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PRIME AND CO-PRIME EDGE ANTI-MAGIC VERTEX LABELING OF FAMILIES OF UNICYCLIC GRAPHS IN SENSOR NETWORK

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Abstract. A finite simple undirected graph containing p vertices and q edges is called as prime anti-magic if it has an injection from the vertex set of integers $\{1, 2, \ldots, p\}$ satisfying that for each edge uv, the labels given to u and v are relatively coprime and the induced edge labels generated by $f(uv) = f(u) + f(v)$ are all distinct. If the vertices have been assigned a prime labeling, then the graph is called co-prime. In this article, we study some new classes of unicyclic graphs that admits prime and co-prime anti-magic labelings. Also, they are satisfied with the condition of odd prime anti-magic labeling. Thus, this prime and co-prime anti-magic labeling could serve as a surveillance or security model for various kinds of buildings.

Keywords: Prime labeling, co-prime labeling, odd-prime labeling, anti-magic labeling, hairy cycles, sensor network.

AMS Subject Classification: 05C78

1. Introduction

All graph in this paper are taken as finite, connected and undirected. p and q denote the number of vertices (order) and the number of edges (size) of the graph. A graph which admits a single cycle is called a unicyclic graph. A graph labeling is the assignment of some numbers which may be positive, odd or even to the vertices or the edges or both the vertices and edges together. If only the vertices received the labeling, then this assignment

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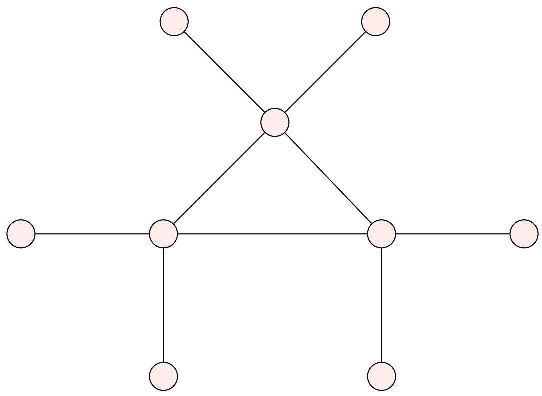

FIGURE 1. $C_3 * S_2$

is called a vertex labeling. If the numbers are assigned to the edges, then it is called an edge labeling. If both the vertices as well as edges are assigned by given numbers, then it is named a total labeling. The detailed study of graph labeling was presented in Gallian's work, [1]. This article is focused on the vertex labeled graphs and on edge anti-magic labeling. The vertices are assigned positive integers and also each adjacent vertex pair are relatively prime. That is, each pair of adjacent vertices satisfies the condition that the greatest common divisor between the labels of adjacent vertices is one. This kind of vertex numbering is called a prime labeling. Entringer invented the prime labeling notation that was later described in details by Tout, [6]. Various types of graphs which admit a prime labeling were discussed in [7] and [8]. Hartsfield and Ringel, [2], gave the new idea of anti-magic labeling of graphs. They said that each vertex labeling f of a graph $G = (V, E)$ from $\{0, 1, 2, \ldots |E(G)|\}$ induces an edge labeling f^* where $f^*(e)$ is sum the labels of the end vertices of an edge e where all the edge labelings are distinct pairwise. Prime anti-magic labeled graphs were presented by Thirugnanasambandam and Chitra, [5]. A prime anti-magic graph is the graph where the vertices are assigned by positive integers and also the labels of adjacent vertices are relatively prime and the edge weights calculated by adding the adjacent vertex weights and all must be distinct. In this work, we discussed prime and co-prime anti-magic labelings of the graphs $C_n * S_3, C_n * S_5, C_n * S_7$ and BW_n. This concept is based on the research paper [4]. The following definitions are necessary for studying the new ideas in Section 2.

Definition 1.1. *For a graph $G = (V, E)$, an edge anti-magic vertex labeling is the one assigning the numbers to the vertices of the graph to get the edge assignment by using the condition of $f(uv) = f(u) + f(v)$ so that all the edge labels are distinct.*

Definition 1.2. *A graph is said to have a prime labeling if there is an injection $f : V \to \{1, 2, \ldots, p\}$ such that for each edge $uv \in E(G), (f(u), f(v)) = 1$. If a graph has a prime vertex labeling, then the graph is called as co-prime.*

Definition 1.3. *For all $m, n \in N$ with $n \geq 3$, an m-hairy n-cycle, denoted by $C_n * S_m$, is the cycle C_n with m pendants attached to each vertex on the cycle.*

Example 1.1. *Figure 1 represents the graph of $C_3 * S_2$.*

For the co-prime labeling, we need some number theoretical result by Bertrand. The Bertrand wheel graph was introduced in [4]:

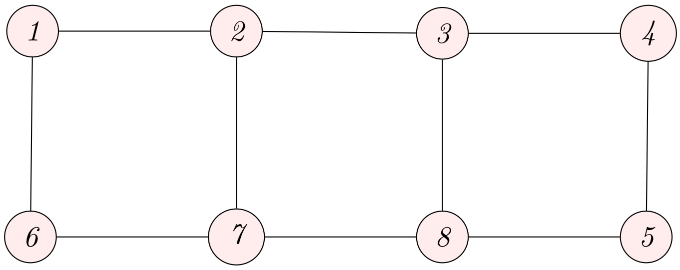

FIGURE 2. An example of a prime labeled graph

Definition 1.4. *For $n \geq 3$, consider the cycle C_n, where the vertices on the cycle are consecutively denoted by u_i and $1 \leq i \leq n$. We define the Bertrand Weed graph, denoted BW_n, to be the non-uniform hairy graph obtained by attaching $2^i - 1$ pendant edges to each u_i.*

The reason of this naming is hidden in the below result:

Definition 1.5 (Bertrand's Postulate)**.** *For every $n \geq 2$, there exists a prime p such that $n < p < 2n$.*

For some graph theoretic notation and terminology, we follow [9].

Definition 1.6. *A prime-anti-magic labeling of a graph G is a bijective function $f : V(G) \to \{1, 2, \ldots, |V|\}$ such that every pair of adjacent vertices $u,\ v$, $gcd(f(u), f(v)) = 1$ and the induced mapping $f* : E(G) \to N$ defined by $f*(e = uv) = \sum f(u, v)$ where $(u, v) \in E(G)$ is injective with the condition that all these edge labelings are distinct.*

Definition 1.7. *A simple graph is said to have a prime vertex labeling (or simply a prime labeling) if there is an injection $f : V \to \{1, 2, \ldots, n\}$ such that for each edge $uv \in E(G), (f(u), f(v)) = 1$. For brevity, if a graph has a prime vertex labeling, we will say that the graph is coprime.*

Example 1.2. *The graph in figure 2 depicts one possible prime labeling.*

When attempting to find or identify prime vertex labelings, the following basic facts from number theory are useful:

(1) All pairs of consecutive integers are relatively prime.
(2) All pairs of consecutive odd integers are relatively prime.
(3) A common divisor of two numbers is also a divisor of their sum and difference.
(4) 1 is relatively prime to all integers.

2. RESULTS ON PRIME AND CO-PRIME ANTI-MAGIC LABELING

In this section, we are focused on prime and co-prime anti-magic labelings of $C_n * S_m$ where m is odd.

Theorem 2.1. *All $C_n * S_3$ admits a co-prime anti-magic labeling.*

Proof. Let $G = C_n * S_3$. It has $4n$ vertices. Let u_i be the vertices on the cycle C_n and u_i^j be the three pendant vertices for $j = 1, 2, 3$ attached to the vertex u_i on the cycle. Define $f : V(G) \to \{1, 2, \ldots, 4n\}$ by

$$f(u_i) = \begin{cases} 1, & \text{for } i = 1 \\ 4i - 1, & \text{for } i \geq 2 \end{cases}$$

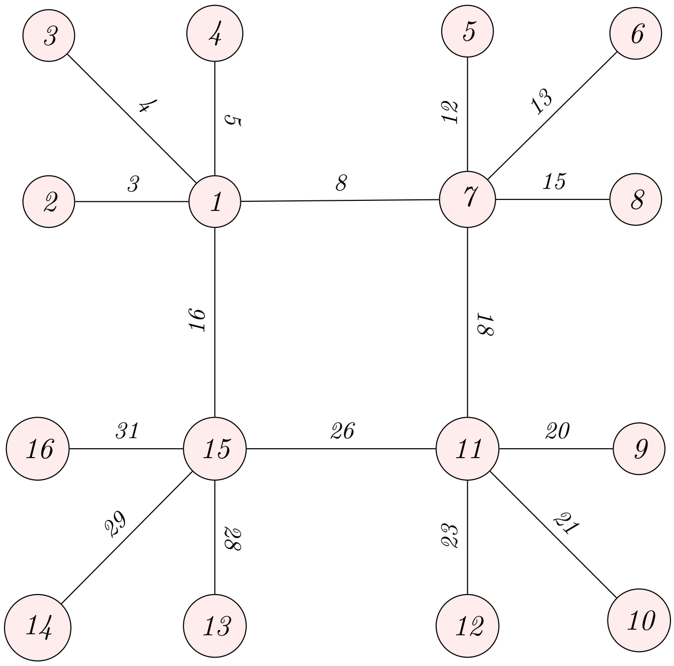

FIGURE 3. Co-prime anti-magic Labeling of $C_4 * S_3$

$$f(u_i^j) = \begin{cases} j+1, & i=1, 1 \leq j \leq 3 \\ 4i-3, & i \geq 2, j=1 \\ 4i-2, & i \geq 2, j=2 \\ 4i, & i \geq 2, j=3. \end{cases}$$

Then for $i=1$ and $1 \leq j \leq 3$ it is clear that $(f(u_1), f(u_i^j)) = (1, 1+j) = 1$. For $i \geq 2$, we will show that the remaining pendant vertices have appropriate labels by checking individual values for j. If $j=1$, then $(f(u_1), f(u_1^i)) = (4i-1, 4i-3) = 1$, if $j=2$, then $(f(u_1), f(u_2^i)) = (4i-1, 4i-2) = 1$, and lastly if $j=3$, then $(f(u_i), f(u_3^i)) = (4i-1, 4i) = 1$. Finally, to see that all adjacent cycle vertices are assigned relatively prime labels, note that $(f(u_1), f(u_n)) = 1$, and $(f(u_i), f(u_1, u_{i+1})) = (4i-1, 4i+3) = 1$. In all the above vertex labeling is relatively prime and the edge labels are identified by $f(uv) = f(u)+f(v)$ and hence all are distinct. Hence, we conclude that 3-hairy n-cycles are co-prime anti-magic. □

Example 2.1. *Figure 3 shows the $C_4 * S_3$ admits co-prime anti-magic labeling.*

Theorem 2.2. *All $C_n * S_5$ admits co-prime anti-magic labeling.*

Proof. Consider the graph $G = C_n * S_5$. It has $6n$ vertices. The vertices of the cycle are denoted as $u_i, i = 1, 2, \ldots, n$. The pendant vertices are denoted as u_i^j and $1 \leq j \leq 5$. Define $f : V(G) \to \{1, 2, \ldots, 6n\}$ by

$$f(u_i) = \begin{cases} 1, & \text{for } i = 1 \\ 6(i-1)+5, & \text{for } i \geq 2 \end{cases}$$

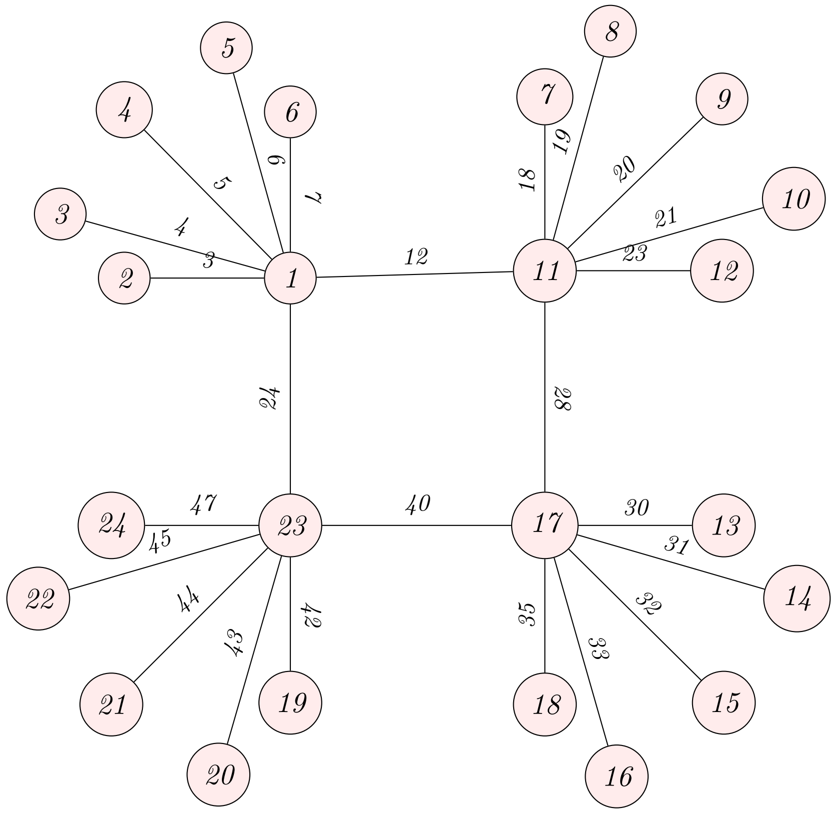

FIGURE 4. Co-prime anti-magic labeling of $C_4 * S_5$

$$f(u_i^j) = \begin{cases} j+1, & i=1, 1 \leq j \leq 5 \\ 6(i-1)+j, & i \geq 2, 1 \leq j \leq 4 \\ 6(i-1)+6, & i \geq 2, j=5 \end{cases}$$

Then for $i=1$ and $1 \leq j \leq 5$ it is clear that $(f(u_1), f(u_j^1)) = (1, j+1) = 1$. Similarly, for $2 \leq i \leq n$ and $1 \leq j \leq 4$, we have $(f(u_i), f(v_j^i)) = (6(i-1)+5, 6(i-1)+j) = 1$, and for $2 \leq i \leq n$ and $j=5, (f(u_i), f(u_5^i)) = (6(i-1)+5, 6(i-1)+6) = 1$. Finally, to see that all adjacent cycle vertices are assigned relatively prime labels, note that $(f(u_1), f(u_2)) = (1, 6i-1) = 1$, and $(f(u_1), f(u_n)) = (1, 6n-1) = 1$. This implies that for $i \geq 2$, we have $(f(u_i), f(u_{i+1})) = (6(i-1)+5, 6(i+1-1)+5) = (6i-1, 6i+5) = 1$. All the vertex labeling are relatively prime and hence the condition of prime is satisfied. The induced edge labels are defined by $f* : E(G) \to N$ such that $f(uv) = f(u) + f(v)$. Hence all are distinct. □

Example 2.2. *Figure 4 exhibits a co-prime anti-magic labeling of $C_4 * S_5$.*

Theorem 2.3. *All $C_n * S_7$ admit co-prime anti-magic labeling.*

Proof. It has $8n$ vertices. Let $u_i, i = 1, 2, \ldots, n$ be the vertices of the cycle and u_i^j and $1 \leq j \leq 7$ be the pendant vertices. Let $f : V(G) \to \{1, 2, \ldots, 8n\}$ by $f(u_1) = 1; f(u_i^j) = j + 1$

$$f(u_i) = \begin{cases} 8i-5, & i \equiv_{15} \ 2,3,6,8,9,11,12,14 \\ 8i-3, & i \equiv_{15} \ 4,5,7,10,13 \\ 8i-1, & i \equiv_{15} \ 0,1. \end{cases}$$

Hence $f(u_i^j)$ is the unique element of $\{8i-7, 8i-6, \ldots, 8i\} \backslash \{f(u_i)\}$ the choice of which is immaterial.

Case 1. $(f(u_i), f(u_{i+1})) = (8i-5, 8(i+1)-5) = (8i-5, 8i+3) = 1,$

Case 2. $(f(u_i), f(u_{i+1})) = (8i-5, 8(i+1)-3) = (8i-5, 8i+5) = 1,$

Case 3. $(f(u_i), f(u_{i+1})) = (8i-5, 8(i+1)-1) = (8i-5, 8i+7) = 1,$

Case 4. $(f(u_i), f(u_{i+1})) = (8i-3, 8(i+1)-5) = (8i-3, 8i+3) = 1,$

Case 5. $(f(u_i), f(u_{i+1})) = (8i-3, 8(i+1)-3) = (8i-3, 8i+5) = 1,$

Case 6. $(f(u_i), f(u_{i+1})) = (8i-1, 8(i+1)-5) = (8i-1, 8i+3) = 1,$

Case 7. $(f(u_i), f(u_{i+1})) = (8i-1, 8(i+1)-1) = (8i-1, 8i+7) = 1.$

Hence, (i) $f(u_i) = 8i-5$, (ii) $f(u_i) = 8i-3$, (iii) $f(u_i) = 8i-1$ and all are relatively prime. The labeling function splits up N into smaller sets of eight consecutive numbers each, such as $\{1, 2, 3, 4, 5, 6, 7, 8\}$. In each set of 8 numbers, the second, third, or fourth odd number is assigned to the cycle vertices. The first set is assigned as labels to the first cycle vertex and its associated pendants, the second set is assigned to the second cycle vertex and its associated pendants, and so on. Also, it satisfies the anti-magic labeling condition. Thus, it is proved. □

Example 2.3. *Figure 5 gives the explanation of the theorem 2.3. It shows the prime and co-prime anti-magic labeling of $C_3 * S_7$.*

Theorem 2.4. *All BW_n are co-prime anti-magic.*

Proof. By the definition of Bertrand Weed graph, each clump has exactly 2^i vertices, specifically $2^i - 1$ from the pendants and 1 from the cycle vertex. Hence the natural numbers are partitioned into sets of size 2, 4, 8 and so on. Then by Bertrand's postulate the adjacent vertex labels are relatively prime to each other and all the edge labels are distinct. Bertrand Weed graphs are coprime graphs, where the natural numbers are partitioned into sets of size 2, 4, 8, and so on. Using Bertrand's Postulate, there is a prime in each set of integers that is assigned to the vertices of each clump. Hence the Bertrand Weed graph is co-prime anti-magic. □

Example 2.4. *In figure 6, we have provided a prime vertex labeling for BW_3.*

3. Odd Co-Prime Anti-Magic Labeling of Unicyclic Graphs

In this part, discussed the graph vertices are admitted by odd numbers by using the condition of co-prime condition as well as the graph edges are satisfying the anti-magic condition.

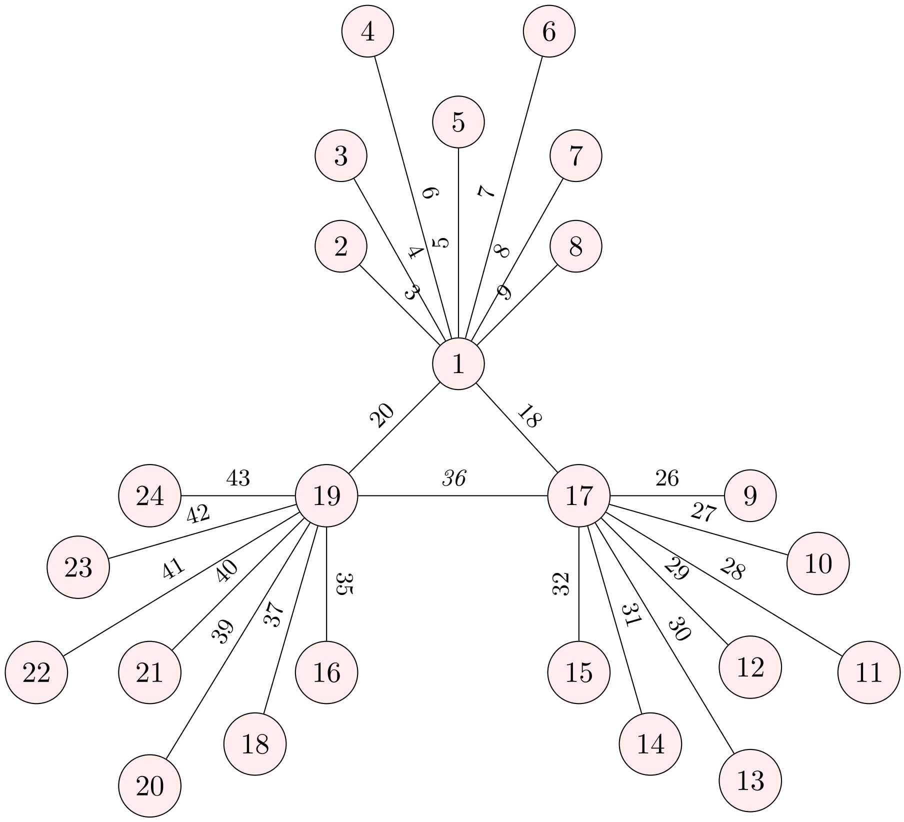

FIGURE 5. Co-prime anti-magic Labeling of $C_3 * S_7$

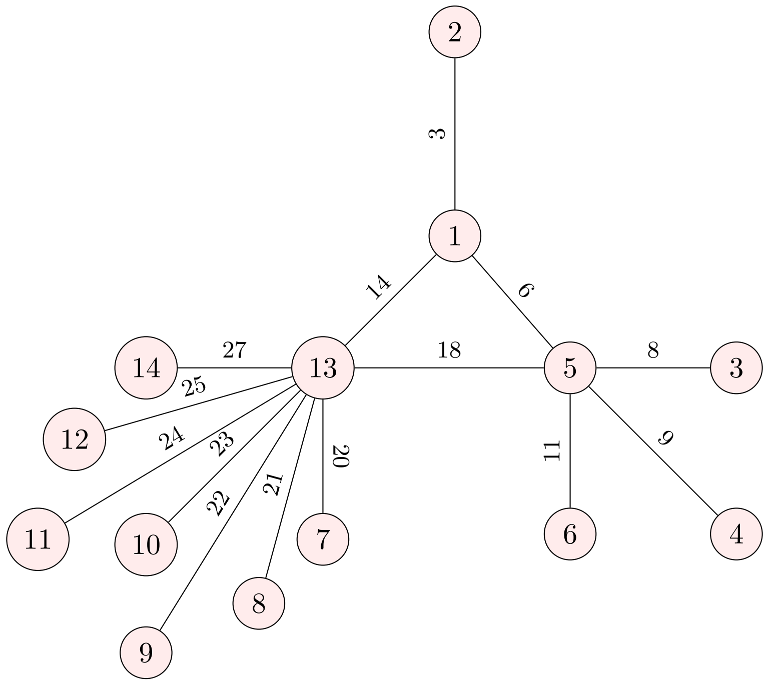

FIGURE 6. Co-prime anti-magic Labeling of BW_3

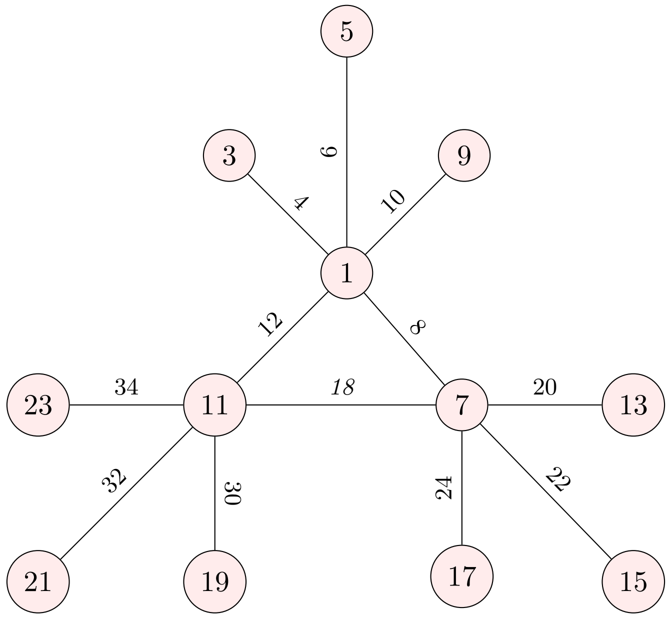

FIGURE 7. Odd co-prime anti-magic labeling of $C_3 * S_3$

Definition 3.1. *A graph with p vertices and q edges is said to have odd co - prime anti-magic labeling if the vertex function* $f : V \to \{1, 3, 5, \ldots, 2p-1\}$ *such that for each edge* $uv \in E(G), (f(u), f(v)) = 1$ *and the induced edge labeling* $f* : E \to N$ *is defined by* $f*(uv) = f(u) + f(v)$ *and also all the edge labeling must be distinct.*

Theorem 3.1. *All* $C_n * S_3$ *admits odd co-prime anti-magic labeling.*

Proof. The cycle vertices are denoted as u_i and $i = 1, 2, \ldots, n$ and the pendant vertices are denoted as u_i^j. The assignment of vertex labeling of the cycle vertices are $f(u_1) = 1; f(u_2) = 7; f(u_3) = 11$. Here all the vertices are assigned by odd integers only. The condition of co-prime is each adjacent vertex are relatively prime. That means the greatest common divisor between the adjacent vertices are one. Hence all the vertex of the given graph admits odd co-prime labeling. The edge assignments are calculated by adding the adjacent vertex labeling. If all the edge labeling are distinct then it is called as anti-magic. Hence the given graph $C_n * S_3$ admits an odd co-prime anti-magic labeling. □

Example 3.1. *The* $C_3 * S_3$ *graph admits odd co-prime anti-magic labeling.*

Theorem 3.2. *All* $C_n * S_5, C_n * S_7$ *admits odd co-prime anti-magic labelings.*

Proof. By using Theorem 3.1, we get the proof. □

Theorem 3.3. *All* BW_n *are odd co-prime anti-magic.*

Proof. The cycle vertices are considered as u_i and $1 \le i \le n$ and attaching $2^i - 1$ pendant to each u_i.
Assignment of cycle vertex labeling: $f(u_1) = 1; f(u_2) = 7; f(u_3) = 11$ and so on. The pendant vertices are assigned as $f(v_1) = 3, f(v_2) = 5, f(v_3) = 9, f(v_4) = 13 f(v_5) = 2i + 5, i = 5, 6, \ldots, n$. The edge labeling was calculated by $f(uv) = f(u) + f(v)$, and all the edge labelings are distinct. Hence the weed graph admits an odd co-prime anti-magic labeling. □

4. Applications

Sensor networks are networks of sensors that sense and control the environment, enabling humans and computers to interact with each other and the surrounding environment. The sensor network connects to the internet or computer networks to transfer data for analysis and use by humans and machines. They enable interaction between persons or computers and the surroundings. Let the sensor node graph be represented by a graph (V, E), where $V = (1, 2, 3, \ldots, N)$ is the set of nodes, each representing a and E is the set of edges connecting the nodes. Two nodes i and j are connected if they share a load. Each node i (vertex labeling) has a scalar l_i representing the load on each node. The amount of load to be transferred from node i to node j is given by the edge labeling. The load balancing schedule should make the load on each node equal to the average load. The conjugate Gradient Algorithm, [3], was used to calculate the average load on each node. The graph of m-hairy n-cycle could serve as a hub for the sensor network. The pendant vertices could serve as a smart city linked to the hub. The edges of the graph will consider as a router between the hub and the smart city. The frequency level could be controlled by the prime and co-prime anti-magic conditions. Using a graph theory approach to load balancing in wireless sensor networks for the smart city, we can see how sensors interact with each other to ensure that they are not switched on or switched off at the same time by different systems running in different parts of the city.

5. Conclusions

The prime anti-magic graphs that are defined as the graph vertices are assigned by positive integers and also the adjacent vertices are relatively prime and hence calculated the edge weight by adding the adjacent vertex weights and all must be distinct. In this research, we mainly discussed prime and co-prime anti-magic labels of the graphs $C_n * S_3, C_n * S_5, C_n * S_7$, and BW_n. Thus, the prime and co-prime anti-magic labelings could serve as a surveillance or security model for various kinds of buildings.

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Prof. Dr. Ismail Naci Cangul - for the photography and short autobiography, see TWMS J. App. and Eng. Math. V.6, N.2