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ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

K. MARIMUTHU[1*], J. UMA[1], §

Abstract. In this study, we establish a relation between particular subclasses of complex order univalent functions and Pascal distribution series. The main aim of the present investigation is to obtain the necessary and sufficient conditions for Pascal distribution series $\mathcal{F}_q^j(\xi)$ belongs to the classes $\mathcal{S}(\delta,\eta,\beta)$ and $\mathcal{R}(\delta,\eta,\beta)$. We also provide some properties of integral operator connected to Pascal distribution series and adequate conditions for $\mathcal{P}_q^j(\xi)$ to be in $\mathcal{K}(\eta,\delta)$ and $\mathcal{R}(\eta,\delta)$. Also, we discuss some corollaries and consequences of the major results.

Keywords: Complex order, Pascal distribution, Integral operator, Univalent function.

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1. Introduction

The class of analytic functions denoted by $\mathcal{A}$ is defined as

$$f(\xi)=\xi+\sum_{s=2}^{\infty}a_s\xi^s, \tag{1}$$

which are analytic in the open unit disc $\mathbb{D}=\{\xi:\xi\in\mathbb{C} \text{ such that } 0<|\xi|<1\}$.

Let us consider the subclass $\mathcal{H}$ of $\mathcal{A}$ consists of functions of the form,

$$f(\xi)=\xi-\sum_{s=2}^{\infty}a_s\xi^s,\quad a_s\geq 0. \tag{2}$$

The following condition for a function $f\in\mathcal{A}$ to be starlike of complex order $\delta(\delta\in\mathbb{C}-\{0\})$ as

$$\mathcal{R}\left\{1+\frac{1}{\delta}\left(\frac{\xi f'(\xi)}{f(\xi)}-1\right)\right\}>0,\quad \xi\in\mathbb{D}. \tag{3}$$

which is both necessary and sufficient when $\frac{f(\xi)}{\xi}\neq 0$.

[1] Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur-603203, Tamilnadu, India.
e-mail: mk1997@srmist.edu.in; ORCID: https://orcid.org/0000-0002-4076-2494.
e-mail: umaj@srmist.edu.in; ORCID: https://orcid.org/0000-0002-1547-823X.
[*] Corresponding author.

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The class of all those functions is denoted by $\mathcal{S}(\delta)$. The class $\mathcal{S}(\delta)$ was proposed by Nasr and Aouf [9].

Remark 1.1. *In the above inequality* (3)*, for different values of* δ*, we obtain the famous class of functions as follows:*

(i) For $\delta = 1$*, we have the class of starlike function* $(\mathcal{S}^*)$*;*
(ii) For $\delta = e^{-i\theta} cos\theta$*,* $(|\theta| < \frac{\pi}{2})$*, we have the class of spirallike function* $(\mathcal{S}^\theta)$*;*
(iii) For $\delta = 1 - \zeta$*,* $(0 \le \zeta < 1)$*, we have the class of starlike function* $(\mathcal{S}^*_\zeta)$ *of order* ζ*;*
(iv) For $\delta = (1-\zeta)e^{-i\theta} cos\theta$*,* $(|\theta| < \frac{\pi}{2})$*, we have the class of spirallike function of order* ζ $(\mathcal{S}^\theta_\zeta)$*;*

The following necessary and sufficient condition gives the function $f \in \mathcal{A}$ to be convex of complex order $\delta\,(\delta \in \mathbb{C} - \{0\})$ when $f'(\xi) \neq 0$, it is given by

$$\mathcal{R}\left\{1 + \frac{1}{\delta}\left(\frac{\xi f''(\xi)}{f'(\xi)}\right)\right\} > 0, \quad \xi \in \mathbb{D}. \tag{4}$$

The class $\mathcal{C}(\delta)$ was introduced by Wiatrowski [13]. It is worth noting that $f \in \mathcal{C}(\delta)$ if and only if $\xi f' \in \mathcal{S}(\delta)$.

Remark 1.2. *In the above inequality* (4)*, for different values of* δ*, we obtain the famous class of functions as follows:*

(i) For $\delta = 1$*, we have the class of convex function* $\mathcal{C}$*;*
(ii) For $\delta = 1 - \zeta$*,* $(0 \le \zeta < 1)$*, we have the class of convex function* $(\mathcal{C}_\zeta)$ *of order* ζ*;*
(iii) For $\delta = e^{-i\theta} cos\theta$*,* $(|\theta| < \frac{\pi}{2})$*, we have the class of* θ*-Robertson function* $(\mathcal{S}_\theta)$*;*

The function $f \in \mathcal{A}$ is known as close to convex function of order $\delta(\delta \in \mathbb{C} - \{0\})$ provided with the following necessary and sufficient condition

$$\mathcal{R}\left\{1 + \frac{1}{\delta}(f'(\xi) - 1)\right\}, \quad \xi \in \mathbb{D}. \tag{5}$$

The class $\mathcal{R}(\delta)$ was introduced by Owa [11] and Addul Halim [1].

The following subclasses of $\mathcal{A}(j)$ were obtained by Altintas et al. [4], and they consist of functions of the form

$$f(\xi) = \xi - \sum_{s=j+1}^{\infty} a_s \xi^s.$$

Definition 1.1. [4] *Let* $\mathcal{S}_j(\delta, \eta, \beta)$ *indicate the subclass of* $\mathcal{A}(j)$ *containing of function* f *that holds the inequality*

$$\left|\frac{1}{\delta}\left(\frac{\xi f'(\xi) + \eta \xi^2 f''(\xi)}{\eta \xi f'(\xi) + (1-\eta) f(\xi)} - 1\right)\right| < \beta,$$

where $\xi \in \mathbb{D}, \delta \in \mathbb{C} - \{0\}, 0 < \beta \le 1$ *and* $0 \le \eta \le 1$.

Also let $\mathcal{R}_j(\delta, \eta, \beta)$ *indicate the subclass of* $\mathcal{A}(j)$ *consisting of functions* f *that holds the inequality*

$$\left|\frac{1}{\delta}(f'(\xi) + \eta \xi f''(\xi) - 1)\right| < \beta.$$

When on the assumption of functions in the above classes, the authors found the following coefficient inequalities in their paper:

Lemma 1.1. [4] *Let the function $f \in \mathcal{A}(j)$, then $f \in \mathcal{S}_j(\delta, \eta, \beta)$ if and only if*

$$\sum_{s=j+1}^{\infty} [\eta(s-1)+1][s+\beta|\delta|-1]a_s \leq \beta|\delta|.$$

Lemma 1.2. [4] *Let the function $f \in \mathcal{A}(j)$, then $f \in \mathcal{R}_j(\delta, \eta, \beta)$ if and only if*

$$\sum_{s=j+1}^{\infty} s[\eta(s-1)+1]a_s \leq \beta|\delta|.$$

If the functions f belongs to the classes $\mathcal{S}_j(\delta, \eta, \beta)$ and $\mathcal{R}_j(\delta, \eta, \beta)$, we assume that $j = 1$, then we write $\mathcal{S}_1(\delta, \eta, \beta) = \mathcal{S}(\delta, \eta, \beta)$ and $\mathcal{R}_1(\delta, \eta, \beta) = \mathcal{R}(\delta, \eta, \beta)$.

Definition 1.2. *Let $\mathcal{K}(\eta, \delta)$ indicate the subclass of $\mathcal{A}$ containing functions of the form* (1) *which holds*

$$\mathcal{R}\left\{1+\frac{1}{\delta}\left(\frac{\xi f'(\xi)+\eta\xi^2 f''(\xi)}{\eta\xi f'(\xi)+(1-\eta)f(\xi)}-1\right)\right\} > 0,$$

where $\xi \in \mathbb{D}, \delta \in \mathbb{C}-\{0\}$ and $0 \leq \eta \leq 1$.

Also, let $\mathcal{R}(\eta, \delta)$ indicate the subclass of $\mathcal{A}$ containing functions of the form (1) *which holds*

$$\mathcal{R}\left\{1+\frac{1}{\delta}(f'(\xi)+\eta\xi f''(\xi)-1)\right\} > 0.$$

Altıntaş, et al.[3] and Aouf [5] investigated and studied the classes $\mathcal{K}(\eta, \delta)$ and $\mathcal{R}(\eta, \delta)$. The following lemmas have been proved by Aouf [5] and given with sufficient conditions for the functions f to be in the classes $\mathcal{K}(\eta, \delta)$ and $\mathcal{R}(\eta, \delta)$.

Lemma 1.3. [5] *Assume that the function f defined in* (1) *satisfies the following inequality:*

$$\sum_{s=2}^{\infty}[\eta(s-1)+1][(s-1)+|2\delta+s-1|]|a_s| \leq 2|\delta|$$

then $f \in \mathcal{K}(\eta, \delta)$.

Lemma 1.4. [5] *Let $f \in A$ then $f \in \mathcal{R}(\eta, \delta)$ if it holds the inequality,*

$$\sum_{s=2}^{\infty} s[\eta(s-1)+1]|a_s| \leq 2|\delta|.$$

We can see that $\mathcal{K}(0, \delta) = \mathcal{S}(\delta), \mathcal{K}(1, \delta) = \mathcal{C}(\delta)$ and $\mathcal{R}(0, \delta) = \mathcal{R}(\delta)$.

A variable x follows Pascal distribution if

$$P(x=m) = \binom{m+j-1}{j-1}(1-q)^j q^m, m = 0, 1, 2, ..$$

El-Deeb, et al.[6] recently studied the following representation whose coefficients are the probabilities from Pascal distribution

$$\mathcal{P}_q^j(\xi) = \xi + \sum_{s=2}^{\infty}\binom{s+j-2}{j-1} q^{s-1}(1-q)^j \xi^s, \quad \xi \in \mathbb{D}, \tag{6}$$

where $j \geq 1$, $0 \leq q < 1$ and by ratio test since the aforementioned series is convergent for all values of s whose radius of convergence is infinity. In addition to that, they defined the following

$$\mathcal{F}_q^j(\xi) = 2\xi - \mathcal{P}_q^j(\xi) = \xi - \sum_{s=2}^{\infty} \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \xi^s, \quad \xi \in \mathbb{D}. \tag{7}$$

As given in [8], we utilise the following notations in this paper for our convenience,

$$\sum_{s=0}^{\infty} \binom{s+j-1}{j-1} q^s = \frac{1}{(1-q)^j}, \quad \sum_{s=0}^{\infty} \binom{s+j-2}{j-2} q^s = \frac{1}{(1-q)^{j-1}},$$

$$\sum_{s=0}^{\infty} \binom{s+j}{j} q^s = \frac{1}{(1-q)^{j+1}}, \quad \sum_{s=0}^{\infty} \binom{s+j+1}{j+1} q^s = \frac{1}{(1-q)^{j+2}},$$

where $j \geq 1$ and $0 \leq q < 1$. We also state the following result,

$$\sum_{s=2}^{\infty} \binom{s+j-2}{j-1} q^{s-1} = \sum_{s=0}^{\infty} \binom{s+j-1}{j-1} q^s - 1,$$

$$\sum_{s=2}^{\infty} (s-1) \binom{s+j-2}{j-1} q^{s-1} = qj \sum_{s=0}^{\infty} \binom{s+j}{j} q^s,$$

$$\sum_{s=2}^{\infty} (s-1)(s-2) \binom{s+j-2}{j-1} q^{s-1} = q^2 j(j+1) \sum_{s=0}^{\infty} \binom{s+j+1}{j+1} q^s.$$

Motivated by the works of Porwal.et.al.[10], Murugusundramoorthy et.al. [7] (see also [2], [12]), we obtain the necessary and sufficient conditions for the Pascal distribution series $\mathcal{F}_q^j(\xi)$ belongs to the classes $\mathcal{S}(\delta, \eta, \beta)$ and $\mathcal{R}(\delta, \eta, \beta)$, and for the Pascal distribution series $\mathcal{P}_q^j(\xi)$ belongs to the classes $\mathcal{K}(\eta, \delta)$ and $\mathcal{R}(\eta, \delta)$ there must be sufficient conditions. In addition, an integral theorem for the series $\mathcal{G}_q^j(\xi)$ and $\mathcal{H}_q^j(\xi)$ are established.

2. Main Results

In this section, we obtain the necessary and sufficient condition for $\mathcal{F}_q^j(\xi) \in \mathcal{S}(\delta, \eta, \beta)$.

Theorem 2.1. *Let $j > 0$. We have $\mathcal{F}_q^j(\xi) \in \mathcal{S}(\delta, \eta, \beta)$ if and only if*

$$\frac{qj}{(1-q)^{j+1}} \left[\eta \left(\frac{q(j+1)}{(1-q)} + \beta|\delta| + 1 \right) + 1 \right] \leq \beta|\delta|. \tag{8}$$

Proof. Let

$$\mathcal{F}_q^j(\xi) = \xi - \sum_{s=2}^{\infty} \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \xi^s.$$

Using lemma 1.1, it sufficies to establish that

$$\sum_{s=2}^{\infty} [\eta(s-1)+1][s+\beta|\delta|-1] \binom{s+j-2}{j-1} (1-q)^j q^{s-1} \leq \beta|\delta|. \tag{9}$$

For, we consider

$$\begin{aligned}
&\sum_{s=2}^{\infty}[\eta(s-1)+1][s+\beta|\delta|-1]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\\
=\;&\sum_{s=2}^{\infty}\left[\eta s^2+s(1-2\eta+\eta\beta|\delta|)+\eta(1-\beta|\delta|)+\beta|\delta|-1\right]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j,\\
=\;&\sum_{s=2}^{\infty}[\eta(s-1)(s-2)+(s-1)(1+\eta+\eta\beta|\delta|)+\beta|\delta|]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j,\\
=\;&\eta\sum_{s=2}^{\infty}(s-1)(s-2)\binom{s+j-2}{j-1}(1-q)^jq^{s-1}+(1+\eta+\eta\beta|\delta|)\sum_{s=2}^{\infty}(s-1)\\
&\binom{s+j-2}{j-1}q^{s-1}(1-q)^j+\beta|\delta|\sum_{s=2}^{\infty}\binom{s+j-2}{j-1}q^{s-1}(1-q)^j,\\
=\;&\eta q^2j(j+1)\sum_{s=0}^{\infty}\binom{s+j+1}{j+1}q^{s-1}(1-q)^j\\
&+(1+\eta+\eta\beta|\delta|)qj\sum_{s=0}^{\infty}\binom{s+j}{j}q^{s-1}(1-q)^j+\beta|\delta|\left(\sum_{s=0}^{\infty}\binom{s+j-1}{j-1}q^s-1\right),\\
=\;&\eta q^2j(j+1)\tfrac{1}{(1-q)^2}+(1+\eta+\eta\beta|\delta|)qj\tfrac{1}{(1-q)}+\beta|\delta|(1-(1-q)^j)
\end{aligned}$$

Therefore the inequality (9) holds if and only if

$$\eta\frac{q^2j(j+1)}{(1-q)^2}+(1+\eta+\eta\beta|\delta|)\frac{qj}{(1-q)}+\beta|\delta|(1-(1-q)^j)\le\beta|\delta|.$$

This completes the proof. □

Theorem 2.2. *The function $\mathcal{F}_q^j(\xi)\in\mathcal{R}(\delta,\eta,\beta)$ if and only if*

$$\frac{qj}{1-q}\left[1+\eta\left(\frac{q(j+1)}{1-q}+2\right)\right]+\left(1-(1-q)^j\right)\le\beta|\delta|. \tag{10}$$

Proof. We establish that in view of lemma 1.2

$$\sum_{s=2}^{\infty}s[\eta(s-1)+1]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\le\beta|\delta|.$$

Now,

$$\begin{aligned}
&\sum_{s=2}^{\infty}s[\eta(s-1)+1]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\\
=\;&\sum_{s=2}^{\infty}[\eta(s-1)(s-2)+(s-1)(1+2\eta)+1]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j,\\
=\;&\eta\sum_{s=2}^{\infty}(s-1)(s-2)\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\\
&+(1+2\eta)\sum_{s=2}^{\infty}(s-1)\binom{s+j-2}{j-1}q^{s-1}(1-q)^j+\sum_{s=2}^{\infty}\binom{s+j-2}{j-1}(1-q)^jq^{s-1},\\
=\;&\eta q^2j(j+1)\sum_{s=0}^{\infty}\binom{s+j+1}{j+1}(1-q)^jq^s\\
&+(1+2\eta)qj\sum_{s=0}^{\infty}\binom{s+j}{j}q^{s-1}(1-q)^j+\sum_{s=0}^{\infty}\binom{s+j-1}{j-1}q^s(1-q)^j-(1-q)^j,\\
=\;&\eta\tfrac{q^2j(j+1)}{(1-q)^2}+(1+2\eta)\tfrac{qj}{(1-q)}-(1-q)^j+1.
\end{aligned}$$

Hence the proof is complete if and only if the inequality (10) holds. □

Theorem 2.3. *A sufficient condition for the function* $\mathcal{P}_q^j(\xi) \in \mathcal{K}(\eta,\delta)$ *is*

$$\frac{2qj}{(1-q)^{j+1}}\left[\eta\left(\frac{q(j+1)}{(1-q)}+|\delta|+1\right)+1\right] \leq 2|\delta|. \tag{11}$$

Proof. It suffices to establish that in light of lemma 1.3

$$\sum_{s=2}^{\infty}[\eta(s-1)+1][(s-1)+|2\delta+s-1|]\left|\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\right| \leq 2|\delta|. \tag{12}$$

The proof is obtained using similar procedure of Theorem 2.1. □

Theorem 2.4. *A sufficient condition for the function* $\mathcal{P}_q^j(\xi) \in \mathcal{R}(\eta,\delta)$ *is*

$$\frac{qj}{(1-q)}\left[\eta\left(\frac{q(j+1)}{(1-q)}+2\right)+1\right]+(1-(1-q)^j) \leq |\delta|. \tag{13}$$

Proof. It suffices to establish that in light of lemma 1.4

$$\sum_{s=2}^{\infty}s[1+\eta(s-1)]\left|\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\right| \leq |\delta|.$$

Now, we can write

$$\begin{aligned}
&\sum_{s=2}^{\infty}s[1+\eta(s-1)]\left|\binom{s+j-2}{j-1}(1-q)^jq^{s-1}\right|\\
=\ &\sum_{s=2}^{\infty}\left[(1-\eta)s+\eta s^2\right]\binom{s+j-2}{j-1}q^{s-1}(1-q)^j,\\
=\ &\eta\sum_{s=2}^{\infty}(s-1)(s-2)\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\\
&+(1+2\eta)\sum_{s=2}^{\infty}(s-1)\binom{s+j-2}{j-1}q^{s-1}(1-q)^j+\sum_{s=2}^{\infty}\binom{s+j-2}{j-1}q^{s-1}(1-q)^j,\\
=\ &\eta q^2j(j+1)\sum_{s=0}^{\infty}\binom{s+j+1}{j+1}q^s(1-q)^j\\
&+(1+2\eta)qj\sum_{s=0}^{\infty}\binom{s+j}{j}q^{s-1}(1-q)^j+\sum_{s=0}^{\infty}\binom{s+j-1}{j-1}q^s(1-q)^j-(1-q)^j,\\
=\ &\eta\tfrac{q^2j(j+1)}{(1-q)^2}+(1+2\eta)\tfrac{qj}{(1-q)}-(1-q)^j+1
\end{aligned}$$

Hence, we obtain the proof if and only if the inequality (13) holds. □

3. Integral Operator

In this section, we define the integral operators as follows:

$$\mathcal{G}_q^j(\xi)=\int_0^{\xi}\frac{\mathcal{F}_q^j(t)}{t}dt, \quad \mathcal{H}_q^j(\xi)=\int_0^{\xi}\frac{\mathcal{P}_q^j(t)}{t}dt. \tag{14}$$

Theorem 3.1. *Let* $j>0$, *then* $\mathcal{G}_q^j(\xi) \in \mathcal{S}(\delta,\eta,\beta)$ *if and only if*

$$\frac{\eta qj}{(1-q)}+(1-\eta)(\beta|\delta|-1)\frac{(1-q)}{q(j-1)}(1-(1-q)^{j-1})-\beta|\delta|(1-q)^j+(1-\eta) \leq (1-\eta)\beta|\delta|. \tag{15}$$

Proof. Since

$$\mathcal{G}_q^j(\xi)=\xi-\sum_{s=2}^{\infty}\binom{s+j-2}{j-1}q^{s-1}(1-q)^j\frac{\xi^s}{s},\ \xi\in\mathbb{D}$$

according to lemma 1.1, we just need to demonstrate that

$$\sum_{s=2}^{\infty} \frac{1}{s}[\eta(s-1)+1][s+\beta|\delta|-1] \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \leq \beta|\delta|. \tag{16}$$

The proof is obtained using similar procedure of Theorem 2.1. □

Theorem 3.2. *Let $j>0$, then $\mathcal{G}_q^j(\xi) \in \mathcal{R}(\delta,\eta,\beta)$ if and only if*

$$\frac{\eta q j}{(1-q)} - (1-q)^j + 1 \leq \beta|\delta|. \tag{17}$$

Proof. We establish that in view of lemma 1.2

$$\sum_{s=2}^{\infty}[\eta(s-1)+1] \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \leq \beta|\delta|.$$

Now,

$$\begin{aligned}
&\sum_{s=2}^{\infty}[\eta(s-1)+1] \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \\
=\;& \eta(1-q)^j \sum_{s=2}^{\infty}(s-1) \binom{s+j-2}{j-1} q^{s-1} + (1-q)^j \sum_{s=2}^{\infty} \binom{s+j-2}{j-1} q^{s-1}, \\
=\;& \eta(1-q)^j qj \sum_{s=0}^{\infty} \binom{s+j}{j} q^s + (1-q)^j \left[\sum_{s=0}^{\infty} \binom{s+j-1}{j-1} q^s - 1\right], \\
=\;& \tfrac{\eta q j}{(1-q)} - (1-q)^j + 1.
\end{aligned}$$

Hence the proof is complete if and only if the inequality (17) holds. □

Theorem 3.3. *Let $j>0$, then $\mathcal{H}_q^j(\xi) \in \mathcal{R}(\eta,\delta)$ if*

$$1-(1-q)^j + \frac{\eta q j}{1-q} \leq |\delta|. \tag{18}$$

Proof. Since

$$\mathcal{H}_q^j(\xi) = \xi + \sum_{s=2}^{\infty} \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \frac{\xi^s}{s},\ \xi \in \mathbb{D}$$

according to lemma 1.4, we just need to demonstrate that

$$\sum_{s=2}^{\infty}[1+\eta(s-1)] \left| \binom{s+j-2}{j-1} q^{s-1}(1-q)^j \right| \leq |\delta|.$$

After simple computation, we obtain the required result. □

4. Special Cases

Considering the above Remarks and Definitions, we obtain the following Corollaries by specialising different values of the parameter η and δ in Theorem 2.1 to Theorem 3.3. For $\eta=0$, we obtain the following result for starlike functions of δ:

Corollary 4.1. *We have $\mathcal{P}_q^j(\xi) \in \mathcal{S}(\delta)$ if*

$$\frac{qj}{(1-q)^{j+1}} \leq |\delta|. \tag{19}$$

For $\eta=1$, we obtain the following result for convex functions of δ:

Corollary 4.2. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{C}(\delta)$ *if*

$$\frac{qj}{(1-q)^{j+1}}\left[\frac{q(j+1)}{(1-q)}+|\delta|+2\right] \leq |\delta|. \tag{20}$$

For $\eta = 1, \delta = 1$, we obtain the following result for the class of convex functions:

Corollary 4.3. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{C}$ *if*

$$\frac{qj}{(1-q)^{j+1}}\left[\frac{q(j+1)}{(1-q)}+3\right] \leq 1. \tag{21}$$

For $\eta = 0, \delta = 1$, we obtain the following result for the class of starlike functions:

Corollary 4.4. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{S}^*$ *if*

$$\frac{qj}{(1-q)^{j+1}} \leq 1. \tag{22}$$

For $\eta = 0, \delta = 1-\zeta$, we have the following result for the class of starlike functions of order ζ:

Corollary 4.5. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{S}^*(\zeta)$ *if*

$$\frac{qj}{(1-q)^{j+1}} \leq 1-\zeta. \tag{23}$$

For $\eta = 0, \;\; \delta = e^{-i\theta}cos\theta$, we have the following result for the class of spirallike functions:

Corollary 4.6. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{S}^\theta$ *if*

$$\frac{qj}{(1-q)^{j+1}} \leq |cos\theta| \tag{24}$$

For $\eta = 0, \;\; \delta = (1-\zeta)e^{-i\theta}cos\theta$, we have the following result for the class of spirallike functions of order ζ:

Corollary 4.7. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{S}^\theta(1-\zeta)$ *if*

$$\frac{qj}{(1-q)^{j+1}} \leq (1-\zeta)|cos\theta|. \tag{25}$$

For $\eta = 0$, we obtain following result:

Corollary 4.8. *We have* $\mathcal{P}_q^j(\xi) \in \mathcal{R}(\eta)$ *if*

$$\frac{qj}{(1-q)^{j+1}} + 1 - (1-q)^j \leq |\eta|. \tag{26}$$

5. Conclusion

This paper deals with the different subclasses of complex order univalent function associted with Pascal distribution series. And, we have obtained the results of integral theorem and many interesting special cases. In future, partial sums, Turan-type inequalities and neighbourhood problems are to be investigated in such classes.

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K. Marimuthu completed his M.Sc., in Mathematics at the University of Madras, Chennai in 2013 and B.Ed., in Tamilnadu Teacher Educational University, Chennai in 2014. He also completed his M.Phil., at the University of Madras in 2015. Later, he worked as an Assistant Professor at Indo-American College of Arts and Science from 2017 to 2019. Presently he is a Ph.D., candidate in the Department of Mathematics, SRM Institute of Science and Technology, Kattankulathur, India. He is doing research in the field of Geometric Function Theory.

Dr. J. Uma obtained her Ph.D. at SRM Institute of Science and Technology in 2018. She is a Senior Assistant Professor in the Department of Mathematics at the same institute. Her research interests are Non-Archimedean Analysis, Complex Analysis and Graph Theory.