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MAPPING PROPERTIES OF HOLOMORPHIC FUNCTION ASSOCIATED WITH GAUSSIAN HYPERGEOMETRIC FUNCTION

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ABSTRACT. This paper aims to present the associated results of holomorphic function $J(z) := J_{\mu,\delta}(p, q; r; z)$ defined on open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, belongs to $\varphi^*(A, B)$ and $\mathcal{K}(A, B)$. This work also consider an integral operator $I(z)$ associated with the hypergeometric functions and identified the necessary and sufficient condition for $I(z)$ belongs to $\varphi^*(A, B)$ as well as $\mathcal{K}(A, B)$.

Keywords: holomorphic function, starlike function, convex function, hypergeometric function.

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1. INTRODUCTION

Complex analysis has many real-life applications including dynamic systems and particle physics. A specific group of complex functions that can be differentiated anywhere in the open subset of a complex plane is known as a holomorphic function. The proof of the Bieberbach conjecture given in 1985 by De Branges [3] involved the use of the holomorphic function. This gave rise to research in the holomorphic functions and their applications to solve 2D problems in physics has received attention from the researchers. In this work, we present some results on associated identities of holomorphic function described in terms of Gaussian hypergeometric function.

The collection of all the holomorphic functions given in the following equation (1) and defined in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ is denoted by $\mathcal{A}$.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

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The subclass of $\mathcal{A}$ consisting of functions of the form (1) which are univalent in U is denoted by S. It is well known that $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ respectively denotes the class of starlike functions of order α and the class of convex functions of order α in U respectively. The subclasses of analytic functions were studied by many researchers (for more details see [1]).

Let T be the subclass of all functions $f(z)$, with non-positive coefficients, in $\mathcal{A}$ of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0). \tag{2}$$

The classes obtained by taking intersections of the classes $S^*(\alpha)$ and $\mathcal{K}(\alpha)$ with T are denoted by $T^*(\alpha)$ and $\mathcal{K}^*(\alpha)$ respectively (for more details see [17]).

Let $H = \{W : W \text{ is analytic}, W(0) = 0, |W(z)| < 1 \text{ in } U\}$ and $\mathcal{P}(A, B)$ denote the class of analytic functions $\mathcal{P}(z)$ in U which are of the form $\mathcal{P}(z) = \dfrac{1 + AW(z)}{1 + BW(z)}$, $W \in H$ and $-1 \leq A < B \leq 1$. The class $\mathcal{P}(A, B)$ was first introduced and studied by Janowski [8].

Define,

$$\varphi^*(A, B) = \left\{ f \in \mathcal{A} : \frac{zf'}{f} \in \mathcal{P}(A, B) \right\}$$

and

$$\mathcal{K}(A, B) = \{ f \in \mathcal{A} : zf' \in \varphi^*(A, B) \}.$$

We observe that

$$\begin{array}{ll} \varphi^*((2\alpha-1)\beta, \beta) \equiv S^*(\alpha, \beta), & \varphi^*((2\alpha-1), 1) \equiv S^*(\alpha), \\ \varphi^*(\alpha, 1) \equiv S^*(\alpha), & \varphi^*(-1, 1) \equiv S^*, \\ \mathcal{K}((2\alpha-1)\beta, \beta) \equiv \mathcal{K}(\alpha, \beta), & \mathcal{K}((2\alpha-1), 1) \equiv \mathcal{K}(\alpha), \\ \mathcal{K}(\alpha, 1) \equiv \mathcal{K}(\alpha) \quad \text{and} & \mathcal{K}(-1, 1) \equiv \mathcal{K}. \end{array}$$

Let $F(p,\ q;\ r;\ z)$ be the Gaussian hypergeometric function defined by

$$F(p, q; r; z) = \sum_{n=0}^{\infty} \frac{(p)_n (q)_n}{(r)_n (1)_n} z^n \quad (z \in U), \tag{3}$$

where, p, q and r be complex numbers with $r \neq 0, -1, -2, \ldots$ and $(p)_n$ is the Pochhammer symbol defined by

$$(p)_n = \begin{cases} 1 & , \text{ if n} = 0, \\ p(p+1)(p+2)\ldots(p+n-1) & , \text{ if n} = 1, 2, 3, \ldots \ . \end{cases}$$

This function is analytic in the unit disc U. We note that for $z = 1$ the equation (3) becomes $F(p, q; r; 1)$ which converges for $\Re(r-p-q) > 0$ and is related to gamma function by

$$F(p, q; r; 1) = \frac{\Gamma(r)\ \Gamma(r-p-q)}{\Gamma(r-p)\ \Gamma(r-q)} \ . \tag{4}$$

Many researchers have studied and investigated the mapping properties of various subclasses of analytic functions associated with hypergeometric functions. Shukla and Shukla [16], Merkes and Scott [12], Ruscheweyh and Singh [15], Joshi *et al* [9], Kwon and Cho [11], Silverman [18], Cho *et al* [5] and Aouf *et al* [2] studied the mapping properties of

$$g(z) := g(p, q; r; z) = zF(p, q; r; z) \tag{5}$$

with the help of elementary results of starlike and convex functions. In 1987, Owa and Srivastava [14] studied the mapping properties of generalized hypergeometric function. In 1984, Carlson and Shaffer [4] investigated certain conditions on starlike and prestarlike hypergeometric functions. Miller and Mocanu [13] obtained the univalency of some hypergeometric function with the help of the method of differential subordination in 1990.

The mapping properties of function

$$h(z) := h_\mu(p, q; r; z) = (1 - \mu)g(z) + \mu z g'(z) \tag{6}$$

were studied by Shukla and Shukla [16] and Aouf *et al* [2], where $z \in U$, $\mu \geq 0$ and $g(z)$ is defined in (5).

Kim and Shon [10] studied the mapping properties for convolution of $h(z)$ and $f(z)$ involving hypergeometric function.

The enormous role played by hypergeometric functions in the proof of the Bieberbach conjecture which is proved by Branges [3] in 1985 has sparked new research interest into the characteristics of these functions.

Consider,

$$J(z) := J_{\mu,\delta}(p, q; r; z) = (1 - \mu + \delta)[g(z)] + (\mu - \delta)z[g(z)]' + \mu\delta z^2[g(z)]'', \tag{7}$$

where $z \in U$; $\mu, \delta \geq 0$; $\mu \geq \delta$ and $g(z)$ is defined in equation (5).
In 2016, some constraints of hypergeometric function $J(z)$ belonged to certain subclasses of analytic functions were studied by Aouf *et al* [2].

In 2019, Damodaran and Keerthi [6] proved some properties of hypergeometric functions of certain subclasses of Sakaguchi Type Functions.

Motivated by aforementioned work of various authors, we examined the mapping characteristics of the function $J(z)$ defined by (7) in the present study. The lemmas attributed to Goel and Sohi, cited in Section 2, at $p = 1$ are actually our primary instruments in the analysis of mapping properties of $J(z)$. The necessary condition for $J(z)$ to be in $\varphi^*(A, B)$ and $\mathcal{K}(A, B)$ is first discovered in Section 3. Further with the proper constraints on p, q, and r, we proved the necessary and sufficient conditions for $J(z)$ to be in $\varphi^*(A, B)$ and $\mathcal{K}(A, B)$. We also explored the mapping properties of the integral operator of the form

$$I(z) := I(p, q; r; z) = \int_0^z \frac{J(t)}{t} dt \; . \tag{8}$$

Furthermore, we have obtained sufficient conditions for $I(z)$ to be in $\varphi^*(A, B)$ and $\mathcal{K}(A, B)$ and also determined the necessary and sufficient condition for $I(z)$ to be in $\varphi^*(A, B)$ and $\mathcal{K}(A, B)$. Our results generalize the corresponding results of Shukla and Shukla [16] and Silverman [18]. Conclusion of the paper is given in Section 4.

2. Preliminary Lemmas

Lemma 2.1 and Lemma 2.2 are directly considered from the work of Goel and Sohi [7] by $\mathfrak{p} = 1$, and Lemma 2.3 is proved for current work. These three lemmas are then used to prove the results in current work.

Lemma 2.1. *(i) A sufficient condition for $f(z)$ given by (1) to be in $\varphi^*(A, B)$ is that*

$$\sum_{n=2}^{\infty} \{(1 + B)n - (A + 1)\}|a_n| \leq (B - A).$$

(ii) A sufficient condition for $f(z)$ given by (1) to be in $\mathcal{K}(A,B)$ is that

$$\sum_{n=2}^{\infty} n\{(1+B)n-(A+1)\}|a_n| \le (B-A).$$

Proof. The proof of Lemma 2.1 is straightforward hence not elaborated here. □

Lemma 2.2. *(i) A necessary and sufficient condition for $f(z)$ given by (2) to be in $\varphi^*(A,B)$, is that*

$$\sum_{n=2}^{\infty}\{(1+B)n-(A+1)\}|a_n| \le (B-A).$$

(ii) A necessary and sufficient condition for $f(z)$ given by (2) to be in $\mathcal{K}(A,B)$, is that

$$\sum_{n=2}^{\infty} n\{(1+B)n-(A+1)\}|a_n| \le (B-A).$$

Proof. The proof of Lemma 2.2 is straightforward and hence not elaborated here. □

Lemma 2.3. *If $I(z)$ and $J(z)$ defined by (8) and (7) respectively, then $I(z)\in\mathcal{K}(A,B)$ if and only if $J(z)\in\varphi^*(A,B)$.*

Proof. We observe that

$$I' = \frac{J}{z}\,, \quad I'' = \frac{zJ'-J}{z^2} \;\; \text{and so} \;\; 1+z\,\frac{I''}{I'} = \frac{zJ'}{J}.$$

Hence any starlikeness property for $J(z)$ will result into the convex property for $I(z)$. □

3. Main Results

Theorem 3.1. *Let $p,\ q>0$ and $r>p+q+3$, then a sufficient condition for $J(z)$ to be in $\varphi^*(A,B)$, $-1\le A<B\le 1$, is that*

$$\frac{\Gamma r\ \Gamma(r-p-q)}{\Gamma(r-p)\ \Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p)_3(q)_3}{(B-A)(r-p-q-3)_3} + \frac{[(1+B)(5\mu\delta+\mu-\delta)-(A+1)\mu\delta](p)_2(q)_2}{(B-A)(r-p-q-2)_2} + \frac{[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)]pq}{(B-A)(r-p-q-1)}+1\right\}\le 2\,. \tag{9}$$

Proof. Since

$$J(z)=J_{\mu,\delta}(p,q;r;z)=z+\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}\ z^n\ , \quad (z\in U).$$

According to (i) of Lemma 2.1, we need to show that

$$T_1 := \sum_{n=2}^{\infty}[(1+B)n-(A+1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}} \le (B-A).$$

Now,

$$T_1 = \sum_{n=2}^{\infty}\Big\{n^2(n-1)\mu\delta(1+B)+n(n-1)[(1+B)(\mu-\delta)-\mu\delta(A+1)] + n(1+B)-(n-1)(A+1)(\mu-\delta)-(A+1)\Big\}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}.$$

Note that $(p)_n = p(p+1)_{n-1}$, $(1)_n = n(1)_{n-1}$ and applying (4), we have

$$\begin{aligned} T_1 &= \mu\delta(1+B)\sum_{n=2}^{\infty} n^2 \frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} + [(1+B)(\mu-\delta)-\mu\delta(A+1)]\sum_{n=2}^{\infty} n\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} \\ &+(1+B)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} + (1+B)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}} \\ &-(A+1)(\mu-\delta)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} - (A+1)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}. \end{aligned}$$

We write $n^2 = (n-2)^2 + 4(n-2) + 4$, $\ n = (n-2) + 2$ and simplifying we get

$$\begin{aligned} T_1 &= \mu\delta(1+B)\frac{(p)_3(q)_3}{(r)_3}\sum_{n=0}^{\infty}\frac{(p+3)_n\ (q+3)_n}{(r+3)_n\ (1)_n} \\ &\quad +[(1+B)(5\mu\delta+\mu-\delta)-\mu\delta(A+1)]\frac{(p)_2(q)_2}{(r)_2}\sum_{n=0}^{\infty}\frac{(p+2)_n\ (q+2)_n}{(r+2)_n\ (1)_n} \\ &\quad +[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)]\frac{pq}{r} \\ &\quad \sum_{n=0}^{\infty}\frac{(p+1)_n\ (q+1)_n}{(r+1)_n\ (1)_n} + (B-A)\sum_{n=0}^{\infty}\frac{(p)_n\ (q)_n}{(r)_n\ (1)_n} - (B-A) \\ &= \frac{\Gamma(r)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p)_3(q)_3}{(r-p-q-3)_3}\right. \\ &\quad +\frac{[(1+B)(5\mu\delta+\mu-\delta)-\mu\delta(A+1)](p)_2(q)_2}{(r-p-q-2)_2} \\ &\quad \left.+\frac{[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)]pq}{(r-p-q-1)} + (B-A)\right\} - (B-A). \end{aligned}$$

The last expression is bounded above by $(B-A)$ provided that (9) holds. This completes the proof of Theorem 3.1. □

Note that, the condition (9) is necessary and sufficient for $J_1(z) = z\left[2 - \frac{J(z)}{z}\right]$, that is,

$$J_1(z) = z - \sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}\ z^n \tag{10}$$

to be in $\varphi^*(A, B)$.

Theorem 3.2. *Let $p, q > -1$, $r > p+q+3$ and $pq < 0$, then a necessary and sufficient condition for $J(z)$ to be in $\varphi^*(A, B)$, is that*

$$\begin{aligned} &\mu\delta(1+B)(p)_3(q)_3 + [(1+B)(5\mu\delta+\mu-\delta)-(A+1)\mu\delta](p)_2(q)_2(r-p-q-3) \\ &\quad +[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)]pq(r-p-q-3)_2 \\ &\quad +(B-A)(r-p-q-3)_3 \geq 0. \end{aligned} \tag{11}$$

Proof. Since,

$$J(z) = z - \frac{|pq|}{r}\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}}\ z^n\ ,\ (z\in U).$$

According to (i) of Lemma 2.2, we need to show that,

$$\begin{aligned} T_2 : &= \sum_{n=2}^{\infty}[(1+B)n-(A+1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}} \\ &\leq \frac{r}{|pq|}(B-A) \end{aligned}$$

Now,

$$\begin{aligned} T_2 = \sum_{n=2}^{\infty} &\{n^2(n-1)\mu\delta(1+B)+n(n-1)[(1+B)(\mu-\delta)-\mu\delta(A+1)] \\ &+n(1+B)-(n-1)(A+1)(\mu-\delta)-(A+1)\} \frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}}. \end{aligned}$$

Noting that $(p)_n = p(p+1)_{n-1}$, $(1)_n = n(1)_{n-1}$ and applying (4), we have

$$\begin{aligned} T_2 &= \mu\delta(1+B)\sum_{n=2}^{\infty} n^2 \frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} \\ &+[(1+B)(\mu-\delta)-\mu\delta(A+1)]\sum_{n=2}^{\infty} n\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} \\ &+(1+B)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} + (1+B)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}} \\ &-(A+1)(\mu-\delta)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} - (A+1)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}}. \end{aligned}$$

Simplifying we get,

$$\begin{aligned} T_2 &= \frac{\Gamma(r+1)\Gamma(r-p-q-3)}{\Gamma(r-p)\Gamma(r-q)}\{\mu\delta(1+B)(p+1)_2(q+1)_2 \\ &+[(1+B)(5\mu\delta+\mu-\delta)-\mu\delta(A+1)](p+1)(q+1)(r-p-q-3) \\ &+[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)](r-p-q-3)_2 \\ &+\left.\frac{(B-A)}{pq}(r-p-q-3)_3\right\} + (B-A)\frac{r}{|pq|}. \end{aligned}$$

But, the last expression is bounded above by $(B-A)\frac{r}{|pq|}$ if and only if condition (11) holds. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $p, q > 0$ and $r > p+q+4$, then a sufficient condition for $J(z)$ to be in $\mathcal{K}(A,B)$, $-1 \leq A < B \leq 1$, is that*

$$\begin{aligned} &\frac{\Gamma r\ \Gamma(r-p-q)}{\Gamma(r-p)\ \Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p)_4(q)_4}{(B-A)(r-p-q-4)_4}\right. \\ &\quad+\frac{[(1+B)(9\mu\delta+\mu-\delta)-\mu\delta(A+1)]\,(p)_3(q)_3}{(B-A)(r-p-q-3)_3} \\ &\quad\quad+\frac{[(1+B)(19\mu\delta+5\mu-5\delta+1)-(A+1)(5\mu\delta+\mu-\delta)](p)_2(q)_2}{(B-A)(r-p-q-2)_2} \\ &\quad\quad\left.+\frac{[(1+B)(8\mu\delta+4\mu-4\delta+3)-(A+1)(4\mu\delta+2\mu-2\delta+1)]pq}{(B-A)(r-p-q-1)}+1\right\} \leq 2. \qquad (12) \end{aligned}$$

Proof. Since

$$J(z) = z + \sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}\ z^n\ , \quad (z \in U).$$

According to (ii) of Lemma 2.1, we need to show that

$$T_3 := \sum_{n=2}^{\infty} n[(1+B)n-(A+1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}} \leq (B-A)$$

Now,

$$T_3 = \sum_{n=2}^{\infty}\{n^3(n-1)\mu\delta(1+B)+n^2(n-1)[(1+B)(\mu-\delta)-\mu\delta(A+1)]$$
$$+n^2(1+B)-n(n-1)(A+1)(\mu-\delta)-n(A+1)\}\ \frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}.$$

Noting that $(p)_n = p(p+1)_{n-1}$, $(1)_n = n(1)_{n-1}$, $n^2 = (n-1)^2+2(n-1)+1$, $n = (n-1)+1$ and applying (4), we have

$$\begin{aligned}
T_3 \quad = \quad & \mu\delta(1+B)\sum_{n=2}^{\infty} n^3\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} \\
& +[(1+B)(\mu-\delta)-\mu\delta(A+1)]\sum_{n=2}^{\infty} n^2\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} \\
& +(1+B)\sum_{n=2}^{\infty}(n-1)\frac{(p)_{n-1}\ q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} + 2(1+B)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} \\
& +(1+B)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}} - (A+1)(\mu-\delta)\sum_{n=2}^{\infty} n\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} \\
& -(A+1)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} - (A+1)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}.
\end{aligned}$$

Simplifying we get,

$$\begin{aligned}
T_3 \quad = \quad & \frac{\Gamma(r)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p)_4(q)_4}{(r-p-q-4)_4}\right. \\
& +\frac{[(1+B)(9\mu\delta+\mu-\delta)-\mu\delta(A+1)]\,(p)_3(q)_3}{(r-p-q-3)_3}+ \\
& \frac{[(1+B)(19\mu\delta+5\mu-5\delta+1)-(A+1)(5\mu\delta+\mu-\delta)](p)_2(q)_2}{(r-p-q-2)_2}+ \\
& \left.\frac{[(1+B)(8\mu\delta+4\mu-4\delta+3)-(A+1)(4\mu\delta+2\mu-2\delta+1)]pq}{(r-p-q-1)}+(B-A)\right\} \\
& -(B-A).
\end{aligned}$$

But, the last expression is bounded above by $(B-A)$, if condition (12) holds. This completes the proof of Theorem 3.3. □

Note that, the condition (12) is necessary and sufficient for $J_1(z)$ defined in equation (10) to be in $\mathcal{K}(A, B)$.

Theorem 3.4. *Let $p, q > -1$, $pq < 0$ and $r > p + q + 4$, then a necessary and sufficient condition for $J(z)$ to be in $\mathcal{K}(A, B)$, is that*

$$\begin{aligned}
&\mu\delta(1+B)(p)_4(q)_4 + [(1+B)(9\mu\delta+\mu-\delta) - \mu\delta(A+1)]\,(p)_3(q)_3(r-p-q-4) \\
&\quad + [(1+B)(19\mu\delta+5\mu-5\delta+1) - (A+1)(5\mu\delta+\mu-\delta)](p)_2(q)_2(r-p-q-4)_2 \\
&\quad + [(1+B)(8\mu\delta+4\mu-4\delta+3) - (A+1)(4\mu\delta+2\mu-2\delta+1)]pq(r-p-q-4)_3 \\
&\ + (B-A)(r-p-q-4)_4 \geq 0.
\end{aligned} \tag{13}$$

Proof. Since

$$J(z) = z - \frac{|pq|}{r}\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}}\ z^n\ , \quad (z \in U).$$

According to (ii) of Lemma 2.2, we need to show that

$$\begin{aligned}
T_4 : &= \sum_{n=2}^{\infty} n[(1+B)n-(A+1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}} \\
&\leq (B-A)\frac{r}{|pq|}.
\end{aligned}$$

Now,

$$\begin{aligned}
T_4 &= \sum_{n=2}^{\infty}\left\{n^3(n-1)\mu\delta(1+B) + n^2(n-1)[(1+B)(\mu-\delta)-\mu\delta(A+1)]\right. \\
&\quad \left. + n^2(1+B) - n(n-1)(A+1)(\mu-\delta) - n(A+1)\right\}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}}.
\end{aligned}$$

Noting that $(p)_n = p(p+1)_{n-1}$, $(1)_n = n(1)_{n-1}$, $n^2 = (n-1)^2 + 2(n-1) + 1$, $n = (n-1) + 1$ and applying (4), we have

$$\begin{aligned}
T_4 &= \mu\delta(1+B)\sum_{n=2}^{\infty} n^3 \frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} \\
&\quad + [(1+B)(\mu-\delta)-\mu\delta(A+1)]\sum_{n=2}^{\infty} n^2 \frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} \\
&\quad + (1+B)\sum_{n=2}^{\infty}(n-1)\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} \\
&\quad + 2(1+B)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} + (1+B)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}} \\
&\quad - (A+1)(\mu-\delta)\sum_{n=2}^{\infty} n \frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} \\
&\quad - (A+1)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} - (A+1)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}}.
\end{aligned}$$

Simplifying we get,

$$\begin{aligned} T_4 &= \frac{\Gamma(r+1)\Gamma(r-p-q-4)}{\Gamma(r-p)\Gamma(r-q)}\Big\{ \mu\delta(1+B)(p+1)_3(q+1)_3 \\ &+ [(1+B)(9\mu\delta+\mu-\delta)-\mu\delta(A+1)]\,(p+1)_2(q+1)_2(r-p-q-4) \\ &+[(1+B)(19\mu\delta+5\mu-5\delta+1)-(A+1)(5\mu\delta+\mu-\delta)](p+1)(q+1)(r-p-q-4)_2 \\ &+[(1+B)(8\mu\delta+4\mu-4\delta+3)-(A+1)(4\mu\delta+2\mu-2\delta+1)](r-p-q-4)_3 \\ &+\frac{(B-A)(r-p-q-4)_4}{pq}\Big\} + (B-A)\frac{r}{|pq|}. \end{aligned}$$

But, the last expression is bounded above by $(B-A)\frac{r}{|pq|}$, if and only if condition (13) holds. This completes the proof of Theorem 3.4. □

Theorem 3.5. *Let $p, q > 0$ and $r > p+q+2$, then a sufficient condition for $I(z)$ defined by (8) to be in $\varphi^*(A,B)$, $-1 \le A < B \le 1$, is that*

$$\begin{aligned} &\frac{\Gamma r\ \Gamma(r-p-q)}{\Gamma(r-p)\ \Gamma(r-q)}\Big\{\frac{\mu\delta(1+B)(p)_2(q)_2}{(B-A)(r-p-q-2)_2} \\ &\quad+\frac{[(1+B)(2\mu\delta+\mu-\delta)-\mu\delta(A+1)]\,pq}{(B-A)(r-p-q-1)} + \frac{[(1+B)-(A+1)(\mu-\delta)]}{(B-A)} \\ &\quad-\frac{(A+1)(1-\mu+\delta)(r-p-q)}{(p-1)(q-1)(B-A)}\Big\} + \frac{(A+1)(1-\mu+\delta)(r-1)}{(p-1)(q-1)(B-A)} \le 2. \end{aligned} \tag{14}$$

Proof. Since

$$I(z) = z + \sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_n}\ z^n\ , \quad (z\in U).$$

According to (i) Lemma 2.1, we need to show that

$$T_5 := \sum_{n=2}^{\infty}[(1+B)n-(A+1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_n} \le (B-A)$$

Now,

$$\begin{aligned} T_5 &= \sum_{n=2}^{\infty}\big\{n^2(n-1)\mu\delta(1+B) + n^2[(1+B)(\mu-\delta)-\mu\delta(A+1)] \\ &+n[(1+B)(1-\mu+\delta)-(A+1)(\mu-\delta-\mu\delta)]-(A+1)(1-\mu+\delta)\big\}\,\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_n}. \end{aligned}$$

Noting that $(p)_n = p(p+1)_{n-1}$, $(1)_n = n(1)_{n-1} = n(n-1)(1)_{n-2}$, and applying (4), we have

$$\begin{aligned} T_5 &= \mu\delta(1+B)\sum_{n=2}^{\infty} n\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-2}} \\ &+[(1+B)(\mu-\delta)-\mu\delta(A+1)]\sum_{n=2}^{\infty} n\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}} \\ &+[(1+B)(1-\mu+\delta)-(A+1)(\mu-\delta-\mu\delta)]\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}} \\ &-(A+1)(1-\mu+\delta)\sum_{n=2}^{\infty}\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n}}. \end{aligned}$$

Simplifing we get

$$\begin{aligned} T_5 &= \frac{\Gamma(r)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p)_2(q)_2}{(r-p-q-2)_2}\right. \\ &+[(1+B)(2\mu\delta+\mu-\delta)-\mu\delta(A+1)]\frac{pq}{(r-p-q-1)} \\ &\left.+[(1+B)-(A+1)(\mu-\delta)]-\frac{(A+1)(1-\mu+\delta)(r-p-q)}{(p-1)(q-1)}\right\} \\ &-(B-A)+\frac{(A+1)(1-\mu+\delta)(r-1)}{(p-1)(q-1)}. \end{aligned}$$

But, the last expression is bounded above by $(B-A)$ if condition (14) holds. This completes the proof of Theorem 3.5. □

Note that, the condition (14) is necessary and sufficient for $I_1(z) = z\left[2-\frac{I(z)}{z}\right]$, that is

$$I_1(z) = z - \frac{|pq|}{r}\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p)_{n-1}\ (q)_{n-1}}{(r)_{n-1}\ (1)_{n-1}}\ z^n \tag{15}$$

to be in $\varphi^*(A,B)$.

Theorem 3.6. *Let $p, q > -1$, $pq < 0$ and $r > p+q+2$, then a necessary and sufficient condition for $I(z)$ to be in $\varphi^*(A,B)$, is that*

$$\begin{aligned} &\frac{\Gamma(r+1)\ \Gamma(r-p-q)}{\Gamma(r-p)\ \Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p)_2(q)_2}{r-p-q-2)_2}\right. \\ &\quad+\frac{[(1+B)(2\mu\delta+\mu-\delta)-\mu\delta(A+1)]\,pq}{(r-p-q-1)}\ +\ [(1+B)-(A+1)(\mu-\delta)] \\ &\left.-\frac{(A+1)(1-\mu+\delta)(r-p-q)}{(p-1)(q-1)}\right\}+\frac{(A+1)(1-\mu+\delta)(r-1)_2}{(p-1)(q-1)} \geq 0. \end{aligned} \tag{16}$$

Proof. Since

$$I(z) = z - \frac{|pq|}{r}\sum_{n=2}^{\infty}[1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_n}\ z^n\ , \quad (z\in U).$$

According to (i) of Lemma 2.2, we need to show that

$$\begin{aligned} T_6: &= \sum_{n=2}^{\infty}[(1+B)n-(A+1)][1+(n-1)(\mu-\delta+n\mu\delta)]\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_n} \\ &\le (B-A)\frac{r}{|pq|}. \end{aligned}$$

Now,

$$\begin{aligned} T_6 = \sum_{n=2}^{\infty} &\{n^2(n-1)\mu\delta(1+B)+n(n-1)[(1+B)(\mu-\delta)-\mu\delta(A+1)] \\ &+n[(1+B)-(A+1)(\mu-\delta)] -(A+1)(1-\mu+\delta)\}\ \frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_n}. \end{aligned}$$

Noting that $(p)_n = p(p+1)_{n-1}$, $(1)_n = n(1)_{n-1} = n(n-1)(1)_{n-2}$, and applying (4), we have

$$\begin{aligned} T_6 &= \mu\delta(1+B)\sum_{n=2}^{\infty} n\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} + [(1+B)(\mu-\delta)-\mu\delta(A+1)] \\ &\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-2}} + [(1+B)-(A+1)(\mu-\delta)] \\ &\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n-1}} - (A+1)(1-\mu+\delta)\sum_{n=2}^{\infty}\frac{(p+1)_{n-2}\ (q+1)_{n-2}}{(r+1)_{n-2}\ (1)_{n}} \end{aligned}$$

Simplifing we get,

$$\begin{aligned} T_6 &= \frac{\Gamma(r+1)\Gamma(r-p-q)}{\Gamma(r-p)\Gamma(r-q)}\left\{\frac{\mu\delta(1+B)(p+1)(q+1)}{(r-p-q-2)_2}\right. \\ &+\frac{[(1+B)(2\mu\delta+\mu-\delta)-\mu\delta(A+1)]}{(r-p-q-1)} + \frac{[(1+B)-(A+1)(\mu-\delta)]}{pq} \\ &\left.-\frac{(A+1)(1-\mu+\delta)(r-p-q)}{(p-1)_2(q-1)_2}\right\} + \frac{(A+1)(1-\mu+\delta)(c-1)_2}{(p-1)_2(q-1)_2} + (B-A)\frac{r}{|pq|}. \end{aligned}$$

But, the last expression is bounded above by $(B-A)\dfrac{r}{|pq|}$ if and only if condition (16) holds. This completes the proof of Theorem 3.6. □

Theorem 3.7. *Let $p, q > 0$ and $r > p+q+3$, then a sufficient condition for $I(z)$ defined by (8) to be in $\mathcal{K}(A, B)$, $-1 \le A < B \le 1$, is that*

$$\begin{aligned} \frac{\Gamma r\ \Gamma(r-p-q)}{\Gamma(r-p)\ \Gamma(r-q)} &\left\{\frac{\mu\delta(1+B)(p)_3(q)_3}{(B-A)(r-p-q-3)_3}\right. \\ &+\frac{[(1+B)(5\mu\delta+\mu-\delta)-\mu\delta(A+1)]\ (p)_2(q)_2}{(B-A)(r-p-q-2)_2} \\ &\left.+\frac{[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)]pq}{(B-A)(r-p-q-1)}+1\right\} \le 2. \quad (17) \end{aligned}$$

Proof. The proof of Theorem 3.7 is trivial due to Lemma 2.3. Hence, the proof of this theorem is omitted. □

Note that, the condition (17) is necessary and sufficient for $I_1(z)$ defined by (15) to be in $\mathcal{K}(A, B)$.

Theorem 3.8. *Let $p, q > -1$, $pq < 0$ and $r > p + q + 3$, then a necessary and sufficient condition for $I(z)$ to be in $\mathcal{K}(A, B)$, is that*

$$\mu\delta(1+B)(p)_3(q)_3 + \left[(1+B)(5\mu\delta+\mu-\delta) - \mu\delta(A+1)\right](p)_2(q)_2(r-p-q-3)$$
$$+[(1+B)(4\mu\delta+2\mu-2\delta+1)-(A+1)(2\mu\delta+\mu-\delta)]pq(r-p-q-3)_2+(B-A)(r-p-q-3)_3 \geq 0. \tag{18}$$

Proof. The proof of Theorem 3.8 is trivial due to Lemma 2.3. Hence, the proof of this theorem is omitted. □

Remarks. (i) If we take $\delta = 0$, our results coincide with the results of Shukla and Shukla [16] for the function $h(z)$ given in (6).
(ii) If we take $\delta = 0,\ \mu = 0,\ A = 2\alpha - 1$ *and* $B = 1$, our results coincide with the results of Silverman [18].

4. Conclusions

In this article, we estimated the conditions for certain hypergeometric functions to be in some subclass of holomorphic functions. Also we derived some results for the particular integral operator acting on hypergeometric functions. Our results generalize the corresponding results of Shukla and Shukla [16] and Silverman [18]. Results derived in this artical are universal in character and expected to find certain applications in the theory of special functions.

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