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OPTIMAL (r, Q) MODELS CONSIDERING INVENTORY SHRINKAGE

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Abstract. Although inventory shrinkage negatively affects firms' operations decisions in nearly every type of industry, it is generally ignored in practice. Most of the available mathematical models are not considering inventory shrinkage caused by lost or misplaced inventory items, because of the mathematical complexity. Therefore, we develop new (r, Q) models addressing shrinkage. Besides misplaced items, we consider perished or lost items, which are common in the retail industry. We further propose an algorithm to solve the models developed. To demonstrate the applicability of our models and solution algorithm, we assume normal and exponential demand distributions. We show that total inventory costs obtained using our models are significantly lower than those from available models, which does not consider misplaced items and the sensitivity analysis leads us to several managerial implications.

Keywords: Optimal Inventory Systems, (r, Q) Models, Misplaced items, Lost-sales, Backorders.

AMS Subject Classification: 90-10, 90B05, 90B06, 93E20, 49K45.

1. Introduction

Inventory shrinkage has always been a serious challenge in supply chain management liable to occur at any phase vulnerable to manual operational mistakes, including manufacturing, distribution, warehousing, and retailing ([27], [1], [26]). It is recognised as one of the two major sources for inventory inaccuracy, i.e., the discrepancy between physical and recorded inventory levels ([23], [7]). Thus, inventory shrinkage not only devastatingly affects the decisions on supply chain operations (e.g., manufacturer's production planning decisions) but also escalate supply chain total costs and stock-out threshold ([8], [6], [23]). Items misplaced in stores are unavailable to customers until found. Misplaced items, consequently, has emerged as a common and expensive problem in the retail setting. As early

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as the 90s, the sales losses due to products being in storage areas rather than the selling floor amount to $560-960 million per year in the United States supermarket industry [5]. Misplaced items also cause 25-30 % of grocery stock-outs, revealed in a study by [9]. In their longitudinal study, [21] showed that one-standard deviation increase in misplaced items leads to a loss of approximately 7 % of the net income generated by an average retail store. Similarly, [17] found that misplaced items reduced profits by 25 % in their case study of a leading retailer. A relatively recent study [2], concluded that there was an estimated 10 % profit reduction due to inventory errors.

Several common reasons cause inventory misplacement in the retail setting, including (i) customers picking up products from one location and later replacing them in another location; (ii) staff not placing products on the correct shelf or at the appropriate time; and (iii) staff losing products in a backroom or another storage area ([18], [31], [21]). Due to the negative influences of misplaced items on the performance of retail stores and the whole supply chains, the use of RFID technology in tracking/locating items has been proposed as a solution ([20], [3], [22]). However, due to the difficulties in fully eliminating the above common execution errors made by either customers or staff, despite the use of RFID, misplaced items become remain the norm rather than an anomaly in the retail settings ([4], [27]). In fact, "implementing RFID could eliminate part of the shrinkage and misplacement errors" ([23]) but the main drawback is that it demands heavy investment costs. Due to the cost requirements, many firms cannot afford it and only large companies implement it [28]. At the end of the day the main aim of this study is to show that instead of using non environmental friendly technologies such as RFIDs, using mathematical approaches can solve these important practical issues in inventory management.

In literature, most of the available inventory models, whether periodic review systems or continuous review systems, do not consider misplaced items; but are based on the common assumption that the actual and recorded inventories are the same, as pointed out in ([7], [18], [6]). Thus, the traditional models may not be able to provide supply chain partners with practical decision-making support and may distort inventory decision-making ([23]). To bridge this gap, in an earlier study we ([29]) proposed some improved (r,Q) models with misplaced items. Whereas, in this study we moved one step forward and investigate (r,Q) policy for continuous review systems while considering inventory shrinkage in more broader perspective. We study the (r,Q) models considering both perished or lost items, besides misplaced items,which is often used for fast moving products and is widely adopted in the retail settings ([13]).

In practice, customers react differently to stock-out situations. For products, such as diapers, cosmetics, and feminine hygiene, customers walk away immediately and purchase from other sources ([10]). In such situations, unsatisfied demand is lost, resulting in the lost sales. For other products (e.g., coffee, shampoo, pet food), customers are more willing to delay their purchase until the products are available in the stores again. Such behaviour allows the stores to back order the products, generating backorders. There are practical situations involving a mixed phenomenon of the above two, as pointed out in ([15], [24]). For example, e-commerce companies usually perform as suppliers for large business customers as well as serving their own individual customers. When the inventory is out of stock, the individual customers will likely switch their purchasing to other websites, whilst the large business customers will wait for reasons including, price discounts and constraints in the contracts. To facilitate inventory decision making as comprehensively as possible, we, thus, develop new (r,Q) models considering misplaced items for all three situations,lost-sales, backorders, and the mixed cases involving both of these.

To summarise, unlike most of the available models [1], which assume the same actual and recorded inventory levels, we develop the (r, Q) models which take into account the inevitability of misplaced items. In addition, the models are developed for all three existing situations in practice. Therefore, this study makes the following four contributions to both the existing literature and practice. First, we develop the inventory profiles involving misplaced items in the three situations. Second, we develop the new (r, Q) models for all the three cases, including lost-sales, back orders, and a mixture of both, thus providing inventory decision-making support for all situations encountered. Third, we develop an algorithm to solve the models developed. Finally, based on the numerical examples and sensitivity analysis, we show much lower total inventory costs obtained based on our models compared to those from the model that does not consider misplaced items. Additionally, we arrive at several managerial implications such as firms' flexibility to allow adjustment of the backordering and shortage costs to minimise inventory costs.

In the following section, we detail the problem context, notations, and the new (r, Q) models involving misplaced items for the three cases. In Section 3, we present the pseudo code algorithm to solve the models. We conduct numerical examples to demonstrate the applicability of our models and solution algorithm in making practical inventory decisions. Sensitivity analysis allows managerial implications and we conclude the paper in Section 5 by highlighting the potential avenues for future research.

2. Notation and models

In an (r, Q) inventory system, the inventory is reviewed continuously, and an order of Q units of products is placed whenever the inventory position falls to or below the reorder point r. In the system, due to misplacement, not all products are available on the shelves for purchase. Moreover, some of the misplaced items perish or are lost, thus not being available for purchase at all ([19], [20]). To model the misplaced and lost items, in this study, we introduce two parameters: θ and β. In such a system, it is very important to determine the values of r and Q to minimise the total inventory cost. In the literature, a continuous demand is frequently used to develop inventory models ([16], [12]) as it is in this model. In addition, we assume at most one outstanding order at any given time. This assumption is met in the practical situations where an ordering cost is relatively high and a lead-time is shorter than an order cycle ([25], [12]). For example, products that are transported by air have a short lead-time but a high ordering cost. As a result, there is often a large difference between order quantity and reorder point, and the inventory system seldom has more than one outstanding order. Last, we assume that the lead-time is the same in any replenishment cycle and that in each cycle, the reorder point is non-negative and constant, as often adopted in literature ([14], [30], [16]). The below notation is used to develop the new (r, Q) models.

r : Reorder point
Q : Order quantity
A : Fixed ordering cost
h : Inventory holding cost per unit per replenishment cycle
s : Shortage cost per unit
o : Backordering cost per unit
c : Unit purchasing price
C_{cycle} : Average inventory cost per replenishment cycle

[1]To the best of our knowledge, ([23]) is the only article where the authors developed a News-vendor model considering misplaced items. The authors considered the lost-sales case, instead of all the three.

D : Expected annual demand
μ : Expected demand during lead-time
$C(r,Q)$: Expected total annual inventory cost
θ, $\theta \in [0,1]$: Percentage of products available for sale
β, $\beta \in [0,1]$: Percentage of misplaced items that are perished or lost
μ_θ : Expected proportion of products available for sale
μ_β : Expected proportion of misplaced items that are perished or lost
$g(\theta)$: The probability distribution of θ.
X : Random variable denoting the total demand during lead-time
$f(X)$: Probability density function of X
$F(X)$: Cumulative distribution function of X
$\bar{a}(r)$: Expected on-hand inventory at the end of each cycle
$\bar{s}(r)$: Expected number of shortages per cycle
$\bar{b}(r)$: Expected number of backordered products per cycle
α : Percentage of back ordered products in the mixed case

In the following sections, we describe the problem contexts for the three cases and develop the corresponding inventory profiles. In view of the similar logic in model development, we present the model details for the mixed case of lost-sales and backordering, whereas the details for the other two cases are provided in Appendixes A and B, respectively.

2.1. **(r,Q) models involving misplaced items for the cases of lost-sales and backordering.** In the case of lost-sales, the excess demand is lost, thus incurring a shortage cost. When Q products are ordered, θQ products are available on the shelves for purchase; $(1-\theta)Q$ products are misplaced and $\beta(1-\theta)Q$ misplaced products are perish or are lost. $(1-\beta)(1-\theta)Q$ products are, thus, misplaced and might reappear on other shelves or in the backroom. The corresponding inventory profile is shown in Fig. 1. At time 0, an order Q is received. When the inventory drops to r at time t_0, the actual inventory level is θr due to the inventory misplacement. An order Q is placed at this time (and received at time t_3). During the delivery lead-time, the actual inventory decreases and reaches 0 at time t_1, which is earlier than time t_2 when the inventory level drops to 0. Thus, the lost sales occur earlier at t_1. Because $\beta(1-\theta)Q$ misplaced products are lost, only $(1-\beta)(1-\theta)Q$ misplaced products reappear later, thus being available for purchase again, as shown in the figure.

Because θ and β can be either deterministic or stochastic, we develop the new (r,Q) models involving misplaced items considering two situations. In the first situation, both θ and β are deterministic, whereas in the second, both θ and β are stochastic with uniform distributions and they are independent. The developed models involving misplaced items considering the above two situations of θ and β are provided in Appendix A.

In the second case, when the system is out of stock, the excess demand is backordered. The inventory profile in this case is similar to the lost-sales case except that backordering takes place at time t_1. Similarly, the (r,Q) models involving misplaced items for the case of backordering are developed considering the same two situations of θ and β (see model details in Appendix B).

2.2. **(r,Q) models involving misplaced items for the mixed case of lost-sales and backordering.** In the mixed case, when the system is out of stock, some excess demand is back ordered, and the rest is lost. In this study, we introduce a parameter α to represent the percentage of backordered demand. $(1-\alpha)$, thus, represents the percentage of lost sales, as shown in the inventory profile in Figure 2. The (r,Q) models for the mixed case

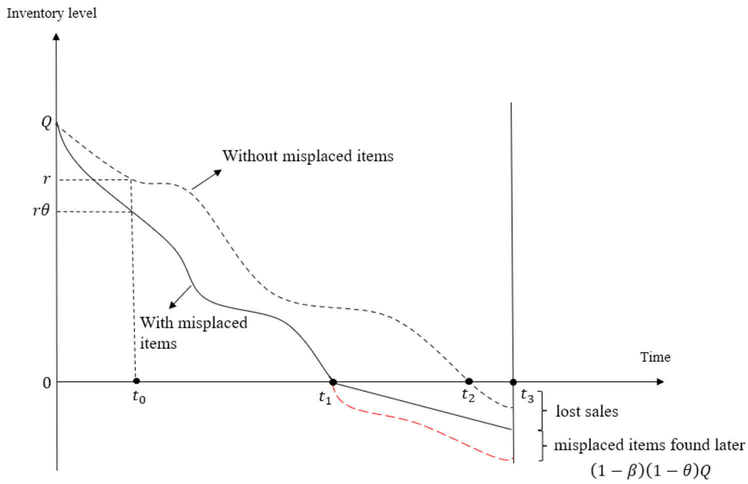


FIGURE 1. Inventory profile in the case of lost-sales.

of lost-sales and backordering are developed considering the same two situations of θ and β.

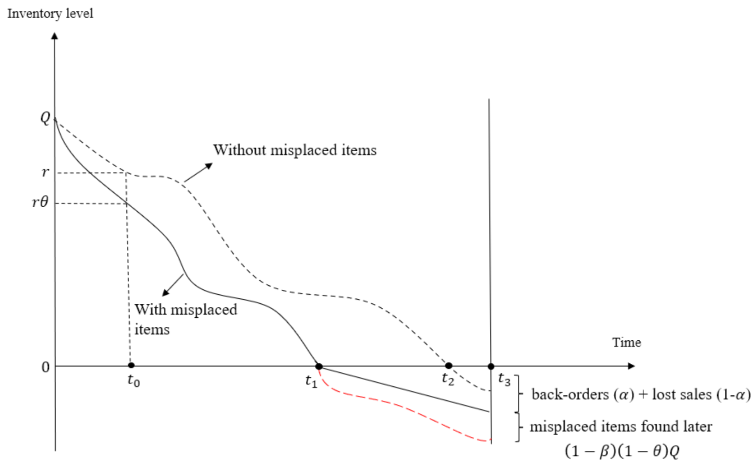


FIGURE 2. Inventory profile in the mixed case.

Situation 1: θ and β are both deterministic.

When unsatisfied demands are all lost, the number of shortages at the end of each cycle is $max[0, X - r\theta]$. When partial unsatisfied demands are lost at a percentage $(1-\alpha)$, the number of shortages at the end of each cycle is $max[0, (1-\alpha)(X - r\theta)]$. With the interval of shortages, the expected number of shortages per cycle $\bar{s}(r)$ is calculated as follows:

$$\bar{s}(r) = \int_{\theta r}^{\infty} (1-\alpha)(X - \theta r) f(X) dX \tag{1}$$

When all unsatisfied demands are backordered, the number of backordered products at the end of each cycle is $max[0, X - r\theta]$. When these demands are partially backordered at a percentage α at a unit backordering cost o. The expected number of backordered products per cycle $\bar{b}(r)$ can be computed as follows:

$$\bar{b}(r) = \int_{\theta r}^{\infty} \alpha(X - \theta r) f(X) dX \tag{2}$$

At the end of each cycle, the on-hand inventory is $max[0, r\theta - X] + (1-\beta)(1-\theta)r$. With the interval of the on-hand inventory, the expected on-hand inventory $\bar{a}(r)$ can be obtained using Eq. (3).

$$\begin{aligned}
\bar{a}(r) &= \int_{0}^{\theta r} (\theta r - X) f(X) dX + (1-\beta)(1-\theta)r \\
&= \int_{0}^{\infty} (\theta r - X) f(X) dX + \int_{\theta r}^{\infty} (1 - \alpha + \alpha)(X - \theta r) f(X) dX + (1-\beta)(1-\theta)r \\
&= \int_{0}^{\infty} (\theta r - X) f(X) dX + \int_{\theta r}^{\infty} (1 - \alpha)(X - \theta r) f(X) dX \\
&+ \int_{\theta r}^{\infty} \alpha(X - \theta r) f(X) dX + (1-\beta)(1-\theta)r \\
&= \bar{s}(r) + \bar{b}(r) - \mu + r - \beta r + \theta\beta r
\end{aligned} \tag{3}$$

We know that when all the lost sales are backordered then the net inventory and the on hand inventory are not same. Since in our model some of the lost sales are backordered and some of them are shortage we should reconsider the net inventory level. Therefore, for the expected number of products which are backordered and for which there is a shortage and the expected on-hand inventory per cycle above, the average cost per cycle C_{cycle} should consider the net inventory level. In our case the net inventory level is equal to the average inventory level (Equation 3) minus the average backorder and can be computed as follows:

$$\begin{aligned}
C_{cycle} &= A + cQ + h\frac{Q}{D}(\frac{Q}{2} + \bar{s}(r) - \mu + r - \beta r + \theta\beta r) + s\bar{s}(r) + o\bar{b}(r) \\
&+ c(1-\theta)\beta Q
\end{aligned} \tag{4}$$

Based on the above average cost per cycle, the average annual cost $C(r,Q)$ can be calculated using Eq. (5) below.

$$\begin{aligned}
C(r,Q) &= \frac{AD}{Q} + cD(1 + \beta - \theta\beta) + h\left(\frac{Q}{2} + \bar{s}(r) - \mu + r - \beta r + \theta\beta r\right) \\
&+ (\frac{sD}{Q})\bar{s}(r) + \left(\frac{oD}{Q}\right)\bar{b}(r)
\end{aligned} \tag{5}$$

$C(r,Q)$ average annual cost is convex when condition (E.8) is satisfied. The proof is presented in Appendix E. To minimise $C(r,Q)$, we first obtain its partial derivatives with respect to r and Q.

$$\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{oD\bar{b}(r)}{Q^2} - \frac{sD\bar{s}(r)}{Q^2} = 0 \tag{6}$$

$$\frac{\partial C}{\partial r} = h\left(\frac{\partial \bar{s}(r)}{\partial r} + 1 - \beta + \theta\beta\right) + \frac{D}{Q}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right) = 0 \tag{7}$$

For a given r, Eq. (6) yields the optimal Q value, as shown in Eq. (8) below.

$$Q = \sqrt{\frac{2D(A + o\bar{b}(r) + s\bar{s}(r))}{h}} \tag{8}$$

For a given r, Eq. (7) yields the complementary cumulative distribution of X evaluated at the optimal r, as shown in the equation below.

$$1 - F(\theta r) = \frac{hQ(1 - \beta + \theta\beta)}{\theta(hQ(1 - \alpha) + Ds(1 - \alpha) - \alpha oD)} \tag{9}$$

Based on Eq. (9), the optimal value of r can be obtained as follows:

$$r = \frac{F^{-1}\left(1 - \frac{hQ(1-\beta+\theta\beta)}{\theta(hQ(1-\alpha)+Ds(1-\alpha)+\alpha oD}\right)}{\theta} \tag{10}$$

Eqs (8 & 10) minimise $C(r, Q)$, thus being the optimal inventory policy in this situation.

Situation 2: θ and β are both stochastic with standard uniform distributions. Being a random variable, θ has an upper bound U_θ, a lower bound L_θ, a probability distribution function $g(\theta)$, and an expected average μ_θ. Similarly, β has an expected average μ_β. In accordance with the fact that some excess demands are backordered, and some are lost, $\bar{b}(r)$ and $\bar{s}(r)$ can be calculated using the equations below.

$$\bar{b}(r) = \int_{L_\theta}^{U_\theta} \int_{\theta r}^{\infty} \alpha(X - \theta r)f(X)g(\theta)dXd\theta \tag{11}$$

$$\bar{s}(r) = \int_{L_\theta}^{U_\theta} \int_{\theta r}^{\infty} (1 - \alpha)(X - \theta r)f(X)g(\theta)dXd\theta \tag{12}$$

Accordingly, at the end of each cycle, the expected on-hand inventory $\bar{a}(r)$ can be computed using Eq. (13). See the derivation details in Appendix C.

$$\bar{a}(r) = \bar{b}(r) + \bar{s}(r) - \mu + r - \mu_\beta r + \mu_\theta\mu_\beta r \tag{13}$$

With the expected on-hand inventory per cycle above, the average cost per cycle C_{cycle} can be obtained as follows:

$$\begin{aligned} C_{cycle} = A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{s}(r) - \mu + r - \mu_\beta r + \mu\theta\mu_\beta r\right) + o\bar{b}(r) \\ + s\bar{s}(r) + c(1 - \theta)\beta Q \end{aligned} \tag{14}$$

Based on the above average cost per cycle, the average annual cost $C(r, Q)$ can be calculated using Eq. (15) below.

$$\begin{aligned} C(r, Q) = \frac{AD}{Q} + cD(1 + \beta - \theta\beta) + h\left(\frac{Q}{2} + \bar{s}(r) - \mu + r - \mu_\beta r + \mu_\theta\mu_\beta r\right) \\ + \left(\frac{oD}{Q}\right)\bar{b}(r) + \frac{sD}{Q}\bar{s}(r) \end{aligned} \tag{15}$$

Similarly, we first obtain the partial derivatives of $C(r, Q)$ with respect to r and Q below.

$$\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{oD\bar{b}(r)}{Q^2} - \frac{sD\bar{s}(r)}{Q^2} = 0 \tag{16}$$

$$\frac{\partial C}{\partial r} = h\left(\frac{\partial s(r)}{\partial r} + 1 - \mu_\beta + \mu_\theta \mu_{beta}\right) + \frac{D}{Q}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right) = 0 \tag{17}$$

For a given r, Eq. (16) yields the optimal Q value, as shown in Eq. (18) below.

$$Q = \sqrt{\frac{2D(A + o\bar{b}(r) + s\bar{s}(r))}{h}} \tag{18}$$

For a given r, Eq. (17). yields the complementary cumulative distribution of X (Eq. (19)), evaluated at the optimal r.

$$\int_{L_\theta}^{U_\theta} \theta(1 - F(\theta r))g(\theta)d\theta = \varphi(r) = \frac{hQ(1 - \mu_\beta + \mu_\beta \mu_\theta)}{hQ(1-\alpha) + Ds(1-\alpha) + \alpha o D} \tag{19}$$

Similarly, the optimal value of r can be obtained using the below equation.

$$r = \varphi^{-1}\left(1 - \frac{hQ(1 - \mu_\beta + \mu_\theta \mu_\beta)}{hQ(1-\alpha) + Ds(1-\alpha) + \alpha o D}\right) \tag{20}$$

Eqs (18 & 20) minimise $C(r, Q)$, thus being the optimal inventory policy in this situation.

As shown in the new (r, Q) models in different situations of the three cases, the optimal order quantity Q is affected by the unit inventory holding cost, shortage (or backordering) cost, expected demand, fixed ordering cost, and expected number of shortages (or backordered products) per cycle. The unit inventory holding cost, shortage (or backordering) cost, expected demand, and fixed ordering cost are given and known, whilst the expected shortages (or backordered products) per cycle is to be calculated and is related to the reorder point r. In the mixed case, the expected shortages (or backordered products) are also affected by the percentage of backordered items α. r in all cases/situations is affected by both θ and β which we introduced to model the proportion of products available for sale and the proportion of perish or are lost misplaced items. Similarly, in the mixed case, r is also affected by α. Thus, we conclude that θ, β, and α are very important in determining the reorder point r, which, in turn, affects the optimal order quantity and total inventory cost.

3. Solution algorithm

We develop a solution algorithm to solve the new (r, Q) models for obtaining optimal order quantities and reorder points. Due to model development similarity, the algorithm is able to solve all the three models corresponding to the three cases. It has three general steps. In the first step, the expected number of shortages, reorder point, and order quantity are all initialised to 0; the other parameter values are set including A, D, h, μ (mean demand), σ (standard deviation of demand), o, s, α, θ, and β. In the second step, random new values are assigned to the initialised reorder point and order quantity, respectively. Subsequently, a new order quantity is computed based on Eq. (8) for deterministic θ and β in the mixed case (or the relevant equations in other cases) with the above newly assigned reorder point value. If the computed order quantity is the same as the above newly assigned one, the computation moves to the third step. If not a new reorder point is computed based on Eq. (10) for deterministic θ and β in the mixed case (or the relevant equations in other cases) with the computed order quantity. The second step repeats until a different newly computed reorder point is identified. In the third step, the newly computed reorder point and order quantity are output as the optimal values. While we summarise the algorithm above, the pseudo code is provided in Algorithm 1.

Algorithm 1 The solution algorithm in pseudo code

Initialise:
$\bar{b}(r_0) = 0$;
$Q_n = 0$;
$r_n = 0$;
Input A,D,h, $\mu, \delta, o, s, \alpha, \theta, \beta$:
Do:
$Q_0 = Q_n$;
$r_0 = r_n$;
$(^*)Q_n = \sqrt{\frac{2D(A+o\bar{b}(r_0)+s\bar{s}(r_0))}{h}}$,
if $(Q_n = Q_0)$ then break;
$(^{**})r_n = F^{-1}\left(1 - \frac{hQ_n(1-\beta+\theta\beta)}{\theta(hQ_n(1-\alpha)+Ds(1-\alpha)+\alpha oD)}\right)/\theta \quad$ while $\quad (r_n \neq r_0)$;
Output:
$Q^* = Q_n$;
$r^* = r_n$;

4. Numerical examples and sensitivity analysis

We carry out numerical examples to demonstrate the applicability of our models considering misplaced items for all three cases. (Because the results for the other two cases have very similar characteristics, for ease of communication without repetition, we present the results pertaining to the mixed case involving both lost-sales and backorders.) In the examples, there are several settings regarding the demand as well as θ and β. More specifically, the demand follows either normal distribution or exponential distribution; θ and β are either deterministic or uniformly distributed. In these examples, the percentage of backordered items α is 0.8; the other common parameters include the expected annual demand: 10000, mean and standard demand distribution: 300 and 40, fixed ordering cost: 70, unit-purchasing cost: 2, inventory holding cost: 0.6, and shortage/backordering cost: 1.5. The source of these parameters and values is ([11]) in which misplaced items were not considered. The solution algorithm is coded in R on an Intel(R) Core(TM) i7-8750H CPU @ 2.20 GHz 2.21 GHz and 24 GB RAM.

4.1. **Results.** When the demand follows the normal distribution, we obtain the results for both deterministic and uniformly distributed θ and β. More specifically, we calculate the optimal order quantity Q^*, reorder point r^*, and total inventory cost TC^* for two random values of θ and β: θ=0.6 and β=0.2 in the deterministic case and two random mean values of θ and β: $\mu_\theta = 0.6$, and $\mu_\beta = 0.2$ in the stochastic case. The results are provided in Table I. We also calculate the optimal order quantity, reorder point, and total inventory cost using the original model without involving misplaced items in (e.g., [11]) and the same common parameter values above. (The derivation of optimal order quantity, reorder point, and total inventory cost from the original model is provided in Appendix D.) The results are provided in Table I as well.

As shown in the table, the total inventory costs from our model when θ and β are either deterministic or stochastic are both lower than the cost obtained from the original model.

Table 1. Results when demand follows the normal distribution.

<table>
<tr><th colspan="2"></th><th>Q^*</th><th>r^*</th><th>TC^*</th><th>$\%TC$ decrease</th></tr>
<tr><td rowspan="2">Our model</td><td>$\theta = 0.6$, $\beta = 0.2$</td><td>1556</td><td>589</td><td>22678.7</td><td>11.7</td></tr>
<tr><td>$\mu_\theta = 0.6$, $\mu_\beta = 0.2$</td><td>2696</td><td>353</td><td>23243.6</td><td>9.5</td></tr>
<tr><td colspan="2">The original model without involving misplaced items</td><td>1545</td><td>362</td><td>25694.4</td><td>-</td></tr>
</table>

Compared with the cost obtained based on the original model, the decreases in the total inventory costs obtained based on our model are 11.7 % and 9.5 % (as shown in the last column) for deterministic and uniformly distributed θ and β, respectively. In view of the significant cost decrease, we stress that it is of paramount importance for practitioners to consider misplaced items when making inventory decisions. This is especially true when the percentage of misplaced items is unknown, thus being stochastic. This is because with an unknown percentage of misplaced items, higher total inventory costs might be incurred, as indicated by the lower cost decrease percentage in the last column in Table I. Our model can guide such practical inventory decisions. We also calculate the results when the demand follows the exponential distribution for both deterministic and stochastic θ and β. These results have similar characteristics to the normal distribution of the demand. Thus, for illustrative simplicity, we present the above results corresponding to the normal distribution of the demand.

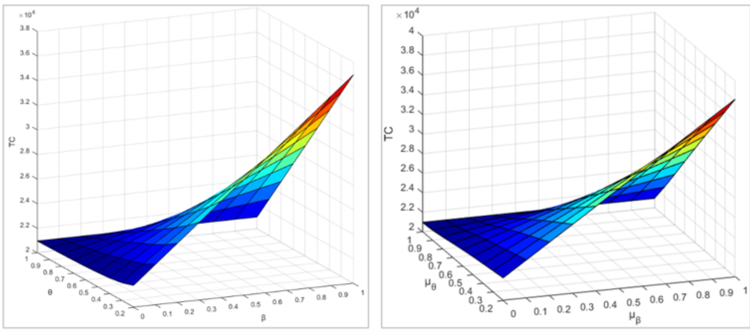
4.2. **Performance analysis of the solution algorithm.** To analyse the performance of our algorithm with respect to robustness, we test 10 times a base example, which includes the above common parameter values, the normal distribution of demand, θ (0.6), and β (0.2). We obtain 10 optimal order quantity values and reorder point values, as shown in Table II. The average, highest, and lowest optimal order quantities are 2694.5, 2703. 8, and 2677.3, respectively. The increase/decrease percentage between the highest/lowest and the average is 0.3 %/0.6 %. Regarding the reorder point, the average, highest, and lowest values are 352.8, 357.1, and 350.6, respectively. The increase/decrease percentage between the highest/lowest and the average is 1.2 %/0.6 %. We also test other base examples for the normal distribution of the demand and exponential distribution of the demand and obtain very small increase/decrease percentages of optimal order quantity and reorder point. These insignificant changes show that our solution algorithm is robust.

Table 2. Robustness analysis result of the solution algorithm.

<table>
<tr><th></th><th>T#1</th><th>T#2</th><th>T#3</th><th>T#4</th><th>T#5</th><th>T#6</th><th>T#7</th><th>T#8</th><th>T#9</th><th>T#10</th></tr>
<tr><td>Q^*</td><td>2694.8</td><td>2695.8</td><td>2692.5</td><td>2699.8</td><td>2700.7</td><td>2698.0</td><td>2687.4</td><td>2677.3</td><td>2694.3</td><td>2703.8</td></tr>
<tr><td>r^*</td><td>352.8</td><td>353.3</td><td>352.5</td><td>351.5</td><td>351.3</td><td>352.0</td><td>354.6</td><td>357.1</td><td>352.9</td><td>350.6</td></tr>
<tr><td></td><td colspan="3">Average</td><td colspan="4">Increase percentage (%)</td><td colspan="3">Decrease percentage (%)</td></tr>
<tr><td>Q^*</td><td colspan="3">2694.5</td><td colspan="4">0.3 %</td><td colspan="3">0.6 %</td></tr>
<tr><td>r^*</td><td colspan="3">352.8</td><td colspan="4">1.2 %</td><td colspan="3">0.6 %</td></tr>
</table>

4.3. **Sensitivity analysis.** We carry out various sensitivity analysis to investigate how the changes in pairs of parameters and in individual parameters affects the total inventory cost. The parameter pairs include θ and β as well as backordering cost (o) and shortage cost

(s); the individual parameters are o, s, α, and β. Based on the results, we draw managerial implications. We first carry out sensitivity analysis to examine the influence of θ and β on the total inventory cost for both types of demand distribution. More specifically, we calculate the total inventory cost for various combinations of different values (or mean values) of θ and β. (Note that all the total inventory cost values are lower than the corresponding costs obtained using the original model.) When θ and β are deterministic, θ values range from 0.2 to 1 at a step of 0.1 and β values from 0 to 1 at the same step. When θ and β are uniformly distributed, μ_θ ranges from 0.2 to 1 at a step of 0.1 and μ_β from 0 to 1 at the same step. In the analysis, we change the values (or mean values) of θ and β, but all the other parameter values are unchanged. Because of the similar patterns in the results of both normal and exponential distributions of the demand, for illustrative simplicity, Figure 3 provides the results corresponding to the normal distribution of the demand and for both deterministic and uniformly distributed θ and β.



(a) When θ and β are deterministic (b) When θ and β are stochastic

FIGURE 3. Total inventory cost changes for deterministic and uniformly distributed θ and β

As shown in the figure, higher total inventory costs are incurred when θ and β are stochastic than when they are deterministic. More specifically, for the same values of $\theta - \beta$ pairs and $\mu_\theta - \mu_\beta$ pairs, there are higher inventory costs corresponding to $\mu_\theta - \mu_\beta$ pairs. If we take a random case: $\theta = 0.7$ and $\beta = 0.3/\mu_\theta = 0.7$ and $\mu_\beta = 0.3$ as an example. The corresponding total inventory costs are 22827 and 23230, respectively. This is understandable because of uncertainties about θ and β values. The managerial implication reached based on this result is as follows. θ and β uncertainties affect the total inventory cost. As a result, to reduce inventory costs, companies should adopt suitable approaches and technologies to better track and control the misplaced and lost items in each replenishment cycle in order to obtain values of θ and β. Additionally, when θ and β are either deterministic or stochastic, we observe a general trend in total inventory cost changes, that is, the total inventory cost increases in β and decreases in θ, as shown in the figure. This trend is reasonable as θ decreases, more items are misplaced; as β increases, more misplaced items are lost.Both these trends result in reduced sales, which, in turn, lead to higher total inventory costs. As shown in the figure, when $\theta = 1/\mu_\theta = 1$ (i.e., all items are available for sale), the total inventory cost: 20964 is the lowest. At the opposite extreme: $\theta = 0.2$ and $\beta = 1/\mu_\theta = 0.2$ and $\mu_\beta = 1$ (i.e., all misplaced items are lost), the corresponding total inventory cost: 36964/38127 is the highest. These results leads to the following implication. θ and β are very important for minimising companies' total inventory costs. In light of their importance, companies should reduce β while increasing θ.

Moreover, as shown in the figure, the total cost does not change equally in θ and β. It changes quicker in β than in θ. For example, the costs are 24250 (when θ=0.6 and β=0.4) vs. 24114 (when θ=0.5 and β=0.3). In another example, the costs are 24823 (when μ_θ=0.6 and μ_β=0.4) vs. 24816 (when μ_θ=0.5 and μ_β=0.3). As shown in these examples, although the values of θ/μ_θ increase (indicating more items are available for sale), the total costs increase due to the increasing values of β/μ_β (indicating more misplaced items are lost). This result is understandable. This is because some of the misplaced items (i.e., $1-\theta$) are found later, thus becoming available for sale again. To conclude, compared with β, θ affects the total inventory cost at a lower magnitude. Regarding this, companies should develop effective solutions to reduce the number of misplaced items and to prevent them from becoming lost or perished. Additionally, we also conduct sensitivity analysis with respect to various combinations of o and s. Similarly, when we conduct the analysis, we fix the values for all the rest parameters and only change the backordering cost and shortage cost when the demand follows either the normal or exponential distribution. In the analysis, both the backordering cost and the shortage cost are changed from 0.5 to 9.5 at a step of 1. We obtain the results for both types of demand distributions. Similarly, for illustrative simplicity, we provide the results when the demand follows the normal distribution for deterministic and uniformly distributed θ and β in Figure 4. Corresponding to the results in these two figures, both the deterministic values of θ and β and the mean values of uniformly distributed θ and β are 0.6 and 0.2. (Note that the other random values (or mean values) lead to very similar patterns of the results.)

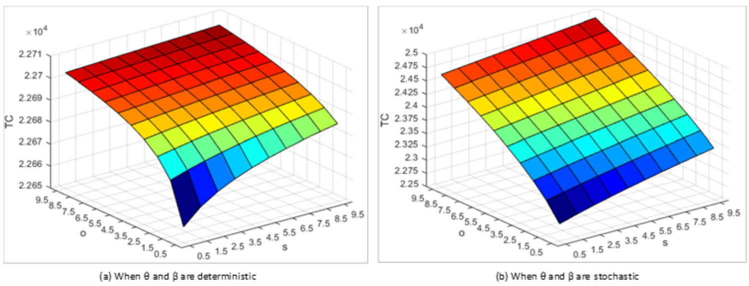


FIGURE 4. Total inventory cost changes caused by different backordering and shortage costs

As shown in the figure, the total cost change has a very similar behaviour for deterministic and stochastic θ and β. That is, the total cost increases in o and s in both situations. The difference is that the costs are higher for stochastic θ and β. In our results, the highest cost values are 22706 for deterministic θ and β and 24922 for stochastic θ and β. The corresponding o and s values (or mean values) are all 9.5. Similarly, this result is explainable in the uncertainty associated with θ and β. Regarding this, a similar managerial implication is reached: companies should aim to obtain values of θ and β so as to incur lower total inventory costs. In addition, as shown in the figure, the total cost changes quicker in o than in s in either situation because this analysis is conducted when $\alpha = 0.8$ indicating that a larger portion of excess demand is back ordered. The sensitivity analysis for different α values indicate that whether the total cost changes in these two parameters is equal, or faster in one than the other. More specifically, when $\alpha = 0.5$, the cost changes equally in two parameters; when $\alpha > 0.5/\alpha < 0.5$, the cost changes quicker/slower in o than in s. This result is understandable because when $\alpha > 0.5/\alpha < 0.5/\alpha = 0.5$ as compared with the total shortage cost, the total backordering cost plays a more/less/equal important role in the total cost. In accordance with this, the total cost changes more quickly/slowly / in o than in s or equally. These results leads to the following implication. Companies should determine suitable backordering and shortage costs to optimise the total inventory costs,regardless of whether they can predetermine the percentage of backordered excess demand. Our models and solution algorithm can facilitate such decision making. A careful examination of the results in either situation leads to the following observations. First, same cost values are incurred for two different combinations of o and s. Second, there is a pattern in these same inventory costs. We provide below some examples of same costs and the corresponding o and s combinations when θ and β are deterministic. The total cost 22676 is incurred when $o = 0.5$ and $s = 4.5$ or when $o = 1.5$ and $s = 0.5$; 22693 is incurred when $o = 3.5$ and $s = 5.5$ or when $o = 4.5$ and $s = 1.5$; 22700 is incurred when $o = 6.5$ and $s = 6.5$ or when $o = 7.5$ and $s = 2.5$; 22703 is incurred when $o = 7.5$ and $s = 9.5$ or when $o = 8.5$ and $s = 5.5$. We describe the pattern as follows: The total inventory cost for a pair of o ($o \neq 9.5$) and s ($s > 3.5$) is the same for the other pair of $o + 1$ and $s - 4$. (Note: For different ranges of the two costs and for different increment steps, the pattern will be different. i.e., the pattern depends on the ranges and increment steps.) These two observations, lead to the following implication: in practice, companies generally have sufficient flexibility to adjust the backordering cost and shortage cost based on their specific situations, and can minimise the total inventory costs.

Finally, we also conduct sensitivity analysis to examine the influence of individual parameters, including o, s, α, and β, on the total inventory cost. Similarly, when we carry out the analysis, we fix all the other parameters except the values of the parameter in consideration. The results obtained are in line with practice. More specifically, the total inventory cost increases with the increase of o or s or β, but decreases with the increase of the percentage of backordered items (i.e., α).

5. Conclusions

Due to manual operational errors, inventory misplacement is inevitable in supply chains, particularly in the retail industry. The inventory decisions which ignore misplaced items deteriorate firms' business performance by increasing supply chain costs and stock-out threshold, as pointed out in (e.g., [23]). Despite this fact, most available inventory models assume identical actual and recorded inventory records,thus ignoring inventory misplacement. In this study, therefore, we developed new (r, Q) inventory models, involving misplaced items for continuous review systems in the retail industry. In practice, some misplaced items inevitably perish or are lost, and thus, we considered this misplaced items that are perished or lost in our model development. Moreover, in our model development context, we considered three situations in which excess demands are lost, backordered, or partially backordered and partially lost, which are the possible situations in the retail industry. Involving all three possible situations increases the applicability of our study. We also proposed a solution algorithm to solve the new models and to obtain optimal order quantities and reorder points in each situations.

In our numerical examples, we considered both normal distribution and exponential distribution of demands, as well as both deterministic and stochastic θ and β. The results from various combinations of different cases and situations are consistent, and all the total inventory costs are lower than in cases, where misplaced items are unaccounted for. This has demonstrated that our theoretical (r, Q) models are valuable in guiding practitioners in inventory decisions, thus lowering inventory costs. The results from analysing the performance of the solution algorithm has highlighted the robustness of our solution algorithm.The sensitivity analysis results, reveal several managerial implications, for example, companies should obtain values of θ and β in order to minimise the total inventory costs.

Our models have demonstrated a superior capability to help practitioners with practical inventory decision making by considering misplaced items, yet these they can be further enhanced. Currently, emission abatement efforts are the focus in many aspects of supply chains. How emission control can be integrated into the models is an interesting and highly topical issue that deserves attention in the future. Furthermore, customer satisfaction is the ultimate goal of many organisations. In this regard, it is important to understand how customer satisfaction can be integrated into the models. Involving emission control and customer satisfaction may require new solution algorithms, to be developed to solve the enhanced models. Additionally, we targeted continuous review systems in this study. Future efforts might be directed to periodic review systems, taking into account the above practical issues, including inventory misplacement, emission reduction, and customer satisfaction. Furthermore, longitudinal studies could provide empirical verification of the enhanced inventory models for different review systems.

Appendix A. (r,Q) models involving misplaced items for the case of lost sales

Situation 1: θ and β are both deterministic.

In this situation, the number of shortages at the end of each cycle is $max[0, X - r\theta]$. With the interval of the shortage, $\bar{s}(r)$ is calculated as follows:

$$\bar{s}(r) = \int_{\theta r}^{\infty} (X - \theta r) f(X) dX \tag{A1}$$

At the end of each cycle, the on-hand inventory is $max[0, r\theta - X]$. With the interval of the on-hand inventory, the expected on-hand inventory: $\bar{a}(r)$ is calculated using Eq. (A2).

$$\begin{aligned}\bar{a}(r) &= \int_{0}^{\theta r} (\theta r - X) f(X) dX + (1-\beta)(1-\theta)r \\ &= \int_{0}^{\infty} (\theta r - X) f(X) dX + \int_{\theta r}^{\infty} (X - \theta r) f(X) dX + (1-\beta)(1-\theta)r \\ &= \bar{s}(r) - \mu + r - \beta r + \theta\beta r\end{aligned} \tag{A2}$$

With the expected number of shortages per cycle and the expected on-hand inventory above, C_{cycle} can be calculated as follows:

$$\begin{aligned}C_{cycle} &= A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{a}(r)\right) + s\bar{s}(r) + c(1-\theta)\beta Q \\ &= A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{s}(r) - \mu + r - \beta r + \theta\beta r\right) + s\bar{s}(r) + c(1-\theta)\beta Q\end{aligned} \tag{A3}$$

With C_{cycle}, the average annual cost: $C(r,Q)$ is obtained using the equation below.

$$
\begin{aligned}
C(r,Q) &= \left(A + cD(1+\beta-\theta\beta) + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{s}(r) - \mu + r - \beta r + \theta\beta r\right) + s\bar{s}(r)\right)\frac{D}{Q} \\
&= \frac{AD}{Q} + cD(1+\beta-\theta\beta) + h\left(\frac{Q}{2} + \bar{s}(r) - \mu + r - \beta r + \theta\beta r\right) + \frac{s\bar{s}(r)D}{Q} \\
&= \frac{AD}{Q} + cD(1+\beta-\theta\beta) + h\left(\frac{Q}{2} - \mu + r - \beta r + \theta\beta r\right) + \left(h + \frac{sD}{Q}\right)\bar{s}(r) \qquad \text{(A4)}
\end{aligned}
$$

To minimise $C(r,Q)$, we first obtain its partial derivatives with respect to r and Q as follows:

$$
\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{sD\bar{s}(r)}{Q^2} = 0 \qquad \text{(A5)}
$$

$$
\frac{\partial C}{\partial r} = h(1-\beta+\theta\beta) + \left(h + \frac{sD}{Q}\right)\frac{\partial \bar{s}(r)}{\partial r} = 0 \qquad \text{(A6)}
$$

For a given r, Eq. (A5) yields the optimal Q value (in Eq. (A7)), and Eq. (A6) yields the complementary cumulative distribution of X (in Eq. (A8)), evaluated at the optimal r.

$$
Q = \sqrt{\frac{2D(A + s\bar{s}(r))}{h}} \qquad \text{(A7)}
$$

$$
1 - F(\theta r) = \frac{hQ(1-\beta+\theta\beta)}{\theta(hQ + sD)} \qquad \text{(A8)}
$$

With Eq. (A8), the optimal value of r can be obtained using the below equation.

$$
r = \frac{F^{-1}\left(1 - \frac{hQ(1-\beta+\theta\beta)}{\theta(hQ+sD)}\right)}{\theta} \qquad \text{(A9)}
$$

Eqs (A7 & A9) minimises $C(r,Q)$, thus being the optimal inventory policy in this situation.

Situation 2: θ and β are both stochastic with standard uniform distributions.

With U_θ, L_θ, and $g(\theta)$, the expected number of shortages per cycle: $\bar{s}(r)$ is calculated as follows:

$$
\bar{s}(r) = \int_{L_\theta}^{U_\theta}\int_{\theta r}^{\infty}(X - \theta r)f(X)g(\theta)dXd\theta \qquad \text{(A10)}
$$

Accordingly, at the end of each cycle, the expected on-hand inventory: $\bar{a}(r)$ can be computed using the equation below.

$$
\begin{aligned}
\bar{a}(r) &= \int_{L_\theta}^{U_\theta}\int_{0}^{\theta r}(\theta r - X)f(X)g(\theta)dXd\theta + (1-\mu_\beta)(1-\mu_\theta)r \\
&= \int_{L_\theta}^{U_\theta}\int_{0}^{\infty}(\theta r - X)f(X)g(\theta)dXd\theta + \int_{L_\theta}^{U_\theta}\int_{\theta r}^{\infty}(X - \theta r)f(X)g(\theta)dXd\theta \\
&+ (1-\mu_\beta)(1-\mu_\theta)r \\
&= \bar{s}(r) - \mu + r - \mu_\beta r + \mu_\theta\mu_\beta r \qquad \text{(A11)}
\end{aligned}
$$

With the expected on-hand inventory above, C_{cycle} can be obtained as follows:

$$C_{cycle} = A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + r - \mu + \bar{s}(r) - \mu_\beta r + \mu_\theta\mu_\beta r\right) + s\bar{s}(r) + c(1-\theta)\beta Q \tag{A12}$$

Based on the above average cost per cycle, the average annual cost: $C(r, Q)$ can be calculated using Eq. (A12) below.

$$C(r,Q) = \frac{AD}{Q} + cD(1+\beta-\theta\beta) + h\left(\frac{Q}{2} + r - \mu - \mu_\beta r + \mu_\theta\mu_\beta r\right) + \left(h + \frac{sD}{Q}\right)\bar{s}(r) \tag{A13}$$

Similarly, we first obtain the partial derivatives of $C(r, Q)$ with respect to r and Q.

$$\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{sD}{Q^2}\bar{s}(r) = 0 \tag{A14}$$

$$\frac{\partial C}{\partial r} = h(1 - \mu_\beta + \mu_\theta\mu_\beta) + \left(h + \frac{sD}{Q}\right)\frac{\partial\bar{s}(r)}{\partial r} = 0 \tag{A15}$$

For a given r, Eq. (A14) yields the optimal Q value (in Eq. (A16)), and Eq. (A15) yields the complementary cumulative distribution of X (in Eq. (A17)), evaluated at the optimal r.

$$Q = \sqrt{\frac{2D(A + s\bar{s}(r))}{h}} \tag{A16}$$

$$\int_{L_\theta}^{U_\theta} \theta(1 - F(\theta r))g(\theta)d\theta = \varphi(r) = \frac{hQ(1 - \mu_\beta + \mu_\beta\mu_\theta)}{hQ + sD} \tag{A17}$$

Similarly, based on Eq. (A17), the optimal value of r can be obtained as follows.

$$r = \varphi^{-1}\left(\frac{hQ(1 - \mu_\beta + \mu_\beta\mu_\theta)}{hQ + sD}\right) \tag{A18}$$

Eqs (A16 & A18) minimises $C(r, Q)$, thus being the optimal inventory policy in this situation.

Appendix B. (r, Q) models involving misplaced items for the case of backorders

When the system is out of stock, the demands are backordered at the unit backordering cost o.

Situation 1: θ and β are both deterministic.

In this situation, the number of backordered products at the end of each cycle is $max[0, X - r\theta]$. $\bar{b}(r)$ can be computed as follows:

$$\bar{b}(r) = \int_{\theta r}^{\infty} (X - \theta r)f(X)dX \tag{B1}$$

At the end of each cycle, $\bar{a}(r)$ is calculated using Eq. (B2) below.

$$\begin{aligned}\bar{a}(r) &= \int_0^{\theta r} (\theta r - X) f(X) dX + (1-\beta)(1-\theta) r \\ &= \int_0^{\infty} (\theta r - X) f(X) dX + \int_{\theta r}^{\infty} (X - \theta r) f(X) dX \\ &+ (1-\beta)(1-\theta) r \\ &= \bar{b}(r) - \mu + r - \beta r + \theta\beta r \end{aligned} \tag{B2}$$

Subsequently, C_{cycle} and $C(r, Q)$ can be obtained as follows:

$$C_{cycle} = A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + r - \mu - \beta r + \theta\beta r\right) + o\bar{b}(r) + c(1-\theta)\beta Q \tag{B3}$$

$$C(r, Q) = \frac{AD}{Q} + cD(1 + \beta - \theta\beta) + h\left(\frac{Q}{2} + r - \mu - \beta r + \theta\beta r\right) + \left(\frac{oD}{Q}\right)\bar{b}(r) \tag{B4}$$

Similarly, to minimise $C(r, Q)$, we obtain its partial derivatives with respect to r and Q.

$$\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{oD\bar{b}(r)}{Q^2} = 0 \tag{B5}$$

$$\frac{\partial C}{\partial r} = h(1 - \beta + \theta\beta) + \left(\frac{oD}{Q}\right)\frac{\partial \bar{b}(r)}{\partial r} = 0 \tag{B6}$$

For a given r, Eq. (B5) yields the optimal Q value (in Eq. (B7)), and Eq. (B6) yields the complementary cumulative distribution of X (in Eq. (B8)), evaluated at the optimal r.

$$Q = \sqrt{\frac{2D(A + o\bar{b}(r))}{h}} \tag{B7}$$

$$1 - F(\theta r) = \frac{hQ(1 - \beta + \beta\theta)}{\theta o D} \tag{B8}$$

With Eq. (B.8), the optimal value of r can be obtained using the below equation.

$$r = \frac{F^{-1}\left(\frac{1 - hQ(1-\beta+\beta\theta)}{\theta o D}\right)}{\theta} \tag{B9}$$

Eqs (B.7 & B.9) minimise $C(r, Q)$, thus being the optimal inventory policy in this situation.

Situation 2: θ and β are both stochastic with standard uniform distributions.

With U_θ, L_θ, and $g(\theta), \bar{b}(r)$ can be calculated in a similar way as in the lost-sales case.

$$\bar{b}(r) = \int_{L_\theta}^{U_\theta} \int_{\theta r}^{\infty} (X - \theta r) f(X) g(\theta) dX d\theta \tag{B10}$$

Accordingly, at the end of each cycle, $\bar{a}(r)$ is computed using the equation below.

$$\begin{aligned}\bar{a}(r) &= \int_{L_\theta}^{U_\theta}\int_0^{\theta r}(\theta r - X)f(X)g(\theta)dXd\theta + (1-\mu_\beta)(1-\mu_\theta)r \\ &= \int_{L_\theta}^{U_\theta}\int_0^{\infty}(\theta r - X)f(X)g(\theta)dXd\theta + \int_{L_\theta}^{U_\theta}\int_{\theta r}^{\infty}(X-\theta r)f(X)g(\theta)dXd\theta \\ &+ (1-\mu_\beta)(1-\mu_\theta)r \\ &= \bar{b}(r) - \mu + r - \mu_\beta r + \mu_\theta\mu_\beta r \end{aligned} \tag{B11}$$

Further, C_{cycle} and $C(r, Q)$ are calculated as follows:

$$C_{cycle} = A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + r - \mu - \mu_\beta r + \mu\theta\mu_\beta r\right) + o\bar{b}(r) + c(1-\theta)\beta Q \tag{B12}$$

$$\begin{aligned}C(r,Q) &= \frac{AD}{Q} + cD(1+\beta-\theta\beta) + h\left(\frac{Q}{2} + r - \mu - \mu_\beta r + \mu_\theta\mu_\beta r\right) \\ &+ \left(\frac{oD}{Q}\right)\bar{b}(r)\end{aligned} \tag{B13}$$

Similarly, we first derive the partial derivatives of $C(r, Q)$ with respect to r and Q.

$$\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{oD\bar{b}(r)}{Q^2} = 0 \tag{B14}$$

$$\frac{\partial C}{\partial r} = h(1-\mu_\beta+\mu_\theta\mu_\beta) + \frac{oD}{Q}\frac{\partial\bar{b}(r)}{\partial r} = 0 \tag{B15}$$

For a given r, Eq. (B14) yields the optimal Q value (in Eq. (B16)), and Eq. (B15) yields the complementary cumulative distribution of X (in Eq. (B17)), evaluated at the optimal r.

$$Q = \sqrt{\frac{2D(A+o\bar{b}(r))}{h}} \tag{B16}$$

$$\int_{L_\theta}^{U_\theta}\theta(1-F(\theta r))g(\theta)d\theta = \varphi(r) = \frac{hQ(1-\mu_\beta+\mu_\beta\mu_\theta)}{oD} \tag{B17}$$

With Eq. (B17), the optimal value of r can be obtained using the below equation.

$$r = \varphi^{-1}\left(\frac{hQ(1-\mu_\beta+\mu_\beta\mu_\theta)}{oD}\right) \tag{B18}$$

Eqs (B16 & B18) minimise $C(r, Q)$, thus being the optimal inventory policy in this situation.

Appendix C. Derivation of Eq. (13)

$$
\begin{aligned}
\bar{a}(r) &= \int_{L_\theta}^{U_\theta}\int_{0}^{\theta r}(\theta r - X)f(X)g(\theta)dXd\theta + (1-\mu_\beta)(1-\mu_\theta)r \\
&= \int_{L_\theta}^{U_\theta}\int_{0}^{\infty}(\theta r - X)f(X)g(\theta)dXd\theta \\
&+ \int_{L_\theta}^{U_\theta}\int_{\theta r}^{\infty}((1-\alpha)+\alpha)(X-\theta r)f(X)g(\theta)dXd\theta + (1-\mu_\beta)(1-\mu_\theta)r \\
&= \int_{L_\theta}^{U_\theta}\int_{0}^{\infty}(\theta r - X)f(X)g(\theta)dXd\theta \\
&+ \int_{L_\theta}^{U_\theta}\int_{\theta r}^{\infty}(1-\alpha)(X-\theta r)f(X)g(\theta)dXd\theta \\
&+ \int_{L_\theta}^{U_\theta}\int_{\theta r}^{\infty}\alpha(X-\theta r)f(X)g(\theta)dXd\theta + (1-\mu_\beta)(1-\mu_\theta)r \\
&= \bar{b}(r) + \bar{s}(r) - \mu + r - \mu_\beta r + \mu_\theta\mu_\beta r
\end{aligned}
\tag{C1}
$$

Appendix D. Derivation of total inventory cost from the original model in the mixed case

When unsatisfied demands are all lost, the number of shortages at the end of each cycle is $max[0, X-r]$. When partial unsatisfied demands are lost at a percentage $(1-\alpha)$, the number of shortages at the end of each cycle is $max[0,(1-\alpha)(X-r)]$. With the interval of shortages, the expected number of shortages per cycle $\bar{s}(r)$ is calculated as follows:

$$\bar{s}(r) = \int_{r}^{\infty}(1-\alpha)(X-r)f(X)dX \tag{D1}$$

When all unsatisfied demands are backordered, the number of backordered products at the end of each cycle is $max[0, X-r]$. When these demands are partially backordered at a percentage α at a unit backordering cost o. The expected number of backordered products per cycle $\bar{b}(r)$ can be computed as follows:

$$\bar{b}(r) = \int_{r}^{\infty}\alpha(X-r)f(X)dX \tag{D2}$$

At the end of each cycle, the on-hand inventory is $max[0, r-X]$. With the interval of the on-hand inventory, the expected on-hand inventory $\bar{a}(r)$ can be obtained using Eq. (D3).

$$\bar{a}(r) = \bar{s}(r) - \mu - r \tag{D3}$$

With the expected number of backordered and shortage products and the expected on-hand inventory per cycle above, the initial average cost per cycle iC_{cycle} can be computed as follows:

$$iC_{cycle} = A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{s}(r) + r - \mu\right) + o\bar{b}(r) + s\bar{s}(r) \tag{D4}$$

Based on the above average cost per cycle, the initial average annual cost iC_{cycle} can be calculated using Eq. (D5) below.

$$iC(r,Q) = \frac{AD}{Q} + cD + h\left(\frac{Q}{2} + \bar{s}(r) + r - \mu\right) + \left(\frac{oD}{Q}\right)\bar{b}(r) + \left(\frac{sD}{Q}\right)\bar{s}(r) \tag{D5}$$

To minimise $iC(r, Q)$, we first obtain its partial derivatives with respect to r and Q.

$$\frac{\partial iC}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{oD\bar{b}(r)}{Q^2} - \frac{sD\bar{s}(r)}{Q^2} = 0 \tag{D6}$$

$$\frac{\partial iC}{\partial r} = h\left(\frac{\partial \bar{s}(r)}{\partial r} + 1\right) + \frac{D}{Q}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right) = 0 \tag{D7}$$

For a given r, Eq. (D6) yields the optimal Q value, as shown in Eq. (D8) below.

$$Q = \sqrt{\frac{2D(A + o\bar{b}(r) + s\bar{s}(r))}{h}} \tag{D8}$$

For a given r, Eq. (D7) yields the complementary cumulative distribution of X evaluated at the optimal r, as shown in the equation below.

$$1 - F(r) = \frac{hQ}{(hQ1-\alpha) + Ds(1-\alpha) - \alpha oD)} \tag{D9}$$

Based on Eq. (D9), the optimal value of r can be obtained as follows:

$$r = F^{-1}\left(1 - \frac{hQ}{(hQ1-\alpha) + Ds(1-\alpha) - \alpha oD)}\right) \tag{D10}$$

Eqs (D8 & D10) minimise the $iC(r, Q)$, thus being the optimal inventory policy in the mixed case when misplaced items are not considered.

At the end of the year, upon realising that they had misplaced items and some of them were lost, a company has to pay an additional cost for the misplaced and lost items. With the additional cost, the final average cost per cycle C_{cycle} and the final average annual cost $C(r, Q)$ are computed as follows:

$$C_{cycle} = A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{s}(r) + r - \mu\right) + o\bar{b}(r) + s\bar{s}(r) + c\theta\mu + cQ\theta\beta \tag{D11}$$

$$\begin{aligned} C(r,Q) = \Bigg(&A + cQ + h\frac{Q}{D}\left(\frac{Q}{2} + \bar{s}(r) - \mu + r\right) + s\bar{s}(r) + o\bar{b}(r) \\ &+ c\theta\mu + cQ\theta\beta\Bigg)\frac{D}{Q} \end{aligned} \tag{D12}$$

Appendix E. Proof of convexity for total inventory cost

Situation 1: θ and β are both deterministic.

$$\begin{aligned} C(r,Q) = &\frac{AD}{Q} + cD(1+\beta-\theta\beta) + h\left(\frac{Q}{2} + \bar{s}(r) + r - \mu - \beta r + \theta\beta r\right) \\ &+ \left(\frac{sD}{Q}\right)\bar{s}(r) + \left(\frac{oD}{Q}\right)\bar{b}(r) \end{aligned} \tag{E1}$$

Taking the first and second partial derivatives of $C(r,Q)$ with respect to r and Q:

$$\frac{\partial C}{\partial Q} = -\frac{AD}{Q^2} + \frac{h}{2} - \frac{oD\bar{b}(r)}{Q^2} - \frac{sD\bar{s}(r)}{Q^2} = 0 \tag{E2}$$

$$\frac{\partial^2 C}{\partial Q^2} = \frac{2AD}{Q^3} + \frac{2oD\bar{b}(r)}{Q^3} + \frac{2sD\bar{s}(r)}{Q^3} > 0 \tag{E3}$$

$$\frac{\partial C}{\partial r} = h\left(\frac{\partial \bar{s}(r)}{\partial r} + 1 - \beta + \theta\beta\right) + \frac{D}{Q}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right) \tag{E4}$$

$$\frac{\partial^2 C}{\partial r^2} = h\left(\frac{\partial^2 \bar{s}(r)}{\partial r^2}\right) + \frac{D}{Q}\left(s\frac{\partial^2 \bar{s}(r)}{\partial r^2} + o\frac{\partial^2 \bar{b}(r)}{\partial r^2}\right) > 0 \tag{E5}$$

and

$$\frac{\partial^2 C}{\partial r \partial Q} = \frac{\partial^2 C}{\partial Q \partial r} = -\frac{D}{Q^2}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right). \tag{E6}$$

With those equations, we obtain the determinant

$$\begin{vmatrix} \frac{\partial^2 C}{\partial Q^2} & \frac{\partial^2 C}{\partial Q \partial r} \\ \frac{\partial^2 C}{\partial r \partial Q} & \frac{\partial^2 C}{\partial r^2} \end{vmatrix} = -\left(-\frac{D}{Q^2}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right)\right)^2$$
$$+\left(\frac{2AD}{Q^3} + \frac{2oD\bar{b}(r)}{Q^3} + \frac{2sD\bar{s}(r)}{Q^3}\right)\left(h\left(\frac{\partial^2 \bar{s}(r)}{\partial r^2}\right) + \frac{D}{Q}\left(s\frac{\partial^2 \bar{s}(r)}{\partial r^2} + o\frac{\partial^2 \bar{b}(r)}{\partial r^2}\right)\right) \tag{E7}$$

This term is non-negative when

$$2(A + o\bar{b}(r) + s\bar{s}(r))\left(h\left(\frac{\partial^2 \bar{s}(r)}{\partial r^2}\right) + \frac{D}{Q}\left(s\frac{\partial^2 \bar{s}(r)}{\partial r^2} + o\frac{\partial^2 \bar{b}(r)}{\partial r^2}\right)\right)$$
$$\geq \frac{D}{Q}\left(s\frac{\partial \bar{s}(r)}{\partial r} + o\frac{\partial \bar{b}(r)}{\partial r}\right)^2 \tag{E8}$$

With $\bar{s}(r) \geq 0$, $\bar{b}(r) \geq 0$, and all positive parameters. Therefore, $C(r,Q)$ is convex function when above condition is satisfied.

Situation 2: θ and β are both stochastic with standard uniform distributions

$$C(r,Q) = \frac{AD}{Q} + cD(1 + \mu_\beta - \mu_\theta\mu_\beta) + h\left(\frac{Q}{2} + \bar{s}(r) + r - \mu - \mu_\beta r + \mu_\theta\mu_\beta r\right)$$
$$+\left(\frac{sD}{Q}\right)\bar{s}(r) + \left(\frac{oD}{Q}\right)\bar{b}(r) \tag{E9}$$

Taking derivatives gives the same expressions, therefore $C(r,Q)$ is convex function when above condition is satisfied.

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