an analytic function $\eta \in \Lambda$ is called in the $k-$symbol Whittaker's class of analytic functions $W^k_{\vartheta,\upsilon}(\rho)$ if and only if $\upsilon^2 \neq 1/4$ and

$$\Theta^{k,\gamma}_{\vartheta,\upsilon} := \tag{8}$$

$$\Big\{\eta : \frac{\left(1+\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]''(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}\right)}{\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}} - \frac{\sigma(\zeta)}{(\upsilon^2-1/4)}\left(\frac{1}{\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}}\right)^2 \prec \gamma(\zeta), \zeta \in \mathcal{O}\Big\}$$

$$:= \Big\{\eta : \frac{\left(1+\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]''(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}\right)}{\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}} - \rho(\zeta)\left(\frac{1}{\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}}\right)^2 \prec \gamma(\zeta), \zeta \in \mathcal{O}\Big\},$$

where $\gamma(0) = 0$. Note that the geometric feature of $\sigma(\zeta)$ is a parabola. Therefore, it is defined in the set $\Sigma := \{\zeta = x + iy \in \mathcal{O} : y < x^2\}$. We request Jack lemma in our investigation [20]

Lemma 2.1. *Assume that ϖ is analytic in $|\zeta| \leq r$ satisfying the properties that $\varpi(0) = 0, \quad |\varpi(\zeta_0)| = \max_{|\zeta|=r} |\varpi(\zeta)|$. Then $\zeta_0\varpi'(\zeta_0) = \kappa\varpi(\zeta_0), \quad \kappa \geq 1$.*

3. Results

We go through some of the geometric habits of the class Θ_β and $\Theta^{k,\gamma}_{\vartheta,\upsilon}$.

3.1. The class Θ_β.

Theorem 3.1. *Consider the class Θ_β such that $\frac{\sigma(\zeta)}{(\upsilon^2-1/4)} \prec \left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2, \quad \zeta \in \mathcal{O}$. If $\upsilon \neq \pm 1/2$ and $\beta = \frac{(1-\delta)}{2\delta^2}, \quad \delta \in [1/2, 1)$ then $\Theta_\beta \subset S^*(\delta)$ (the class of starlike of order δ)*

Proof. Let $\eta \in \Theta_\beta$. By using the analytic function $w(\eta) \in \mathcal{O}$ with $w(0) = 0$, the real

$$\Re\left(\frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}\right) > \delta, \quad \zeta \in \mathcal{O}$$

if and only if $|w(\zeta)| < 1$ (Schwarz function). Moreover, the Schwarz function satisfies the equality

$$\frac{\sigma(\zeta)}{(\upsilon^2-1/4)} = (\rho(w(\zeta)))^2, \quad \zeta \in \mathcal{O},$$

where

$$\rho(w(\zeta)) := \frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}.$$

Also, we have

$$\frac{\zeta\eta'(\zeta)}{\eta(\zeta)} = \frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)},$$

consequently, we obtain

$$\begin{aligned}1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)} &= \rho(\zeta)+\frac{\zeta\rho'(\zeta)}{\rho(\zeta)}\\ &= \frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}+\frac{\zeta[\frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}]'}{\frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}}\\ &= \frac{(-2\delta w(\zeta)+w(\zeta)+1)^2-2(\delta-1)\zeta w'(\zeta)}{(w(\zeta)-1)((2\delta-1)w(\zeta)-1)}\\ &:= \Phi(w(\zeta))\end{aligned}$$

and

$$\begin{aligned}&\left|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}-\frac{\sigma(\zeta)}{(\upsilon^2-1/4)}\left(\frac{1}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)^2\right|\\ &=\left|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}-\left(\frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}\right)^2\left(\frac{1}{\frac{1+(1-2\delta)w(\zeta)}{1-w(\zeta)}}\right)^2\right|\\ &=\left|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}-1\right|\\ &=\left|\frac{\zeta\rho'(\zeta)}{\rho^2(\zeta)}\right|=\left|\frac{2(1-\delta)\zeta w'(\zeta)}{(1+(1-2\delta)w(\zeta))^2}\right|.\end{aligned}$$

Suppose that $\eta\notin S^*(\delta)$. Then according to Jack Lemma 2.1, there occurs a point $\zeta_0\in\mathcal{O}$ for which $|w(\zeta_0)|=1$ and $\zeta_0 w'(\zeta_0)\geq w(\zeta_0)$. As a conclusion, we get the inequality

$$\left|\frac{\zeta_0\rho'(\zeta_0)}{\rho_0^2(\zeta)}\right|\geq\frac{2(1-\delta)}{(2\delta)^2}.$$

It runs counter to our theory. Hence, $\eta\in S^*(\delta)$, $\delta\in[1/2,1)$. The evidence is now complete. □

Corollary 3.1. *Let the assumptions of Theorem 3.1 be held. Then $\Theta_\beta\subset S^*(1/2)$. Moreover, if*

$$\Re\left(\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)>1/2$$

then $\eta\in S^*(1/2)$.

Proof. Let $\beta=1$ in Theorem 3.1. By the first part and the fact that

$$|\ell-1|<1\Leftrightarrow\Re(1/\ell)>1/2,$$

we have the second assertion. □

Corollary 3.2. *Let the assumptions of Theorem 3.1 be held. If* $\eta \in S^*(1/2)$*. Then*

$$\Big|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}} - \frac{\sigma(\zeta)}{(\upsilon^2-1/4)}\Big| < 1, \quad |\zeta| < 0.68...$$

and the result is sharp.

Proof. By the assumption of the corollary, we have

$$\Big|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}} - \frac{\sigma(\zeta)}{(\upsilon^2-1/4)}\left(\frac{1}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)^2\Big| \le \Big|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}} - 1\Big|.$$

Thus, by the outcome [21]-Theorem 2, we arrive at the request result. □

The class Θ_β can be considered for general formula by using the analytic function $\rho \in \mathcal{P}$, where $\rho(0) = 1$, as follows

$$\Theta_\beta(\rho) := \Big\{\eta : \Big|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}} - \rho(\zeta)\left(\frac{1}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)^2\Big| < \beta, \zeta \in \mathcal{O}, \beta \in (0,1)\Big\}. \tag{9}$$

Theorem 3.2. *Consider the class* $\Theta_\beta, \beta \le \sqrt{2}/2$ *such that*

$$\frac{\sigma(\zeta)}{(\upsilon^2-1/4)} \prec \left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2, \quad \zeta \in \mathcal{O}.$$

If $\upsilon \neq \pm 1/2$ *then* $\Theta_\beta \subset \mathcal{C}$ *(the class of convex functions).*

Proof. Let $\eta \in \Theta_\beta$. From the proof of Theorem 3.1, we have

$$\Big|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}} - \frac{\sigma(\zeta)}{(\upsilon^2-1/4)}\left(\frac{1}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}}\right)^2\Big| = \Big|\frac{\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right)}{\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}} - 1\Big|.$$

Now let ς be a Schwarz function, such that

$$\frac{\zeta\eta'(\zeta)}{\eta(\zeta)} = \frac{1}{1-\varsigma(\zeta)}, \quad \zeta \in \mathcal{O}.$$

From Corollary 3.1, we have $\Theta_\beta \subset S^*(1/2)$ then we have $|\zeta\varsigma'(\zeta)| < 2\sqrt{2}$ and, consequently $|\varsigma(\zeta)| < 2\sqrt{2}$. A computation leads to

$$\Re\left(1+\frac{\zeta\eta''(\zeta)}{\eta'(\zeta)}\right) = \Re\left(\frac{1+\zeta\varsigma'(\zeta)}{1-\varsigma(\zeta)}\right) > 0,$$

where

$$\Big|\arg\left(\frac{1+\zeta\varsigma'(\zeta)}{1-\varsigma(\zeta)}\right)\Big| \le |\arg 1+\zeta\varsigma'(\varsigma)| + |\arg 1+\zeta\varsigma'(\zeta)| \le 2(\frac{\pi}{4}) = \frac{\pi}{2}.$$

Hence, the proof. □

Theorem 3.3. *Consider the class* $\Theta_\beta(\rho)$ *such that*

$$\rho(\zeta) \prec \left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2, \quad \zeta \in \mathcal{O}.$$

If

$$\beta = \frac{(1-\delta)}{2\delta^2}, \quad \delta \in [1/2, 1)$$

then $\Theta_\beta(\rho) \subset S^*(\delta)$.

Corollary 3.3. [21] *Let* $\rho \in \mathcal{P}$. *If* $\rho(\zeta) = \left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2, \quad \zeta \in \mathcal{O}$, *then* $\beta = \frac{(1-\delta)}{2\delta^2}, \quad \delta \in [1/2, 1)$ *implies* $\Theta_\beta(\left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2) \subset S^*(\delta)$.

Similarly, for convexity in Theorem 3.2, we have

Theorem 3.4. *Consider the class* $\Theta_\beta(\rho), \beta \leq \sqrt{2}/2$ *such that* $\rho(\zeta) \prec \left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2, \quad \zeta \in \mathcal{O}$. *Then* $\Theta_\beta(\rho) \subset \mathcal{C}$.

Corollary 3.4. [21] *If* $\rho(\zeta) = \left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2, \quad \zeta \in \mathcal{O}$. *Then* $\Theta_\beta(\left(\frac{\zeta\eta'(\zeta)}{\eta(\zeta)}\right)^2) \subset \mathcal{C}$.

3.2. **The class** $\Theta^{k,\gamma}_{\vartheta,\upsilon}$**.** In this part, we deal with the convoluted operator (6). Assume that

$$\Psi(\zeta) := \frac{\left(1 + \frac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]''(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}\right)}{\frac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)}} - \rho(\zeta)\left(\frac{1}{\frac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)}}\right)^2.$$

To obtain our final result, we need the following outcome [22]-Proposition 1.1.

Lemma 3.1. *Let* g *be analytic in* $\mathcal{O}$. *Then for any* $|\zeta| = r < 1$ *and any real function* ϵ *on* $\mathcal{O}$ *the following inequality holds*

$$\Re\left(e^{i\epsilon(\zeta)}g(\zeta)\right) \leq \left(\frac{2r(1 - r\cos(\epsilon(\zeta)))}{1 - r^2}\right) \max_{\zeta\in\mathcal{O}} \Re(g(\zeta)).$$

The result is sharp.

Theorem 3.5. *Let* $\eta \in \Theta^{k,\gamma}_{\vartheta,\upsilon}$. *Then the convoluted operator* (6) *is bounded in* $\mathcal{O}$.

Proof. For the proof, it is sufficient to show that Ψ is bounded in $\mathcal{O}$. Since $\eta \in \Theta^{k,\gamma}_{\vartheta,\upsilon}$ then there is an analytic function according to the specification of the subordination $\varpi \in \mathcal{O}$ with the properties $\varpi(0) = 0$ and $r := |\varpi| \leq |\zeta| < 1$ such that $\Psi(\zeta) = \gamma(\varpi(\zeta)), \quad \zeta \in \mathcal{O}$. It is clear that $\gamma(0) = \Psi(0) = 0$, thus it is enough to show that γ is bounded in $\mathcal{O}$. Assume that $U_\gamma := \sup_{\zeta\in\mathcal{O}} \Re(\gamma(\zeta))$. Then, by the maximum principle for analytic functions, $U_\gamma > \gamma(0) = 0$. Now, define the function $\varpi(\zeta)$, as follows:

$$u := \omega(\zeta) := -2U_\gamma\left(\frac{\zeta}{1-\zeta}\right), \quad \zeta \in \mathcal{O},$$

where $\omega(0) = 0$. Moreover, define the inverse function $\wp(u) := \dfrac{u}{u - 2U_\gamma}$. Consider the function

$$w(\zeta) = \wp(\gamma(\zeta)) := \frac{\gamma(\zeta)}{\gamma(\zeta) - 2U_\gamma}.$$

In view of the conformal mapping theorem [23], the function w is analytic in $\mathcal{O}$ and $|w| \le 1$ and $w(\zeta) \neq 0, \zeta \neq 0$ and $w(0) = 0$. If $\gamma(\zeta) = \gamma_1 + i\gamma_2$, then we have the inequality $-2U_\gamma + \gamma_1 \le \gamma_1 \le 2U_\gamma - \gamma_1$, which implies that $|\gamma_1| \le 2U_\gamma - \gamma_1$ and thus

$$|w(\zeta)|^2 = \frac{\gamma_1^2 + \gamma_2^2}{(2U_\gamma - \gamma_1)^2 + \gamma_2^2} \le 1.$$

Hence, in view of Schwarz lemma, we obtain $|w(\zeta)| \le r$. Consequently, we have

$$|\gamma(\zeta)| = \left| \frac{2U_\gamma w(\zeta)}{1 - w(\zeta)} \right| \le \frac{2U_\gamma r}{1-r};$$

which gives

$$|\gamma(\zeta)| \le \frac{2r}{1-r} \max_{\zeta \in \mathcal{O}} \Re(\gamma(\zeta)).$$

That is γ is bounded and consequently, we obtain the boundedness of Ψ. This completes the proof. □

For the next result, we aim to find the coefficient bound of the function $\Psi(\zeta) = \sum_{n=0}^{\infty} \psi_n \zeta^n$ using the upper bound of $\gamma(\zeta) = \sum_{n=0}^{\infty} \varphi_n \zeta^n$.

Theorem 3.6. *Let $\eta \in \Theta_{\vartheta,\upsilon}^{k,\gamma}$. Then $|\psi_n| \le 2 \max_{\zeta \in \mathcal{O}} \Re(\gamma(\zeta))$, $\quad \zeta \in \mathcal{O}$.*

Proof. Since $\eta \in \Theta_{\vartheta,\upsilon}^{k,\gamma}$ then, according to the subordination's inclusion, there is an analytic function $\varpi \in \mathcal{O}$ with the properties $\varpi(0) = 0$ and $R := |\varpi| \le |\zeta| < 1$ such that

$$\Psi(\zeta) = \sum_{n=0}^{\infty} \psi_n \zeta^n = \gamma(\chi(\zeta)) = \sum_{n=0}^{\infty} \varphi_n \zeta^n.$$

Define the functional $\ell(R) = \max_n |\varphi_n| R^n$. This yields that

$$|\gamma(\zeta)| \le \ell(R) \sum_{n=0}^{\infty} r^n = \frac{r}{1-r} \ell(R), \quad |\zeta| = r < 1. \tag{10}$$

From the Maz'ya and Shaposhnikova estimation of coefficient [24] of the series (10), we obtain

$$\begin{aligned}
|\varphi_n(\varpi)| &= \left| \frac{1}{\pi} \int_0^{2\pi} \gamma_1(\theta) e^{-in\theta} \, d\theta \right| \\
&= \frac{1}{\pi} \max_{\xi \in [0,2\pi]} \int_0^{2\pi} \gamma_1(\theta) \left(1 + \cos(n\theta - \xi)\right) \, d\theta \\
&= \frac{1}{\pi} \max_{\xi \in [0,2\pi]} \int_0^{2\pi} \gamma_1(\theta) \, d\theta + \frac{1}{\pi} \max_{\xi \in [0,2\pi]} \int_0^{2\pi} \gamma_1(\theta) \cos(n\theta - \xi) \, d\theta \\
&\le \frac{1}{\pi} \max_{\xi \in [0,2\pi]} \int_0^{2\pi} \gamma_1(\theta) \, d\theta + \max_{\xi \in [0,2\pi]} \gamma_1(\theta) \left(\frac{1}{\pi} \int_0^{2\pi} \cos(n\theta - \xi) \, d\theta \right) \\
&= 2 \max_{\xi \in [0,2\pi]} \gamma_1(\theta) = 2 \max_{\zeta \in \mathcal{O}} \Re(\gamma(\zeta)),
\end{aligned}$$

where $\int_0^{2\pi} \gamma_1(\theta)\, d\theta = 0$, and $\dfrac{1}{\pi}\int_0^{2\pi} \cos(n\theta - \xi)\, d\theta = 2$ such that $|\zeta| = \max_{\xi\in[0,2\pi]} \Re(e^{i\xi}\zeta)$. Thus, we conclude that $|\psi_n| \leq 2\max_{\zeta\in\mathcal{O}} \Re(\gamma(\zeta)), \quad \zeta \in \mathcal{O}$. □

Theorem 3.7. *Let $\eta \in \Theta^{k,\gamma}_{\vartheta,\upsilon}$. Then for all $|\zeta| = r < 1$ and for arbitrary real valued function ϵ on $\mathcal{O}$, the inequality $\Re\left(e^{i\epsilon(\zeta)}\Psi(\zeta)\right) \leq c\sup_{|\zeta|=r<1}\left(\dfrac{r}{1-r}\right)^2, \quad c > 0$ occurs.*

Proof. By Theorem 3.6, we have

$$\Re(\gamma(\zeta)) \leq |\gamma(\zeta)| \leq \ell(R)\sum_{n=0}^{\infty} r^n = \frac{r}{1-r}\ell(R), \quad |\zeta| = r < 1.$$

That is $\gamma(\zeta)$ is bounded from above. Thus, in view of Lemma 3.1, we have

$$\begin{aligned}\Re\left(e^{i\epsilon(\zeta)}\gamma(\zeta)\right) &\leq \left(\frac{2r(1 - r\cos(\epsilon(\zeta)))}{1-r^2}\right)\max_{\zeta\in\mathcal{O}}\Re(\gamma(\zeta))\\ &\leq \left(\frac{2r(1 - r\cos(\epsilon(\zeta)))}{1-r^2}\right)\sup_{|\zeta|=r<1}\frac{r}{1-r}\ell(R).\end{aligned}$$

But, for analytic function $\chi, |\chi| = R < 1$, we have $\Psi(\zeta) = \gamma(\chi(\zeta)), \quad \zeta \in \mathcal{O}$. By letting $\cos(\epsilon(\zeta)) \to^- 1$, we have

$$\begin{aligned}\Re\left(e^{i\epsilon(\zeta)}\gamma(\chi(\zeta))\right) &\leq \left(\frac{2r(1 - r\cos(\epsilon(\zeta)))}{1-r^2}\right)\max_{\zeta\in\mathcal{O}}\Re(\gamma(\zeta))\\ &\leq \left(\frac{2r(1 - r\cos(\epsilon(\zeta)))}{1-r^2}\right)\sup_{|\zeta|=r<1}\frac{r}{1-r}\ell(R)\\ &= \left(\frac{2r(1+r)}{1-r^2}\right)\sup_{|\zeta|=r<1}\frac{r}{1-r}\ell(R)\\ &\leq 2\ell(R)\sup_{|\zeta|=r<1}\left(\frac{r}{1-r}\right)^2 := c\sup_{|\zeta|=r<1}\left(\frac{r}{1-r}\right)^2.\end{aligned}$$

Hence, the inequality $\Re\left(e^{i\epsilon(\zeta)}\Psi(\zeta)\right) \leq c\sup_{|\zeta|=r<1}\left(\dfrac{r}{1-r}\right)^2, \quad c > 0$ occurs. □

The next result shows the sufficient conditions to obtain a univalent starlike solution of the generalized Whittaker equation in terms of the convoluted operator

$$\frac{\left(1 + \dfrac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]''(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}\right)}{\dfrac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)}} - \rho(\zeta)\left(\frac{1}{\dfrac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)}}\right)^2 = \gamma(\zeta). \tag{11}$$

Let

$$p(\zeta) := \frac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)}.$$

A computation implies that

$$1 + \frac{\zeta[\Psi^k_{\vartheta,\upsilon} * \eta]''(\zeta)}{[\Psi^k_{\vartheta,\upsilon} * \eta]'(\zeta)} = p(\zeta) + \frac{\zeta p'(\zeta)}{p(\zeta)}.$$

Thus, Eq.(11) becomes

$$\frac{p(\zeta)+\dfrac{\zeta p'(\zeta)}{p(\zeta)}}{p(\zeta)}-\frac{\rho(\zeta)}{p^2(\zeta)}=\gamma(\zeta). \tag{12}$$

Which is equivalent to

$$1+\frac{\zeta p'(\zeta)}{p^2(\zeta)}-\frac{\rho(\zeta)}{p^2(\zeta)}-\gamma(\zeta)=0. \tag{13}$$

Rearrange Eq.(13), we obtain

$$\zeta p'(\zeta)\left(\frac{1}{p^2(\zeta)}\right)+p^2(\zeta)\left(\frac{-\rho(\zeta)}{p^4(\zeta)}\right)+1=\gamma(\zeta). \tag{14}$$

By the following assumptions

$$A(\zeta):=\frac{1}{p^2(\zeta)},\quad B(\zeta):=\frac{-\rho(\zeta)}{p^4(\zeta)},\quad D(\zeta):=1,$$

we have the following equation

$$A(\zeta)\zeta p'(\zeta)+B(\zeta)p^2(\zeta)+D(\zeta)=\gamma(\zeta). \tag{15}$$

We formulate the next result.

Theorem 3.8. *Consider* (15). *If the inequalities are satisfied* $\Re(\gamma(\zeta))>0, \Re(A(\zeta))>0, \Re\left(A(\zeta)+2B(\zeta)\right)\Re\left(A(\zeta)-2\right)\geq 0$ *then* $[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)$ *is starlike. Also, it satisfies*

$$\frac{1-r}{1+r}\leq\Re\left(\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}\right)\leq\frac{1+r}{1-r},\quad |\zeta|=r<1 \tag{16}$$

and

$$\left|\Im\left(\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}-1\right)\right|\leq\frac{2r}{1-r^2}\sup_{|\zeta|<1}\Re\left(\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}-1\right). \tag{17}$$

Proof. By the first assumption, we have the following real inequality

$$\Re\left(A(\zeta)\zeta p'(\zeta)+B(\zeta)p^2(\zeta)+D(\zeta)\right)>0.$$

A direct application of [13]-Example 2.4(P43), we obtain that

$$\Re\left(\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}\right)>0$$

and hence the operator $[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)$ is univalent starlike. Since

$$\Re\left(\frac{\zeta[\Psi^k_{\vartheta,\upsilon}*\eta]'(\zeta)}{[\Psi^k_{\vartheta,\upsilon}*\eta](\zeta)}\right)>0,$$

then by the classical Harnack inequality (see [22]-P13), we obtain Inq. (16). For Inq. (17), it is an application of the Lindelof inequality [25]. □

The next result shows that Eq.(12) can be reduced into the well known Briot-Bouquet differential equation, as follows:

$$p(\zeta)+\frac{\zeta p'(\zeta)}{p(\zeta)}=R_{\gamma,\rho}(\zeta), \tag{18}$$

where

$$R_{\gamma,\rho}(\zeta) := p(\zeta)\gamma(\zeta) + \frac{\rho(\zeta)}{p(\zeta)}.$$

Where the result shows the sufficient condition to obtain a univalent solution for the generalized Whittaker equation in terms of the convoluted operator.

Theorem 3.9. *Let $R_{\gamma,\rho}(\zeta)$ be an open door function (see [13]-Eq.2.5.5) and $\frac{\zeta p'(\zeta)}{p(\zeta)}$ be starlike. Then the Briot-Bouquet differential equation (18) has a univalent solution with positive real part. Moreover, it achieves the following inequality*

$$\frac{-2r}{1-r}\sup_{|\zeta|<1}\Re\left(\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right) \le \Re\left(\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right) \le \frac{2r}{1+r}\sup_{|\zeta|<1}\Re\left(\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right) \quad (19)$$

and

$$\left|\left(\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right)'\right| \le \frac{1}{1-|\zeta|^2}\sup_{|\zeta|<1}\left|\left(\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right)\right|. \quad (20)$$

Proof. Clearly that the solution $[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)$ of the generalized Whittaker equation is analytic in the open unit disk. Since

$$P(\zeta) := \frac{\zeta p'(\zeta)}{p(\zeta)}$$

is starlike, then according to [13]-Theorem 3.2f, we obtain the univalency of the operator $[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)$ with $\Re\left([\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right) > 0$. Now to obtain Inq.(19), since $[\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)$ with $\Re\left([\Psi^k_{\vartheta,\upsilon} * \eta](\zeta)\right) > 0$, then Harnack's inequality yields Inq.(19). Finally, Inq.(20) comes as an application of the Carath' eodory inequality estimate. □

4. Conclusion

In this work, we have generalized the well known special Whittaker function utilizing the concept of the $k-$symbol calculus. The proposed $k-$symbol Whittaker function is normalized in the open unit disk to obtain a convoluted operator with the subclass of normalized function $\eta(0) = 0$ and $\eta'(0) = 1$. We formulated the convoluted operator in a class of analytic functions, preparing upon the structure of the Whittaker differential equation (1). We delivered sufficient conditions on the class for being starlike and convex. A univalent solution with double bounded positive real part, is given in Theorem 3.9. Connections of the real and imaginary parts of the operator is studied in Theorem 3.8, as well. For future works, one suggest another types of classes involving the $k-$symbol Whittaker convoluted operator. Also, it can be extended into different classes of analytic functions such as multi-valent and meromorphic analytic functions. By doing such an extension, one can discover some geometric properties.

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