TWMS J. App. and Eng. Math. V.14, N.4, 2024, pp. 1578-1589

WEAK $(\psi - \varphi)$ CONTRACTIONS FOR $\mathscr{F}-$ CLASS FUNCTIONS IN *multiplicative* $\boldsymbol{b}-$METRIC SPACES

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Abstract. The objective of this study is to introduce $\mathscr{F}$- class function with weak $(\psi - \varphi)$ Contractions for two mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ using a pair of $\mathscr{F}$- class function with weak $(\psi - \varphi)$ Contractive maps in the framework of *multiplicative* $\boldsymbol{b}-$ metric spaces In addition, with convergence analysis, the CR iteration algorithm is introduced to this space.

Keywords: Weak $(\psi - \varphi)$ Contractions, $\mathscr{F}-$ class functions, *multiplicative* $\boldsymbol{b}-$metric spaces, fixed points.

AMS Subject Classification: 47H10, 54H25, 47J25.

1. Introduction

In 1922, Banach [1] proved that a mapping $\mathscr{T}$ has a unique fixed point in $\mathscr{U}$ for mapping $\mathscr{T} : \mathscr{U} \to \mathscr{U}$, for a complete metric space $(\mathscr{U}, \Xi)$ satisfying the condition

$$\Xi(\mathscr{T}\kappa, \mathscr{T}\varsigma) \leq \alpha\Xi(\kappa, \varsigma). \tag{1}$$

for all $\kappa, \varsigma \in \mathscr{U}$ and $0 \leq \alpha \leq 1$, which is referred as the Banach contraction principle (or $\mathscr{BCP}$). Since then, many authors have proved various generalizations of the $\mathscr{BCP}$ by examining various contractive conditions. By significantly altering the so-called altering distance functions, which were first introduced by Delbosco [2] and Skof [3], for self maps of complete metric space in 1984, Khan et al. [4] approximated the $\mathscr{BCP}$, and defined as follows:

Definition 1.1. *Let $\mathbb{R}^+$ be the set of all non-negative real numbers ($\mathbb{R}^+ = [0, +\infty)$). A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an altering distance function if the conditions mentioned below are satisfied:*

(1) *ψ is a continuous mapping,*

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Manuscript received: February 02, 2023; accepted: August 18, 2023.

(2) *ψ is a nondecreasing mapping,*

(3) $\psi(\overset{\circ}{\varpi}) = 0$ *iff* $\overset{\circ}{\varpi} = 0$, *for* $\overset{\circ}{\varpi} \in \mathbb{R}^+$.

The set of all altering distance functions will be referred to throughout the entire paper as $\overset{\circ}{\psi}$. In the year 1984, using ψ function, Khan et al. [4] proved that if the metric space $(\mathscr{U}, \Xi)$ is complete and $\psi \in \Psi$ with $0 \leq \alpha \leq 1$, then a mapping $\mathscr{T} : \mathscr{U} \to \mathscr{U}$ satisfying

$$\psi(\Xi(\mathscr{T}\kappa, \mathscr{T}\varsigma)) \leq \alpha(\psi(\Xi(\kappa, \varsigma))), \tag{2}$$

for all $\kappa,\ \varsigma \in \mathscr{U}$ has a unique fixed point. It is significant to mention that $\psi(\overset{\circ}{\varpi}) = \overset{\circ}{\varpi}$, $\overset{\circ}{\varpi} \in \mathbb{R}^+$ in (2), we attain (1). Alber and Guerre-Delabriere [5] introduced generalized results known as single-valued weakly contractive mappings in Hilbert spaces in the year 1997, and proved some fixed point theorems in this framework. Alber and Guerre-Delabriere [5] also acknowledged that their results are true for uniformly smooth and uniformly convex Banach spaces. Further, Rhoades [6] proved that various results obtained in [5] are also satisfied in Banach spaces as well.

Definition 1.2. *Let $(\mathscr{U}, \Xi)$ is a metric space and a mapping $\mathscr{T} : \mathscr{U} \to \mathscr{U}$. Then $\varphi-$ weak contraction is mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:*

(1) *φ continuous nondecreasing function,*
(2) $\varphi(\overset{\circ}{\varpi}) > 0$ *for all* $\overset{\circ}{\varpi} > 0$,
(3) $\varphi(0) = 0$ *and* $\lim_{\overset{\circ}{\varpi} \to +\infty} \varphi(\overset{\circ}{\varpi}) = +\infty$,

such that for all $\kappa,\ \varsigma \in \mathscr{U}$ also satisfies

$$\Xi(\mathscr{T}\kappa, \mathscr{T}\varsigma) \leq \Xi(\kappa, \varsigma) - \varphi(\Xi(\kappa, \varsigma)).$$

The non-newtonian version of above mentioned definition is defined as follows:

Definition 1.3. *Let $(\mathscr{U}, \Xi)$ is a metric space and a mapping $\mathscr{T} : \mathscr{U} \to \mathscr{U}$. Then $\varphi-$ weak contraction is mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:*

(1) *φ continuous nondecreasing function,*
(2) $\varphi(\overset{\circ}{\varpi}) > 1$ *for all* $\overset{\circ}{\varpi} > 1$,
(3) $\varphi(1) = 1$ *and* $\lim_{\overset{\circ}{\varpi} \to +\infty} \varphi(\overset{\circ}{\varpi}) = +\infty$,

such that for all $\kappa,\ \varsigma \in \mathscr{U}$ also satisfies $\Xi(\mathscr{T}\kappa, \mathscr{T}\varsigma) \leq \dfrac{\Xi(\kappa, \varsigma)}{\varphi(\Xi(\kappa, \varsigma))}$.

Note. It is worth noting that $\psi \subset \varphi$, where ψ set of all altering distance functions (ψ) and φ is set of all φ functions. Instead of utilising the constraint $\lim_{\overset{\circ}{\varpi} \to +\infty} \varphi(\overset{\circ}{\varpi}) = +\infty$, Rhoades generalised the $\mathscr{BCP}$ to $\varphi-$ weak contraction mappings in his article [6], and the Theorem is as follows:

Theorem 1.1. *Let $(\mathscr{U}, \Xi)$ is a complete metric space. If a function φ is a non decreasing continuous function satisfying $\varphi(\overset{\circ}{\varpi}) > 0$ for all $\overset{\circ}{\varpi} > 0$ and $\varphi(0) = 0$. In such case, if a $\varphi-$ weak contraction $\mathscr{T} : \mathscr{U} \to \mathscr{U}$, then $\mathscr{T}$ has a unique fixed point.*

Ansari [7] introduced the concept of $\mathscr{C}-$ class functions in 2014, as follows:

Definition 1.4. *A continuous mapping $\mathscr{F} : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a $\mathscr{F}-$ class function if it satisfies the following conditions:*

(1) $\mathscr{F}(\kappa, \varsigma) \leq \kappa$ *forall* $\kappa, \varsigma \in \mathbb{R}^+$, *and*
(2) $\mathscr{F}(\kappa, \varsigma) = \kappa$ *implies that either* $\kappa = 0$ *or* $\varsigma = 0$.

We denote the family of all $\mathscr{C}-$ class functions by $\mathscr{C}_{\mathscr{F}}$.

The $\mathscr{F}-$ class functions are now defined as follows:

Definition 1.5. *A continuous mapping $\mathscr{F}:\mathbb{R}^{+}\times\mathbb{R}\to\mathbb{R}$ is said to be $\mathscr{F}-$ class function if $\mathscr{F}(\kappa,\varsigma)<\kappa$ for all $\kappa>0$ and $\varsigma>0$. We denote the family of $\mathscr{F}-$ class functions by $\mathscr{F}_{\mathscr{C}}$.*

It is important to note that the classes $\mathscr{C}$ and $\mathscr{F}$ both are same [7]. Also, $\mathscr{F}\in\mathscr{C}_{\mathscr{F}}$, $\mathscr{F}(0,0)$ may note be equal to $(0,0)$. Dutta and Choudhury [8] extended two existing generalisations of the $\mathscr{BCP}$, and validated the results by example as example. They proved the following result using the altering distance functions in the year 2008, which is as follows:

Theorem 1.2. *Let $(\mathscr{U},\Xi)$ is a complete metric space and let $\mathscr{T}:\mathscr{U}\to\mathscr{U}$ be a self-mapping satisfying the inequality*

$$\psi(\Xi(\mathscr{T}\kappa,\mathscr{T}\varsigma)\leq\psi(\Xi(\kappa,\varsigma))-\varphi(\Xi(\kappa,\varsigma)),$$

where $\psi,\ \varphi:[0,\infty)\to[0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(\mathring{\varpi})=0=\varphi(\mathring{\varpi})$ if and only if $\mathring{\varpi}=0$.

Then the mapping $\mathscr{M}$ has a unique fixed point.

Abbas and Khan [9] established the existence of a $\mathscr{CFP}$ for mappings $\mathscr{T}_1$, and $\mathscr{T}_2$ (also known as $(\psi-\varphi)$) that satisfy a generalised weak contractive condition in their work in 2009. As a result, a $\mathscr{CFP}$ result for mappings satisfying a contractive condition of integral type was achieved. Their analyses generalise, extend, and integrate some of the most well comparable results from the previous research. The set of coincidence points for two mappings $\mathscr{T}_1,\ \mathscr{T}_2$ defined from $\mathscr{U}$ to $\mathscr{U}$ is denoted by $\mathscr{U}_{\mathcal{CP}}$ and has the following definition:

$$\mathscr{U}_{\mathcal{CP}}(\mathscr{T}_1,\mathscr{T}_2)=\{\gamma\in\mathscr{U}\ :\ \mathscr{T}_1\kappa=\mathscr{T}_2\kappa\};$$

and well defined collection of point of coincidence of mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ is denoted by $\mathscr{U}_{\mathcal{PC}}$ and has the following definition:

$$\mathscr{U}_{\mathcal{PC}}(\mathscr{T}_1,\mathscr{T}_2)=\{\gamma\in\mathscr{U}\ :\ \mathscr{T}_1\kappa=\mathscr{T}_2\kappa,\ \text{for some}\ \kappa\in\mathscr{U}\}.$$

In the year 2006, Jungck and Rhoades [10] obtained two fixed point theorems for a class of operators called occasionally weakly compatible maps, which were defined on a symmetric space. Their results established two of the most general fixed point theorems for four maps. Here, it is important to note that $\mathscr{T}_1$ and $\mathscr{T}_2$ are said to be (nontrivially) weakly compatible [10], when $\kappa\in\mathscr{U}_{\mathcal{CP}}(\mathscr{T}_1,\mathscr{T}_2)$ implies that $\mathscr{T}_1\mathscr{T}_2\kappa=\mathscr{T}_2\mathscr{T}_1\kappa$ and the theorem is as follows:

Theorem 1.3. *Let $(\mathscr{U},\Xi)$ be a metric space and $\mathscr{T}_1$ and $\mathscr{T}_2$ are two mappings satisfying*

$$\psi(\Xi(\mathscr{T}_1\kappa),\Xi(\mathscr{T}_1\varsigma))\leq\psi(\Xi(\mathscr{T}_2\kappa),\Xi(\mathscr{T}_2\varsigma))-\varphi(\Xi(\mathscr{T}_2\kappa),\Xi(\mathscr{T}_2\varsigma)),$$

for all $\kappa,\ \varsigma\in\mathscr{U}$ and $\varphi,\ \psi\in\Psi$. If $\mathscr{T}_1\mathscr{U}\subset\mathscr{T}_2\mathscr{U}$ and $\mathscr{T}_2\mathscr{U}\subset\mathscr{U}$ is a complete subspace, then the mappings $\mathscr{T}_2\mathscr{T}_1$ and $\mathscr{T}_2$ have a unique point of coincidence in $\mathscr{U}$. In addition, if the mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ are weakly compatible, then $\mathscr{T}_1$ and $\mathscr{T}_2$ have $\mathscr{CFP}$ $\alpha\in\mathscr{U}$. Also, if there exists $\beta\in\mathscr{U}$ which is also a $\mathscr{CFP}$ of $\mathscr{T}_1$ and $\mathscr{T}_2$, then $\alpha=\beta$.

Numerous authors have broadened and generalised the approach of weak contraction mappings, as well as analysed the existence of $\mathscr{CFP}$ for this class of mappings in distinctive metric-like spaces. We request the readers to refer [15]-[21] for relavant literature.

Effective and efficient algorithms for computing approximate $\mathscr{CFP}$, on the other hand, are essential tool in metric fixed point theory. The basic, one-step Picard iteration process can be used to approximate the fixed point and the same is guaranteed by the $\mathscr{BCP}$ and some of its extensions. The Krasnoselksii (one-step), Mann (one-step), and Ishikawa (two-step), CR, S, K, K^*, S, SP, N_v^1, M, P, D iterations as well as their extensions and generalisations are among the many more effectual iterative methods for computing estimated fixed points fixed points (and $\mathscr{CFP}$) for contractive maps in linear spaces. We recommend the seminal monograph [22], as well as its citations and references, for a more comprehensible analysis of those algorithms. In this context, Chug et al. [23] suggestest a novel three step iterative scheme called the CR iterative scheme and studied the strong convergence of CR iterative scheme for a certain class of quasi-contractive operators in Banach spaces Fix κ_0 and define the sequence $\{\kappa_n\}_{n=0}^{\infty}$ by

$$\begin{aligned} \kappa_{n+1} &= (1-\mathfrak{a}_n)\varsigma_n + \mathfrak{a}_n \mathscr{T}\varsigma_n, \\ \varsigma_n &= (1-\mathfrak{b}_n)\mathscr{T}\kappa_n + \mathfrak{b}_n \mathscr{T}\rho_n, \\ \rho_n &= (1-\mathfrak{c}_n)\kappa_n + \mathfrak{c}_n \mathscr{T}\kappa_n, \end{aligned}$$

where $\{\mathfrak{a}_n\}$, $\{\mathfrak{b}_n\}$ and $\{\mathfrak{c}_n\}$ are sequences of positive numbers in $[0,1]$ with $\{\mathfrak{a}_n\}$ satisfying $\sum_{n=0}^{\infty} \mathfrak{a}_n = \infty$. In 2006, Bnouhachem et al. [24] proved that three-step iterative methods converges better than two-step and one-step iterative scheme for solving variational issues of inequality. CR iteration algorithm has been generalised in multiple means, and its convergence analysis has been studied for various classes. See, for example, ([25], [32]) and the references therein for examples of contractive-type mappings. The following definitions play important role in our our major results.

Definition 1.6. *Let $(\mathscr{U}, \Xi)$ be a metric space. A mapping $\mathscr{T} : \mathscr{U} \times \mathscr{U} \times [0,1] \to \mathscr{U}$ is said to be a convex structure on $\mathscr{U}$ if for each $(\kappa, \varsigma, \lambda) \in \mathscr{U} \times \mathscr{U} \times [0,1]$ and $\rho \in \mathscr{U}$,*

$$\Xi(\rho, \mathscr{W}(\kappa, \varsigma, \lambda)) \leq \lambda \Xi(\rho, \kappa) + (1-\lambda)\Xi(\rho, \varsigma).$$

Definition 1.7. *A metric space $(\mathscr{U}, \Xi)$ equipped with the convex structure $\mathscr{W}$ is called a convex metric space, and it is denoted by $(\mathscr{U}, \Delta, \mathscr{W})$.*

Convexity in normed linear spaces is a generalisation of this concept (see [11]) and the one of the example is $CAT(0)$ space. As a result, as previously stated, convex metric spaces provide an appropriate setting for defining and extending iterative schemes established with convex combinations. Some studies on the application and convergence of vaious interative methods in convex metric spaces are available in existing literature, readers are requested to refer [12].

Let $\mathscr{W}(\kappa, \varsigma, \lambda)$ be a convex metric space and two self mappings let $\mathscr{T}_1, \ \mathscr{T}_2 : \mathscr{U}_1 \to \mathscr{U}$ on a subset $\mathscr{U}_1$ of $\mathscr{U}$ such that $\mathscr{T}_1(\mathscr{U}_1) \subset \mathscr{T}_2(\mathscr{U}_1)$, where $\mathscr{T}_2(\mathscr{U}_1)$ is a complete subspace of $\mathscr{U}$. For any $\kappa_0 \in \mathscr{U}$, the CR iterative procedure is then defined by the sequence $\{\kappa_n\}$ is generated as:

$$\begin{aligned} \mathscr{T}_2\kappa_{n+1} &= \mathscr{W}(\mathscr{T}_2\varsigma_n, \mathscr{T}_1\rho_n, \mathfrak{c}_n), \\ \mathscr{T}_2\rho_n &= \mathscr{W}(\mathscr{T}_2\kappa_n, \mathscr{T}_1\varsigma_n, \mathfrak{a}_n), \\ \mathscr{T}_2\varsigma_n &= \mathscr{W}(\mathscr{T}_1\varsigma_n, \mathscr{T}_1\kappa_n, \mathfrak{b}_n), \end{aligned} \tag{3}$$

with the sequences $\{\mathfrak{a}_n\}, \ \{\mathfrak{b}_n\}, \ \{\mathfrak{a}_n\} \subset [0,1]$.

2. Preliminaries in *multiplicative* $\boldsymbol{b}-$ metric spaces

To make this article self-contained, this section discusses some *multiplicative* $\boldsymbol{b}-$ metric space definitions and results. Recall that Bakhtin [26] presented the notion of *multiplicative* $\boldsymbol{b}-$ metric spaces, also known as quasimetric spaces, in the theory of metric fixed point in 1989 as a generalisation of conventional metric spaces and demonstrated the $\mathscr{BCP}$ in this context. Later on, Czerwik and others made extensive use of the *multiplicative* $\boldsymbol{b}-$ metric spaces ([27], [28], [29], [30]).

Definition 2.1. *Let $\mathscr{U} \neq 0$ and a real number $\mathfrak{s} \geq 1$ function $\Delta : \mathscr{U} \times \mathscr{U} \to \mathbb{R}^+$ is known as multiplicative $\boldsymbol{b}-$ metric iff for all $\kappa,\ \varsigma,\ \rho \in \mathscr{U}$, the following properties are satisfied:*

(1) $\Xi(\kappa, \varsigma) = 1$ *iff* $\kappa = \varsigma$,
(2) $\Xi(\kappa, \rho) = \Xi(\varsigma, \kappa)$,
(3) $\Xi(\kappa, \varsigma) \leq (\Xi(\kappa, \varsigma) \times \Xi(\varsigma, \rho))^{\mathfrak{s}}$.

The pair $(\mathscr{U}, \Xi)$ is called a multiplicative $\boldsymbol{b}-$ metric space with coefficient of $(\mathscr{U}, \Xi)$ as $\mathfrak{s}$.

The following simple lemma is multiplicative version of lemma given by Aghajani et al. [14] about *multiplicative* $\boldsymbol{b}-$ convergent sequences is worth noting.

Lemma 2.1. Let $(\mathscr{U}, \Xi)$ be a *multiplicative* $\boldsymbol{b}-$ metric space with $s \geq 1$, and let $\{\kappa_n\} \to \kappa$ and $\{\varsigma_n\} \to \varsigma$ are *multiplicative* $\boldsymbol{b}-$ convergent sequences. Then, we have

$$(\Xi(\kappa, \varsigma))^{\frac{1}{\mathfrak{s}^2}} \leq \lim_{n\to+\infty} inf\ \Xi(\kappa_n, \varsigma_n) \leq \lim_{n\to+\infty} sup\ \Xi(\kappa_n, \varsigma_n) \leq (\Xi(\kappa, \varsigma))^{\mathfrak{s}^2}.$$

We now discuss certain definitions and properties that a pair of mappings may satisfy and which are necessary for the proof of our major findings.

Definition 2.2. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Then mappings $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ are said to compatible if for a sequence $\{\kappa_n\}$ is a sequence in $\mathscr{U}$ such that $\lim_{n\to+\infty} \mathscr{T}_1\kappa_n = \lim_{n\to+\infty} \mathscr{T}_2\kappa_n = \mathring{\varpi}$, for some $\mathring{\varpi} \in \mathscr{U}$, we have*

$$\lim_{n\to+\infty} \Xi(\mathscr{T}_1\mathscr{T}_2\kappa_n, \mathscr{T}_2\mathscr{T}_1\kappa_n) = 1.$$

Definition 2.3. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Then mappings $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ are said to non- compatible if for a sequence $\{\kappa_n\}$ is a sequence in $\mathscr{U}$ such that $\lim_{n\to+\infty} \mathscr{T}_1\kappa_n = \lim_{n\to+\infty} \mathscr{T}_2\kappa_n = \mathring{\varpi}$, for some $\mathring{\varpi} \in \mathscr{U}$, we have $\lim_{n\to+\infty} \Xi(\mathscr{T}_1\mathscr{T}_2\kappa_n, \mathscr{T}_2\mathscr{T}_1\kappa_n)$ is either non-unity or non-existent.*

Definition 2.4. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Then mappings $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ are said to satisfy multiplicative $\boldsymbol{b}-$property (EA) if there exists a sequence $\{\kappa_n\}$ in $\mathscr{U}$ such that $\lim_{n\to+\infty} \mathscr{T}_1\kappa_n = \lim_{n\to+\infty} \mathscr{T}_2\kappa_n = \mathring{\varpi}$, for some $\mathring{\varpi} \in \mathscr{U}$.*

Definition 2.5. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Then mappings $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ are said to satisfy property common $\boldsymbol{b}-CLR_T$ if there exists a sequence $\{\kappa_n\}$ in $\mathscr{U}$ such that*

$$\lim_{n\to+\infty} \mathscr{T}_1\kappa_n = \lim_{n\to+\infty} \mathscr{T}_2\kappa_n = \mathscr{T}_2\mathring{\varpi},$$

for some $\mathring{\varpi} \in \mathscr{U}$.

Note: It is worth noting that the mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ are weakly compatible if the mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ are compatible, and commute at their coincidence points. the mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ satisfy the *multiplicative* $\boldsymbol{b}-$ property (EA) if $\mathscr{T}_1$ and $\mathscr{T}_2$ are non-compatible. Also, Weak compatibility and *multiplicative* $\boldsymbol{b}-$ property (EA) are independent to each other. The the condition of closedness of the ranges for $\boldsymbol{b} - CLR_T$

property isn't necessary. Readers are requested to refer [34, 33] for more literature. The *multivalued* $R-$ weakly commuting mapping, *multivalued* $R-$ weakly commuting type *multivalued* $(\mathscr{A}_{\mathscr{T}_1})$, *multivalued* $R-$ weakly commuting type *multivalued* $(\mathscr{A}_{\mathscr{T}_2})$ and *multiplicative* $R-$ weakly commuting type *multivalued* $\mathscr{A}_P$ are as follow:

Definition 2.6. *Let a multiplicative* $\boldsymbol{b}-$ *metric space* $(\mathscr{U}, \Xi)$ *with* $\mathfrak{s} \geq 1$. *The mappings* $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ *are said to be, multiplicative* $R-$ *weakly commuting if there exists some multiplicative* $R > 1$ *such that*

$$\Xi(\mathscr{T}_1\mathscr{T}_2\kappa, \mathscr{T}_2\mathscr{T}_1\kappa) \leq (\Xi(\mathscr{T}_1\kappa, \mathscr{T}_2\kappa))^R,$$

for all $\kappa \in \mathscr{U}$.

Definition 2.7. *Let a multiplicative* $\boldsymbol{b}-$ *metric space* $(\mathscr{U}, \Xi)$ *with* $\mathfrak{s} \geq 1$. *The mappings* $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ *are said to be, multiplicative* $R-$ *weakly commuting type* $(\mathscr{A}_{\mathscr{T}_2})$ *if there exists some multiplicative* $R > 1$ *such that*

$$\Xi(\mathscr{T}_1\mathscr{T}_2\kappa, \mathscr{T}_1\mathscr{T}_1\kappa) \leq (\Xi(\mathscr{T}_1\kappa, \mathscr{T}_2\kappa))^R,$$

for all $\kappa \in \mathscr{U}$.

Definition 2.8. *Let a multiplicative* $\boldsymbol{b}-$ *metric space* $(\mathscr{U}, \Xi)$ *with* $\mathfrak{s} \geq 1$. *The mappings* $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ *are said to be, multiplicative* $R-$ *weakly commuting type* $(\mathscr{A}_{\mathscr{T}_1})$ *if there exists some multiplicative* $R > 1$ *such that*

$$\Xi(\mathscr{T}_1\mathscr{T}_2\kappa, \mathscr{T}_2\mathscr{T}_2\kappa) \leq (\Xi(\mathscr{T}_1\kappa, \mathscr{T}_2\kappa))^R,$$

for all $\kappa \in \mathscr{U}$.

Definition 2.9. [33] *Let a multiplicative* $\boldsymbol{b}-$ *metric space* $(\mathscr{U}, \Xi)$ *with* $\mathfrak{s} \geq 1$. *The mappings* $\mathscr{T}_1,\ \mathscr{T}_2 : \mathscr{U} \to \mathscr{U}$ *are said to be, multiplicative* $R-$ *weakly commuting type* $\mathscr{A}_P$ *if there exists some multiplicative* $R > 1$ *such that*

$$\Xi(\mathscr{T}_1\mathscr{T}_1\kappa, \mathscr{T}_2\mathscr{T}_2\kappa) \leq (\Xi(\mathscr{T}_1\kappa, \mathscr{T}_2\kappa))^R,$$

for all $\kappa \in \mathscr{U}$.

Now, the definition of $\mathscr{F}-$ class function with weak $(\psi-\varphi)$ Contractions type mappings is as follows:

Definition 2.10. *Let a multiplicative* $b-$ *metric space* $(\mathscr{U}, \Xi)$ *with* $\mathfrak{s} \geq 1$ *and* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be self mapings of* $\mathscr{U}$. *Then, the mappings* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *are said to be of* $\mathscr{F}-$ *class function with weak* $(\psi - \varphi)$ *Contractions type mappings if there there exists* $\psi \in \Psi$ *and* $\varphi \in \Phi$ *such that for all* $\kappa,\ \varsigma \in \mathscr{U}$,

$$\psi[(\Xi(\mathscr{T}_1\kappa, \mathscr{T}_1\varsigma))^{\mathfrak{s}}] \leq \mathscr{F}(\psi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma)), \varphi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma))), \tag{4}$$

where $\mathscr{F}$ *is called a* $\mathscr{F}-$ *class function i.e., if* $\mathscr{F}(\kappa, \varsigma) \leq \kappa$ *for all* $\kappa,\ \varsigma > 0$.

Now, we prove that the CR process for $\mathscr{F}-$ class function with weak $(\psi - \varphi)$ Contractions mappings type is *multiplicative* $b-$ convergent.

Proposition 2.1. *Let a multiplicative* $b-$ *metric space* $(\mathscr{U}, \Xi)$ *with* $\mathfrak{s} \geq 1$ *and let* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *be self mappings of* $\mathscr{U}$ *with the condition* $\mathscr{T}_1\mathscr{U} \subset \mathscr{T}_2\mathscr{U}$. *If* $\mathscr{T}_1$ *and* $\mathscr{T}_2$ *are* $\mathscr{F}-$ *class function with weak* $(\psi-\varphi)$ *Contractions type mappings, then for any* $\kappa_0 \in \mathscr{U}$ *the sequence such that*

$$\varsigma_n = \mathscr{T}_1\kappa_n = \mathscr{T}_2\kappa_{n+1}, \tag{5}$$

$n = 0, 1, 2, \cdots$ *satisfies the following conditions*

(1) $\lim_{n\to+\infty} \Xi(\varsigma_n, \varsigma_{n+1}) = 1$.

(2) $\{\varsigma_n\} \subset \mathscr{U}$ *is a multiplicative* $b-$ *Cauchy sequence.*

Proof. To prove the condition (1), suppose that there exists an arbitrary point $\kappa_0 \in \mathscr{U}$. As, it is given that $\mathscr{T}_1\mathscr{T} \subset \mathscr{T}_2\mathscr{T}$ therefore we can define CR sequence as $\varsigma_n = \mathscr{T}_1\kappa_n = \mathscr{T}_2\kappa_{n+1},\ n = 0, 1, 2, \cdots$. Now, let $\varsigma_{n-1} = \mathscr{T}_2\kappa_n = \mathscr{T}_2\kappa_{n+1} = \varsigma_n$, for some $n \in \mathbb{N}$. Because $\{\varsigma_n\}$ would otherwise be a constant sequence for all $n \geq n_0$. By condition (4) we obtain:

$$\begin{aligned}
\psi[\Xi(\varsigma_n, \varsigma_{n+1})] &\leq \psi[(\Xi(\varsigma_n, \varsigma_{n+1}))^{\mathfrak{s}}] \\
&= \psi[(\Xi(\mathscr{T}_1\kappa_n, \mathscr{T}_1\kappa_{n+1}))^{\mathfrak{s}}] \\
&\leq \mathscr{F}(\psi(\Xi(\mathscr{T}_2\kappa_n, \mathscr{T}_2\kappa_{n+1})), \varphi(\Xi(\mathscr{T}_2\kappa_n, \mathscr{T}_2\kappa_{n+1}))) \\
&\leq \mathscr{F}(\psi(\Xi(\varsigma_{n-1}, \varsigma_n)), \varphi(\Xi(\varsigma_{n-1}, \varsigma_n))) \\
&\leq \psi(\Xi(\varsigma_{n-1}, \varsigma_n)). \qquad (6)
\end{aligned}$$

As, we know that $\psi \in \Psi$, we have

$$\Xi(\varsigma_n, \varsigma_{n+1}) < \Xi(\varsigma_{n-1}, \varsigma_n).$$

As, the sequence $\{\Xi(\varsigma_n, \varsigma_{n+1})\}$ is a monotonically decreasing, therefore there exists $\theta \geq 1$ in such a way $\lim_{n\to+\infty} \Xi(\varsigma_n, \varsigma_{n+1}) = \theta$. Now, it is to show that $\theta = 1$. Now, let $\theta > 1$. Hence, by using equation (6), we have

$$\begin{aligned}
\lim_{n\to+\infty} sup\ \psi(\Xi(\varsigma_n, \varsigma_{n+1})) &\leq \mathscr{F}(\lim_{n\to+\infty} sup\ \psi(\Xi(\varsigma_n, \varsigma_{n+1})), \lim_{n\to+\infty} inf\ \varphi(\Xi(\varsigma_n, \varsigma_{n+1}))) \\
&\leq \lim_{n\to+\infty} sup\ \psi(\Xi(\varsigma_n, \varsigma_{n+1})). \qquad (7)
\end{aligned}$$

It results that $\psi(\theta) \leq \mathscr{F}(\psi(\theta), \varphi(\theta)) < \psi(\theta)$. Therefore $\theta = 1$, by contradiction. It further results that $\lim_{n\to+\infty} \Xi(\varsigma_n, \varsigma_{n+1}) = 1$.

Now, we shall prove the condition (2) where it is to prove that the sequence $\{\varsigma_n\} \in \mathscr{U}$ is *multiplicative* $b-$ Cauchy. Contrarily, let $\exists\ \varepsilon > 1$ such that there are two sequences $\{\mu_1(\vartheta)\}$ and $\{\mu_2(\vartheta)\}$ with $\mu_2(\vartheta) > \mu_1(\vartheta) > \vartheta$ in such a way that

$$\Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_2(\vartheta)}) \geq \varepsilon,\ \textit{and}\ \Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_2(\vartheta)-1}) < \varepsilon,$$

for all $\vartheta \in \mathbb{N}$. It results that

$$\begin{aligned}
\varepsilon \leq \Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_2(\vartheta)}) &\leq (\Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_1(\vartheta)-1}))^{\mathfrak{s}} \times (\Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)}))^{\mathfrak{s}} \\
&\leq (\Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_1(\vartheta)-1}))^{\mathfrak{s}} \times (\Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)-1}))^{\mathfrak{s}^2} \times (\Xi(\varsigma_{\mu_2(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)}))^{\mathfrak{s}^2}.
\end{aligned}$$

By the condition (1) and Lemma 2.1, we have

$$\begin{aligned}
(\varepsilon)^{\frac{1}{\mathfrak{s}^2}} &\leq \lim_{k\to+\infty} \Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)-1}) \\
&\leq \lim_{k\to+\infty} sup\ \Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)-1}) \\
&\leq (\lim_{k\to+\infty} sup\ \Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_1(\vartheta)}))^{\mathfrak{s}} \times (\lim_{k\to+\infty} sup\ \Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_2(\vartheta)-1}))^{\mathfrak{s}} \leq (\varepsilon)^{\mathfrak{s}}. \qquad (8)
\end{aligned}$$

It results

$$\begin{aligned}
(\varepsilon)^{\frac{1}{\mathfrak{s}^2}} &\leq \lim_{k\to+\infty} inf\ \Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)-1}) \\
&\leq \lim_{k\to+\infty} sup\ \Xi(\varsigma_{\mu_1(\vartheta)-1}, \varsigma_{\mu_2(\vartheta)-1}) \leq (\varepsilon)^{\mathfrak{s}}. \qquad (9)
\end{aligned}$$

Now suppose that $\kappa = \kappa_{\mu_1(\vartheta)}$ and $\varsigma = \kappa_{\mu_2(\vartheta)}$ in (4), we have

$$\lim_{k\to\infty} sup\ \psi[(\Xi(\varsigma_{\mu_1(\vartheta)}, \varsigma_{\mu_2(\vartheta)}))^{\mathfrak{s}}] \geq \psi((\varepsilon)^{\mathfrak{s}}) > 1$$
$$\lim_{k\to\infty} sup\ \psi[(\Xi(\mathscr{T}_1\kappa_{\mu_1(\vartheta)}, \mathscr{T}_1\varsigma_{\mu_2(\vartheta)}))^{\mathfrak{s}}] \geq \psi((\varepsilon)^{\mathfrak{s}}) > 1$$

$$\mathscr{F}(\lim_{k\to\infty} sup\ \psi(\Xi(\varsigma_{\mu_1(\vartheta)-1},\varsigma_{\mu_2(\vartheta)-1}),\lim_{k\to\infty} inf\ \varphi(\Xi(\varsigma_{\mu_1(\vartheta)-1},\varsigma_{\mu_2(\vartheta)-1})))) \geq \psi((\varepsilon)^{\mathfrak{s}}) > 1$$
$$\mathscr{F}(\lim_{k\to\infty} sup\ \psi(\Xi((\varepsilon)^{\mathfrak{s}}),\lim_{k\to\infty} inf\ \varphi(\Xi((\varepsilon)^{\mathfrak{s}}))) \geq \psi(\mathfrak{s}\varepsilon) > 1$$
$$\psi((\varepsilon)^{\mathfrak{s}}) \geq \psi(\Xi(\varsigma_{\mu_1(\vartheta)-1}) \geq \psi((\varepsilon)^{\mathfrak{s}}) > 1.$$

Sequence $\{\varsigma_n\}$ is *multiplicative**b**−* Cauchy, by contradiction.The following result proves the uniqueness of $\mathscr{U}_{\mathcal{PC}}(\mathscr{T}_1,\mathscr{T}_2)$ for $\mathscr{F}-$ class function with weak $(\psi-\varphi)$ Contractions type mappings.

Proposition 2.2. *Let $(\mathscr{U},\Xi)$ be a multiplicative **b**− metric space, $\mathfrak{s}\geq 1$ and let $\mathscr{T}_1$ and $\mathscr{T}_2$ be self mappings of $\mathscr{U}$. Suppose that that $\mathscr{T}_1$ and $\mathscr{T}_2$ are $\mathscr{F}-$ class function with weak $(\psi-\varphi)$ Contractions type mappings. Now, if $\mathscr{T}_1$ and $\mathscr{T}_2$ have a point $\zeta\in\mathscr{U}_{\mathcal{PC}}(\mathscr{T}_1,\mathscr{T}_2)$, then ζ is unique.*

Proof. Let $\zeta_1,\zeta_2\in\mathscr{U}_{\mathcal{PC}}(\mathscr{T}_1,\mathscr{T}_2)$. In other words, let there exists $\gamma_1,\ \gamma_2\in\mathscr{U}$ in such a manner that $\mathscr{T}_1\gamma_1=\mathscr{T}_2\gamma_1=\zeta_1$ and $\mathscr{T}_1\gamma_2=\mathscr{T}_2\gamma_2=\zeta_2$. Now, using inequality (4), we have

$$\begin{aligned}\psi(\Xi(\zeta_1,\zeta_2)) &\leq \psi((\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_1\gamma_2))^{\mathfrak{s}})\\ &\leq \mathscr{F}(\psi(\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_2\gamma_2)),\varphi(\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_2\gamma_2)))\\ &\leq \mathscr{F}(\psi(\Xi(\zeta_1,\zeta_2)),\varphi(\Xi(\zeta_1,\zeta_2)))\\ &\leq \psi(\Xi(\zeta_1,\zeta_2)). \end{aligned} \tag{10}$$

Hence $\zeta_1=\zeta_2$ by contradiction.

The following result proves the existence of $\mathscr{U}_{\mathcal{PC}}(\mathscr{T}_1,\mathscr{T}_2)$ for $\mathscr{F}-$ class function with weak $(\psi-\varphi)$ Contractions type mappings.

Proposition 2.3. *Let a multiplicative **b**− metric space $(\mathscr{U},\Xi)$ with $\mathfrak{s}\geq 1$. Suppose two $\mathscr{F}-$ class function with weak $(\psi-\varphi)$ Contractions type self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ such that $\mathscr{T}_1\mathscr{U}\subset\mathscr{T}_2\mathscr{U}$. For a complete subspace $\mathscr{T}_2\mathscr{U}\subset\mathscr{U}$, for any $\kappa_0\in\mathscr{U}$ the CR sequence defined by 5 is convergent to $\zeta\in\mathscr{U}$, $\mathscr{U}_{\mathcal{CP}}(\mathscr{T}_1,\mathscr{T}_2)\neq\emptyset$ and $\zeta\in\mathscr{U}_{\mathcal{PC}}(\mathscr{T}_1,\mathscr{T}_2)$ is unique.*

Proof. We can use Proposition 2.1 to consider that the sequence $\{\varsigma_n\}$ is a *multiplicative **b**−* Cauchy in $\mathscr{U}$. Therefore the sequence $\{\mathscr{T}_2\kappa_{n+1}\}$ is *multiplicative **b**−* Cauchy in $\mathscr{U}$. As, it is given that $\mathscr{T}_2\mathscr{U}\subset\mathscr{U}$ is complete and $\{\mathscr{T}_2\kappa_{n+1}\}\subset\mathscr{T}_2\mathscr{U}$. for some $\gamma\in\mathscr{U}$, we have

$$\lim_{n\to+\infty}\mathscr{T}_2\kappa_{n+1}=\zeta=\mathscr{T}_2\gamma_1.$$

Now, to show that $\mathscr{T}_1\gamma_1=\mathscr{T}_2\gamma_1$ let $\mathscr{T}_1\gamma_1\neq\mathscr{T}_2\gamma_1$. By the inequality 4 and Lemma 2.1 we have,

$$\begin{aligned}\psi(\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_2\gamma_2)) &= \psi\left((\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_2\gamma_2))^{\frac{\mathfrak{s}}{\mathfrak{s}}}\right) \leq \lim_{n\to+\infty} sup\ \psi((\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_2\kappa_{n+1}))^{\mathfrak{s}})\\ &\leq \lim_{n\to+\infty} sup\ \psi((\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_1\kappa_n)^{\mathfrak{s}})\\ &\leq \mathscr{F}(\lim_{n\to+\infty} sup\ \psi(\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_2\kappa_n)),\lim_{n\to+\infty} inf\ \varphi(\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_2\kappa_n)))\\ &\leq \mathscr{F}(\psi((\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_1\gamma))^{\mathfrak{s}}),\varphi((\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_2\gamma_1)))^{\mathfrak{s}})\\ &\leq \psi((\Xi(\mathscr{T}_2\gamma_1,\mathscr{T}_1\gamma)))^{\mathfrak{s}})=1,\end{aligned}$$

which results that $\psi(\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_2\gamma_1))=1$. Therefore, $\Xi(\mathscr{T}_1\gamma_1,\mathscr{T}_2\gamma_1)=1$, that is $\mathscr{T}_1\gamma_1=\mathscr{T}_2\gamma_1$. Therefore, $\mathscr{U}_{\mathcal{CP}}\neq\emptyset$. As, since $\mathscr{T}_1\gamma_1=\mathscr{T}_2\gamma_1=\zeta$, $\zeta\in\mathscr{U}_{\mathcal{PC}}$. Furthermore, Proposition

2.2 confirms the uniqueness of ζ.

The results discussed in this section pertain to the existence of $\mathscr{CFP}$ for $\mathscr{F}-$ class function with weak $(\psi - \varphi)$ Contractions type mappings that meet various non-commutative conditions.

Lemma 2.2. [10] Let $\mathscr{U} \neq \emptyset$ and weak compatible self maps $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$. Let unique $\gamma \in \mathscr{U}_{\mathcal{PC}}$, that is $\gamma = \mathscr{T}_1\mu = \mathscr{T}_1\mu$, then γ is the unique $\mathscr{CFP}$ of $\mathscr{T}_1$ and $\mathscr{T}_2$.

Weak compatibility is a bare minimum requirement for the existence of $\mathscr{CFP}$ in contractive type mappings (refer, [35]). As a result, the existence of a unique $\mathscr{CFP}$ is reduced to the existence of a unique $\mathscr{U}_{\mathcal{PC}}$ for weakly compatible maps.

Theorem 2.1. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Also, assume two $\mathscr{F}-$ class function with weak $(\psi - \varphi)$ Contractions type self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$ such that $\mathscr{T}_1\mathscr{U} \subset \mathscr{T}_2\mathscr{U}$ and subspace $\mathscr{T}_2\mathscr{U} \subset \mathscr{U}$ is complete, then $\mathscr{U}_{\mathcal{CP}} \neq \emptyset$ and, additionally, there is existence of unique $\mathscr{CFP}$.*

Proof. As in Proposition 2.3, $(\mathscr{T}_1, \mathscr{T}_2)$ has a fixed point which is unique. Furthermore, weak compatibility of $(\mathscr{T}_1, \mathscr{T}_2)$ and Lemma 2.2 provides the conclusion.

Theorem 2.2. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Also, assume two $\mathscr{F}-$ class function with weak $(\psi - \varphi)$ Contractions type self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$ which satisfy multiplicative $\boldsymbol{b}-$ property (EA) and closed subspace $\mathscr{T}_1\mathscr{U} \subset \mathscr{U}$. Then there exists a unique $\gamma \in \mathscr{U}_{\mathcal{PC}}$ of mappings $\mathscr{T}_1$ and $\mathscr{T}_2$. Also, if the mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ are weak compatible, then there exists unique $\mathscr{CFP}$ $\gamma \in \mathscr{U}$.*

Proof. By definition of *multiplicative* $\boldsymbol{b}-$ property (EA), there exists a sequence $\{\kappa_n\} \in \mathscr{U}$, we have

$$\lim_{n \to +\infty} \mathscr{T}_1\kappa_n = \lim_{n \to +\infty} \mathscr{T}_2\varsigma_n = \gamma,$$

for some $\gamma \in \mathscr{U}$. As, we have closed subspace $\mathscr{T}_1\mathscr{U}$ as $\mathscr{T}_1\mathscr{U} \subset \mathscr{U}$. It results that

$$\lim_{n \to +\infty} \mathscr{U}\kappa_n = \gamma = \mathscr{T}_1\alpha_1 = \lim_{n \to +\infty} \mathscr{U}\varsigma_n,$$

for some $\alpha_1 \in \mathscr{U}$. Now by proposition 2.3,

$$\gamma = \mathscr{T}_1\alpha_1 = \mathscr{T}_2\alpha_1$$

and unique $\gamma \in \mathscr{U}_{\mathcal{PC}}$. By Lemma 2.2, γ is the only common fixed point of $\mathscr{T}_1$ and $\mathscr{T}_2$.

Corollary 2.1. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Also, assume two non-compatible and weak compatible $\mathscr{F}-$ class function with weak $(\psi - \varphi)$ Contractions type self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$. If $\mathscr{T}_1$ is a closed subspace of $\mathscr{U}$. Then, there exists unique $\mathscr{CFP}$, $\gamma \in \mathscr{U}$.*

Theorem 2.3. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Also, assume two $\mathscr{F}-$ class function with weak $(\psi - \varphi)$ Contractions type self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$ which satisfy $\boldsymbol{b} - CLR_T$ property. Then there exists a unique $\gamma \in \mathscr{U}_{\mathcal{PC}}$ of mappings $\mathscr{T}_1$ and $\mathscr{T}_2$. Also, if the mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ are weak compatible, then there exists unique $\mathscr{CFP}$, $\gamma \in \mathscr{U}$.*

Proof. By definition of *multiplicative* $\boldsymbol{b}-$ property (EA), there exists a sequence $\{\kappa_n\} \in \mathscr{U}$, we have

$$\lim_{n \to +\infty} \mathscr{T}_1\kappa_n = \lim_{n \to +\infty} \mathscr{T}_2\varsigma_n = \mathscr{T}_2\alpha_1,$$

for some $\alpha \in \mathscr{U}$. Therefore, there exists $\gamma \in \mathscr{U}$ so that $\gamma = \mathscr{T}\alpha_1$.

Theorem 2.4. *Theorems 2.1, 2.2, and 2.3 as well as Corollary 2.1 are still valid if the weakly compatible property is substituted to one of the following whilst retaining the other suppositions:*

(1) *Mappings satisfying multiplicative Compatibility,*
(2) *multiplicative R− weakly commuting property,*
(3) *multiplicative R− weakly commuting property of type $\mathscr{A}_P$,*
(4) *multiplicative R− weakly commuting property of type $\mathscr{A}_{\mathscr{T}_1}$,*
(5) *multiplicative R− weakly commuting property of type $\mathscr{A}_{\mathscr{T}_2}$.*

Proof. For proof of this Theorem, readers are requested to refer Theorem 4.6 of [36].

Now on wards, our main objective is to explain some extensions and generalisations of well-known results that have previously been presented in the context of conventional metric spaces. We exclude the condition $\mathscr{T}_1\mathscr{U} \subset \mathscr{T}_2\mathscr{U}$ in the following result using the *multiplicative* $\boldsymbol{b}-$ property (EA).

Theorem 2.5. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Also, assume two self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$ satisfying*

$$\psi(\Xi(\mathscr{T}_1\kappa, \mathscr{T}_1\varsigma)^{\mathfrak{s}}) \leq \alpha_\delta \psi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma)), \tag{11}$$

for all κ, $\varsigma \in \mathscr{U}$, $0 \leq \alpha_\delta < 1$ and $\psi \in \Psi$. Also, if $\mathscr{T}_1\mathscr{U} \subset \mathscr{T}_2\mathscr{U}$ and $\mathscr{T}_2\mathscr{U} \subset \mathscr{U}$ is complete, then $\mathscr{U}_{\mathcal{CP}} \neq \emptyset$ and, additionally, there is existence of unique $\mathscr{C}\mathscr{F}\mathscr{P}$ if the mappings $\mathscr{T}_1$, $\mathscr{T}_2$ are weakly compatible.

Proof. For $\psi \in \Psi$ and $0 \leq \alpha_\delta < 1$, If we consider $\varphi(\mathring{\varpi}) = (1 - \alpha_\Xi)\psi(\mathring{\varpi})$ we have $\varphi \in \Phi$. Therefore by the (11), we have that

$$\psi(\Xi(\mathscr{T}_1\kappa, \mathscr{T}_1\varsigma)^{\mathfrak{s}}) \leq \mathscr{F}(\psi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma)), \varphi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma))).$$

The following Theorem is inferred from the Theorem 4.2 by Morales and Rojas [36].

Theorem 2.6. *Let a multiplicative $\boldsymbol{b}-$ metric space $(\mathscr{U}, \Xi)$ with $\mathfrak{s} \geq 1$. Also, assume two self mappings $\mathscr{T}_1$ and $\mathscr{T}_2$ of $\mathscr{U}$ satisfying*

$$(\Xi(\mathscr{T}_1\kappa, \mathscr{T}_1\varsigma))^{\mathfrak{s}} \leq \mathscr{F}(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma), \varphi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma))), \tag{12}$$

for all κ, $\varsigma \in \mathscr{U}$, $\psi \in \psi$ and $\varphi \in \Phi$. Also, if $\mathscr{T}_1\mathscr{U} \subset \mathscr{T}_2\mathscr{U}$ and $\mathscr{T}_2\mathscr{U} \subset \mathscr{U}$ is complete, then $\mathscr{U}_{\mathcal{CP}} \neq \emptyset$ and, additionally, there is existence of unique $\mathscr{C}\mathscr{F}\mathscr{P}$ if the mappings $\mathscr{T}_1$, $\mathscr{T}_2$ are weakly compatible.

Proof. If we consider $\psi = Identity$. Then by the (12), we have that

$$\psi((\Xi(\mathscr{T}_1\kappa, \mathscr{T}_1\varsigma))^{\mathfrak{s}}) \leq \mathscr{F}(\psi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma)), \varphi(\Xi(\mathscr{T}_2\kappa, \mathscr{T}_2\varsigma))).$$

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