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SOME RESULTS OF THE NONLINEAR HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH ALGEBRA

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Abstract. In this paper, a fixed point theorem of Dhage is used to prove under mixed Lipschitz and Caratheodory conditions the existence of solutions for a nonlinear hybrid fractional differential equation in Banach algebra with two-point integral hybrid boundary conditions. Furthermore, sufficient conditions for existence and uniqueness of mild solutions are derived. In addition, the Ulam-Hyers types of stability of solutions are established. Finally, a numerical example is given to clarify the acquired outcomes.

Keywords: Dhage fixed point theorem, Hybrid fractional integro-differential equations, Existence results, Banach Algebra, Boundary value problems, Green's function, Hyers-Ulam stability.

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1. Introduction

Fractional Calculus is a field of mathematical studies that grows out of the traditional definitions of the calculus integral and derivative operators in much the same way fractional exponents is an outgrowth of exponents with integer value. It is also considered as a new branch of mathematical physics that deals with integro-differential equations, where integrals are of convolution type and exhibit mainly singular kernels of power law or logarithm type. It has gained considerably attractiveness and reputation in the past few decades in diverse fields of science and engineering. Efficient analytical and numerical methods have been developed but still need precise consideration. Although researchers have already reported many excellent results in several seminal monographs and review articles, there are still a

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large number of non-local phenomena unexplored and waiting to be discovered. Therefore, year by year, we can discover new aspects of the fractional modeling and applications. The differential equations involving fractional derivatives in time,compared with those of integer order in time, are more realistic to describe many phenomena in nature (for instance,to describe the memory and hereditary properties of various materials and processes), the study of such equations has become an object of extensive study during recent years. The quadratic perturbations of nonlinear differential equations, which are known as the fractional hybrid differential equations, have been recently given much attention. In literature, there are many articles on the theory of hybrid differential equations,see ([5], [11], [14], [21]-[23]). The importance of the investigations of hybrid differential equations lies in the fact that they include several dynamic systems as special cases. For examples, we refer the interested reader to the monographs ([2], [5], [8]-[12], [16], [17], [23] and [24]), and the references therein.

In the following, we give a brief review of some distinguished researches about in the hybrid differential equations. Consider the functions $f \in C(I \times R, R\backslash\{0\})$ and $g \in C(I \times R, R)$ with $I = [0, T]$, and let $\alpha \in (0, 1]$. In 2010, Dhage and Lakshmikantham [6] established the existence uniqueness results, and some fundamental differential inequalities for hybrid differential equations of the form $\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t, x(t))$ a.e. $t \in I$, with $x(t_0) = x_0 \in R$. In 2011, Sun *et al.* [22] discussed the fractional hybrid differential equations involving Riemann-Liouville differential operators $D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = g(t, x(t))$ a.e. $t \in I$, with $x(0) = 0$. In 2015, Hilal *at el.* [13] studied the existence results (under mixed Lipschitz and Carathéodory conditions) of the boundary value problems for hybrid differential equations with fractional orders: $D^\alpha\left(\frac{x(t)}{-f(t,x(t))}\right) = g(t, x(t))$ a.e. $t \in I$, with $a\frac{x(0)}{-f(0,x(0))} + b\frac{x(T)}{-f(T,x(T))} = c$, where a, b, and c are real constants with $a + b \neq 0$. In 2018, Ullah at el. [21] established the sufficient conditions for the existence of at least one solution for the hybrid fractional differential equation to boundary condition involving ordinary derivatives $D^\alpha\left(\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right) = h(t, x(t))$, $t \in I$, with $\left.\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right|_{t=0} = 0$, and $\left.\frac{x(t)-f(t,x(t))}{g(t,x(t))}\right|_{t=1} = 0$. Recently, Al Issa at el. [1] studied the existence results using a fixed-point theorem in a Banach algebra due to Dhage [10] for the nonlinear quadratic functional-integral equations of fractional order $x(t) = k(t, x(\varphi_1(t))) + g(t, x(\varphi_2(t)))\ I^\alpha f(t, I^\beta u(t, x(\varphi_3(t))))$, where $\alpha\ ,\beta \in (0, 1)$, with $g \in C([0, T] \times R, R\backslash\{0\})$, $f, u \in C([0, T] \times R, R)$, and $k \in C([0, T] \times R, R)$. Then, as an application, they established the existence results for the fractional hybrid differential equations involving the Riemann-Liouville differential operators $D^\alpha\left(\frac{x(t)-k(t,x(\varphi_1(t)))}{g(t,x(\varphi_2(t)))}\right) = f(t, I^\beta u(t, x(\varphi_3(t))))$, for $t \in I = [0, T]$, and with $x(0) = k(0, x(0))$.

Inspired by the above works, consider the following integral boundary fractional hybrid differential equations (IBFHDE for short) involving fractional differential operators of order $1 < \alpha < 2$.

$$\begin{cases} D^{\alpha}\left(\dfrac{x(t) - f(t, x(t))}{g(t, x(t))}\right) = h(t, I^{\beta}\ u(t, x(t)),\ t \in I = [0, T], \\ \left.\dfrac{x(t) - f(t, x(t))}{g(t, x(t))}\right|_{t=0} = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} h_1(s, x(s))\, ds, \\ \left.\dfrac{x(t) - f(t, x(t))}{g(t, x(t))}\right|_{t=T} = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} h_2(s, x(s))\, ds, \end{cases} \tag{1}$$

where $\alpha \in (1, 2)$, D^{α} denotes the Liouville-Caputo fractional derivative, $\Gamma(.)$ is the classical Gamma function, I^{β} is the Riemann-Liouville fractional integral of order $\beta \in (0, 1)$, with $g \in C(I \times R, R \setminus \{0\})$, $h, u \in C(I \times R, R)$, and $f \in C(I \times R, R)$.
This paper is organized as follows: In Section 2, we recall some useful preliminaries that will be used through our work. In Section 3, the main results are presented as follows: First, we state and prove the sufficient conditions that guarantee the existence of solutions for the problem under study. These existence results are studied by using a fixed-point theorem for three operators in a Banach algebra due to Dhage [10], and under mixed Lipschitz and Caratheodory conditions. In the second part, we prove that these mild solutions are unique solutions. While, in the third part, we study the Hyers-Ulam stability of problem (1). Finally, in Section 4, a numerical example is given to demonstrate the application of the main results obtained.

2. Preliminaries

In this section, we introduce some basic definitions and preliminaries that will be used throughout our work.

Definition 2.1. *The Riemann-Liouville fractional integral I^{α} of the function $f \in L^1(I, R)$ of order $\alpha > 0$ is defined by:*

$$I_{0^+}^{\alpha}\ f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\ f(s)\ ds,$$

where $\Gamma(.)$ is Euler's Gamma function.

Definition 2.2. *The Caputo fractional derivative D^{α} of the absolutely continuous function $f(t) \in C(I, R)$, where $\alpha > 0$ and $I = [0, T]$ is defined as:*

$${}^{c}D_{0^+}^{\alpha}\ f(t) = \frac{d}{dt} \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\ f(s)\ ds.$$

For further properties of fractional calculus operators (see [18]-[20]).

Now, denote by $X = C(I, R)$ to be the Banach algebra of all real-valued continuous functions from $I = [0, T]$ into R with the norm $\|x\| = \sup\{|x(t)| : t \in I\}$. Moreover, by $L^1(I, R)$, we denote by the space of Lebesgue integrable real-valued functions on I equipped with the L^1-norm $\|x\|_{L^1} = \int_0^T |x(s)|ds$.

Definition 2.3. [9] *(normed algebra) If A is an algebra and $\|.\|$ is a norm on A satisfying $||x.y|| \leq ||x||.||y||$ for all $x, y \in A$, then $\|.\|$ is called an algebra norm and $(A, \|.\|)$ is called a normed algebra. A complete normed algebra is called a Banach algebra.*

Definition 2.4. [9] *Let X be a normed vector space. A mapping $T : X \to X$ is called D-Lipschitzian, if there exists a continuous and nondecreasing function $\phi : R^+ \to R^+$, such that*

$$\|Tx - Ty\| \leq \phi(\|x - y\|)$$

for all $x, y \in X$, where $\phi(0) = 0$.

Further, if $\phi(r) < r$, then T is called nonlinear contraction on X.

Definition 2.5. [9] *A mapping $f : I \times R \to R$ is said to satisfy a condition of L^1-Caratheodory or simply is called L^1-Caratheodory if*

(1) *$t \to f(t, x)$ is measurable for each $x \in R$,*
(2) *$x \to f(t, x)$ is continuous almost everywhere for $t \in I$, and*
(3) *for each real number $r > 0$ there exists a function $g \in L^1(I, R)$ such that $|f(t, x)| \leq g(t)$ a.e. $t \in I$ for all $x \in R$ with $|x| \leq r$.*

Lemma 2.1. [15] *Consider the real number $\alpha \in (0, 1,$ and let $w \in L^1(0, 1)$, Then,*

(1) *$D^\alpha I^\alpha w(t) = w(t)$ for all $t \in I$.*
(2) *$I^\alpha D^\alpha w(t) = w(t) - \dfrac{I^{1-\alpha} w(t)|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1}$ almost everywhere $t \in I$.*

Lemma 2.2. *Let $\alpha > 0$, if we assume that $f \in C(0, R)$, then*

(1) *The fractional differential equation $D^\alpha_{0^+} f(t) = 0$ has $f(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1}$, where $n = [\alpha] + 1$, and $c_i \in R$ for all $i = 1, 2, ..., n$.*
(2) *$I^\alpha_{0^+} D^\alpha_{0^+} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{\alpha-n}$ as a unique solution, where n is the smallest positve integer greater than or equal to α.*

Theorem 2.1. [10] *Let S be a nonempty, closed convex and bounded subset of a Banach algebra X and let $\mathcal{A} : X \to X$, $\mathcal{B} : S \to Y$, and $\mathcal{C} : X \to X$ be three operators such that:*

(1) *$\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian with Lipschitz constants μ and σ, respectively,*
(2) *B is completely continuous.*
(3) *$x = \mathcal{A}x\mathcal{B}y + \mathcal{C}x$ implies that $x \in S$, for all $y \in S$.*
(4) *$\eta\mathcal{N} + \mathcal{R} < \rho$ for $\rho > 0$, where $\mathcal{N} = \|\mathcal{B}(S)\|$.*

Then the operator equation $\mathcal{A}x\mathcal{B}y + \mathcal{C}x = x$ has a solution in S.

Definition 2.6. *The solutuion of the (IBFHDE) (1) is the continuous function $x \in C(I, R)$ such that $t \to \dfrac{x(t) - f(t, x(t))}{g(t, x(t))}$ is continuous function for each $x \in R$, and x satisfies the boundary values problem (1).*

3. Main Results

In this section we will study the existence of solutions of the integral boundary value problem for hybrid differential equation with fractional order $\alpha \in (1,2)$ (IBFHDE) (1) on the closed interval $[0,T]$ under mixed Lipschitz and Caratheodory conditions on the nonlinearities involved in it using the a fixed point theorem for three operators in a Banach algebra X, due to Dhage[10].

First of all, consider the following assumptions:

(A_0) The function $x \to \dfrac{x(t) - f(t,x(t))}{g(t,x(t))}$ is continuous and increasing in R almost every where for $t \in I$.

(A_1) The functions $g : I \times R \to R \setminus \{0\}$, $f : I \times R \to R$, and $h_i : I \times R \to R$ for $i = 1,2$ are continuous and there exist two positive functions $\mu(t)$, $\sigma(t)$, with bounds $\|\mu\| = \sup\{\mu(t) \,|t \in I\}$ and $\|\sigma\| = \sup\{\sigma(t)\,|t \in I\}$ respectively, such that

$$|g(t,x) - g(t,y)| \leq \mu(t)\ |x-y|,$$
$$|f(t,x) - f(t,y)|\ \leq \sigma(t)\ |x-y|,$$

and there exist constants $k_i \in [0,1)$ such that

$$|h_i(t,x) - h_i(t,y)| \leq k_i\ |x-y|$$

for every $t \in I$, and $x, y \in R$.

(A_2) The functions $h : [0,T] \times R \to R$ and $u : [0,T] \times R \to R$ satisfy Caratheodory conditions, i.e, if h and u are measurables in t for any $x \in R$ and continuous in x for almost all $t \in [0,T]$, then there exist three functions $t \to \boldsymbol{a}(t), t \to \boldsymbol{b}(t)$ and $t \to \boldsymbol{m}(t)$ such that

$$|h(t,x)| \leq\ \boldsymbol{a}(t) + \boldsymbol{b}(t)\ |x|,\ \forall\ (t,x)\ \in I \times R,$$

$$|u(t,x)| \leq \boldsymbol{m}(t),\ \forall\ (t,x)\ \in I \times R,$$

where $\boldsymbol{a}(.), \boldsymbol{m}(.) \in L^1$ and $\boldsymbol{b}(.)$ are measurable and bounded, and $I_c^{\gamma}\boldsymbol{m}(.) \leq \mathcal{M}, \forall \gamma \leq \alpha, c \geq 0$.

(A_3) There exists a number $\rho > 0$ such that

$$\rho \geq \frac{1}{2\,\|\boldsymbol{a}\|\,\|\mu\|\,\aleph}\left(1 - \|\boldsymbol{a}\|\,H_0\aleph - \frac{T^{\gamma}}{\Gamma(\gamma+1)}\,\|\mu\| - \|\boldsymbol{b}\|\,\|\mu\|\,\aleph - \sqrt{\Delta}\right), \quad (2)$$

where

$$H_0 = \sup_{t \in J} |g(t,0)|,$$

$$\aleph = G_0 T\left[\|\boldsymbol{a}\| + \mathcal{M}\,\|\boldsymbol{b}\|\,\frac{T^{\beta-\gamma}}{\Gamma(^{\beta-\gamma}+1)}\right],$$

and

$$\begin{aligned}\Delta \ &= \ -4\,\|\boldsymbol{a}\|\,\aleph H_0\left(\frac{T^{\gamma}}{\Gamma(\gamma+1)} + \|\boldsymbol{b}\|\,\aleph\right)\|\mu\| \\ &+ \left[\left((\|\boldsymbol{b}\|\,\|\mu\|\,\aleph) + \frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)\|\mu\| + \|\boldsymbol{a}\|\,H_0\aleph - 1\right]^2\end{aligned}$$

(A_5) There exist a real number $G_0 \in R^+$, such that

$$G_0 = \max \{|G(t,s)|, \ (t,s) \in I \times I\}$$

It is clear that $G_0 = |G(s,s)|$.

(A_6) $h_i : I \times R \to R, i = 1,2$ are continuous and there exist constants $k_i \in [0,1)$ such that

$$|h_i(t, u(t)) - h_i(t, v(t))| \le k_i |u - v|.$$

Remark 3.1. *From assumption (A_1), we dedcue for $i = 1,2$ that*

$$|h_i(t, x(t))| - |h_i(t,0)| \le |h_i(t, x(t)) - h_i(t,0)| \le k_i |x - 0|,$$

which implies that

$$|h_i(t,x)| \le H_i + k_i |x(t)|, \text{ where } H_i = \sup_{t \in I} |h_i(t,0)|.$$

Similary, we obtain that $|g(t,x)| \le \widetilde{G} + \|\mu\| \, |x(t)|$, *where* $\widetilde{G} = \sup_{t\in I} |g(t,0)|$.

Lemma 3.1. *Let $h\left(t, I^\beta u(t, x(t))\right) \in C(I,R)$. Then, the (IBFHDE) (1) is equivalent to the following integral equation*

$$x(t) = v(t, x(t)) + g(t, x(t)) \int_0^T G(t,s) \ h\left(s, I^\beta u(s, x(s))\right) \ ds, \qquad (3)$$

where $v(t, x(t))$ is a continuous function in X such that

$$\begin{aligned} v(t, x(t)) \quad &= \quad f(t, x(t)) + \frac{g(t,x(t))}{\Gamma(\gamma)} \left[\int_0^1 (1-s)^{\gamma-1} h_1(s, x(s)) \, ds \right. \\ &\qquad \left. + \frac{t}{T} \int_0^1 (1-s)^{\gamma-1} \left[h_2(s, x(s)) - h_1(s, x(s)) \right] ds \right], \end{aligned}$$

and $G(t,s)$ is the Green's fucntion defined by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t}{T\ \Gamma(\alpha)} (T-s)^{\alpha-1}, & 0 \le s \le t \le T \\ -\dfrac{t}{T\ \Gamma(\alpha)} (T-s)^{\alpha-1}, & 0 \le t \le s \le T, \end{cases}$$

Proof. If we apply operation I^α on the (IBFHDE) (1), then by Lemma (2.2), we obtain that

$$\frac{x(t) - f(t, x(t))}{g(t, x(t))} = I^\alpha_{0^+} \ h(t, I^\beta \ u(t, x(t)) + c_0 + c_1 t,$$

where $c_0, c_1 \in R$. Hence, integral solution of (1) is

$$\begin{aligned} x(t) \quad &= \quad f(t, x(t)) \\ &\qquad + g(t, x(t)) \left[\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h\left(s, I^\beta u(s, x(s))\right) \ ds + c_0 + c_1 t \right]. \end{aligned}$$

By applying the integral boundary of (1), we get

$$c_0 = \frac{1}{\Gamma(\gamma)} \int_0^1 (1-s)^{\gamma-1} h_1(s, x(s)) \, ds,$$

and

$$c_1 = \frac{1}{T}\left\{\frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\left[h_2(s,x(s)) - h_1(s,x(s))\right]ds - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} h\left(s, I^\beta u(s,x(s))\right) ds\right\}.$$

This implies that

$$\begin{aligned} x(t) &= f(t,x(t)) + g(t,x(t))\left[\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} h\left(s, I^\beta u(s,x(s))\right) ds \right.\\ &\quad + \frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_1(s,x(s))\,ds \\ &\quad + \frac{t}{T}\left\{\frac{1}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\left[h_2(s,x(s)) - h_1(s,x(s))\right]ds \right.\\ &\quad \left.\left. - \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} h\left(s, I^\beta u(s,x(s))\right) ds\right\}\right] \\ &= f(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\gamma)}\left[\int_0^1 (1-s)^{\gamma-1} h_1(s,x(s))\,ds \right.\\ &\qquad \left. + \frac{t}{T}\int_0^1 (1-s)^{\gamma-1}\left[h_2(s,x(s)) - h_1(s,x(s))\right]ds\right] \\ &\quad + \frac{g(t,x(t))}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} h\left(s, I^\beta u(s,x(s))\right) ds \right.\\ &\qquad \left. - \frac{t}{T}\left(\int_0^T (T-s)^{\alpha-1} h\left(s, I^\beta u(s,x(s))\right) ds\right]\right. \\ &= v(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\alpha)}\int_0^T G(t,s)\ h\left(s, I^\beta u(s,x(s))\right) ds, \end{aligned}$$

where

$$\begin{aligned} v(t,x(t)) &= f(t,x(t)) + \frac{g(t,x(t))}{\Gamma(\gamma)}\left[\int_0^1 (1-s)^{\gamma-1} h_1(s,x(s))\,ds \right.\\ &\qquad \left. + \frac{t}{T}\int_0^1 (1-s)^{\gamma-1}\left[h_2(s,x(s)) - h_1(s,x(s))\right]ds\right], \end{aligned}$$

and $G(t,s)$ is the Green's fucntion defined by

$$G(t,s) = \begin{cases} \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{t}{T\ \Gamma(\alpha)}(T-s)^{\alpha-1}, & 0 \le s \le t \le T \\ -\dfrac{t}{T\ \Gamma(\alpha)}(T-s)^{\alpha-1}, & 0 \le t \le s \le T, \end{cases}$$

□

From Lemma (3.1), we obtain the following definition:

Definition 3.1. *By a mild solution of the (IBFHDE) (1), we mean a function $x \in C(I,R)$ satisfying integral equation (3) for all $t \in I$.*

3.1. **Existence Of Solution.** In the following, we prove the following existence theorem:

Theorem 3.1. *Assume that the hypotheses* $(A_0)-(A_3)$ *&* (A_6) *hold. Then, the quadratic functional integral equation (3) has at least one mild solution defined in* I.

Proof. Set $X = C(I, R)$ and define a subset S of X as

$$S = \{x \in X,\ \|x\| \le \rho\},$$

where ρ satisfies inequality (2).
It is clearly that S is closed, convex, and bounded subset of the Banach space X. Now, define the following three operators: $\mathcal{A} : X \to X$, $\mathcal{B} : S \to X$ and $\mathcal{C} : X \to X$ defined by:

$$\mathcal{A}x(t) = g(t, x(t)),\ t \in I \tag{4}$$

$$\mathcal{B}x(t) = \int_0^T G(t,s)\ h(s, I^\beta u(s, x(s))\ ds,\ \text{for}\ (t,s) \in I \times I, \tag{5}$$

$$\mathcal{C}x(t) = v(t, x(t)),\ \text{for}\ t \in I. \tag{6}$$

Then the integral equation (3) is transformed into the following integral operator equation:

$$x(t) = \mathcal{A}x(t).\mathcal{B}x(t) + \mathcal{C}x(t),\ \text{for all}\ t \in I. \tag{7}$$

In the following, we show that the operators $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ satisfy all the conditions of Lemma 2.1. The proof is accomplished in the following steps.

Step 1. Operators $\mathcal{A}$ and $\mathcal{C}$ are Lipschitzian on X.

Let $x, y \in X$, then from assumption (A_1) we have

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &= |g(t, x(t)) - g(t, y(t))| \\ &\le \mu(t)\ |x(t) - y(t)| \le \|\mu\|\ \|x - y\| \end{aligned}$$

which implies $\|\mathcal{A}x - \mathcal{A}y\| \le \|\mu\|\ \|x - y\|$ for all $x, y \in X$. Therefore, $\mathcal{A}$ is a Lipschitzian on X with Lipschitz constant μ.
Also, we have $|\mathcal{C}x(t) - \mathcal{C}y(t)| = |v(t, x(t)) - v(t, y(t))|$, such that

$$\begin{aligned} v(t, x(t)) - v(t, y(t)) &= f(t, x(t)) - f(t, y(t)) \\ &\quad + \frac{g(t, x(t))}{\Gamma(\gamma)} \left[\int_0^1 (1-s)^{\gamma-1} h_1(s, x(s))\, ds \right. \\ &\quad \left. + \frac{t}{T} \int_0^1 (1-s)^{\gamma-1} \left[h_2(s, x(s)) - h_1(s, x(s)) \right] ds \right] \\ &\quad - \frac{g(t, y(t))}{\Gamma(\gamma)} \left[\int_0^1 (1-s)^{\gamma-1} h_1(s, y(s))\, ds \right. \\ &\quad \left. + \frac{t}{T} \int_0^1 (1-s)^{\gamma-1} \left[h_2(s, y(s)) - h_1(s, y(s)) \right] ds \right] \\ &= f(t, x(t)) - f(t, y(t)) + E_1(t) + \frac{t}{T} E_2(t) \end{aligned}$$

where

$$\begin{aligned}
E_1(t) &= \frac{g(t,x(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_1(s,x(s))\,ds \\
&\quad -\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_1(s,y(s))\,ds \\
&= \frac{g(t,x(t))-g(t,y(t))+g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_1(s,x(s))\,ds \\
&\quad -\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_1(s,y(s))\,ds \\
&= \frac{g(t,x(t))-g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} h_1(s,x(s))\,ds \\
&\quad +\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \left[h_1(s,x(s))-h_1(s,y(s))\right]ds
\end{aligned}$$

and

$$\begin{aligned}
E_2(t) &= \frac{g(t,x(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \left[h_2(s,x(s))-h_1(s,x(s))\right]ds \\
&\quad -\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \left[h_2(s,y(s))-h_1(s,y(s))\right]ds \\
&= \frac{\begin{pmatrix} g(t,x(t))-g(t,y(t)) \\ +g(t,y(t)) \end{pmatrix}}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \begin{bmatrix} h_2(s,x(s)) \\ -h_1(s,x(s)) \end{bmatrix} ds \\
&\quad -\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \left[h_2(s,y(s))-h_1(s,y(s))\right]ds \\
&= \frac{g(t,x(t))-g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \left[h_2(s,x(s))-h_1(s,x(s))\right]ds \\
&\quad +\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} \begin{bmatrix} h_2(s,x(s))-h_2(s,y(s)) \\ +h_1(s,y(s))-h_1(s,x(s)) \end{bmatrix} ds
\end{aligned}$$

Thus, by assumption (A_1) and Remark 3.1 we have

$$\begin{aligned}
|E_1(t)| &\leq \frac{|g(t,x(t))-g(t,y(t))|}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} |h_1(s,x(s))|\,ds \\
&\quad +\frac{|g(t,y(t))|}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} |h_1(s,x(s))-h_1(s,y(s))|\,ds \\
&\leq \frac{\mu(t)\ |x-y|}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} (H_1+k_1|x(t)|)\,ds \\
&\quad +\frac{\widetilde{G}+\|\mu\|\,|y|}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1} k_1\,|x-y|\,ds \\
&\leq \frac{\|\mu\|\,(H_1+k_1\,\|x\|)+k_1(\widetilde{G}+\|\mu\|\,\|y\|)}{\Gamma(\gamma+1)}\ |x-y| \\
&\leq c_1(t)\ |x-y|,
\end{aligned}$$

and

$$\begin{aligned}
|E_2(t)| &\leq \frac{|g(t,x(t))-g(t,y(t))|}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\left[|h_2(s,x(s))|+|h_1(s,x(s))|\right]ds \\
&\quad +\frac{g(t,y(t))}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\left[\begin{array}{c} h_2(s,x(s))-h_2(s,y(s)) \\ +h_1(s,y(s))-h_1(s,x(s)) \end{array}\right]ds \\
&\leq \frac{\mu(t)\ |x-y|}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\left[(H_1+H_2)+(k_1+k_2)|x|\right]ds \\
&\quad +\frac{(\widetilde{G}+\|\mu\|\,|y|)}{\Gamma(\gamma)}\int_0^1 (1-s)^{\gamma-1}\left[k_2|x-y|+k_1|y-x|\right]ds \\
&\leq \frac{\mu(t)\ |x-y|}{\Gamma(\gamma+1)}\left[(H_1+H_2)+(k_1+k_2)|x|\right] \\
&\quad +\frac{(\widetilde{G}+\|\mu\|\,|y|)}{\Gamma(\gamma+1)}(k_1+k_2)|x-y| \\
&\leq \frac{\|\mu\|\left[(H_1+H_2)+(k_1+k_2)|x|\right]+(\widetilde{G}+\|\mu\|\,|y|)(k_1+k_2)}{\Gamma(\gamma+1)}\ |x-y| \\
&\leq c_2(t)\ |x-y|
\end{aligned}$$

Hence,

$$\begin{aligned}
|v(t,x(t))-v(t,y(t))| &\leq |f(t,x(t))-f(t,y(t))|+|E_1(t)|+\frac{t}{T}|E_2(t)| \\
&\leq \left(\sigma(t)+c_1(t)+\frac{t}{T}c_2(t)\right)\ |x-y|
\end{aligned}$$

Taking supremum for all $t \in [0,T]$, we get

$$\begin{aligned}
|\mathcal{C}x(t)-\mathcal{C}y(t)| &\leq (\|\sigma\|+\|c_1\|+\|c_2\|)\ \|x-y\| \\
&\leq \left(\|\sigma\|+\frac{\begin{array}{c}\|\mu\|\left[(2H_1+H_2)+(2k_1+k_2)\rho\right] \\ +(\widetilde{G}+\|\mu\|\rho)(2k_1+k_2)\end{array}}{\Gamma(\gamma+1)}\right)\|x-y\|,
\end{aligned}$$

Therefore, operator $\mathcal{C}$ is a Lipschitzian mapping on X.

Step 2. Operator $\mathcal{B}$ is a compact and continuous on S into S.
First, we show that operator $\mathcal{B}$ is continuous on X as follows:

Let $\{x_n\}$ be a sequence in S that converges to point $x \in S$. Since $u(t,x(t))$ is continuous in X for all $t \in T$, then, by assumption (A_2), we have $u(t,x_n(t))$ converges to $u(t,x(t))$. Hence, by applying the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} I^{\beta}u(s,x_n(s)) = I^{\beta}u(s,x(s)).$$

In addition, since $h(t,x(t))$ is continuous in x, then by the fractional-order integral properties, and by applying Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{n\to\infty}\mathcal{B}x_n(t) &= \lim_{n\to\infty}\int_0^T G(t,s)\,h(s,I^\beta u(s,x_n(s))ds\\ &= \int_0^T G(t,s)\lim_{n\to\infty}h(s,I^\beta u(s,x_n(s))ds\\ &= \int_0^T G(t,s)\,h(s,I^\beta u(s,x(s))ds\\ &= \mathcal{B}x(t).\end{aligned}$$

Thus, $\mathcal{B}x_n \to \mathcal{B}x$ as $n\to\infty$ uniformly on R_+. This implies for all $t\in I$ that the operator $\mathcal{B}$ is a continuous operator on S into S.
Next, we show that operator $\mathcal{B}$ is a compact operator on S. To do this, it is enough to prove that $\mathcal{B}(S)$ is a uniformly bounded and equicontinuous set in X.

Let $x\in S$ be arbitrary. Then by assumption (A_2), we have

$$\begin{aligned}|\mathcal{B}x(t)| &= \left|\int_0^T G(t,s)\,h(s,I^\beta u(s,x(s))ds\right|\\ &\le \int_0^T |G(t,s)|\,\left|h(s,I^\beta u(s,x(s))\right|ds\\ &\le G_0\int_0^T\Big(\boldsymbol{a}(s)+\boldsymbol{b}(s)\,I^\beta\,|u(s,x(s))|\Big)ds\\ &\le G_0\int_0^T|\boldsymbol{a}(s)|\,ds+G_0\int_0^T|\boldsymbol{b}(s)|\,I^\beta\,|u(s,x(s))|\,ds\\ &\le G_0\,\|\boldsymbol{a}\|\int_0^T ds+G_0\,\|\boldsymbol{b}\|\int_0^T I^\beta\,|u(s,x(s))|\,ds\\ &\le G_0\,\|\boldsymbol{a}\|\,T+G_0\,\|\boldsymbol{b}\|\int_0^T I^\beta\boldsymbol{m}(s)\,ds\\ &\le G_0\,\|\boldsymbol{a}\|\,T+G_0\,\|\boldsymbol{b}\|\int_0^T I^{\beta-\gamma}I^\gamma\boldsymbol{m}(s)\,ds\\ &\le G_0T\ \|\boldsymbol{a}\|+G_0\,\|\boldsymbol{b}\|\ \mathcal{M}\int_0^T\int_0^t\frac{(t-u)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}du\ ds\\ &\le G_0T\ \|\boldsymbol{a}\|+G_0\,\|\boldsymbol{b}\|\,T\ \mathcal{M}\int_0^t\frac{(t-u)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)}du\\ &\le G_0T\left(\|\boldsymbol{a}\|+\|\boldsymbol{b}\|\ \mathcal{M}\frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right).\end{aligned}$$

Taking supermom over all $t\in I$, we get for all $x\in S$

$$\|\mathcal{B}x\|\le G_0T\left(\ \|\boldsymbol{a}\|+\|\boldsymbol{b}\|\,\mathcal{M}\frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right).$$

Thus, operator $\mathcal{B}$ is uniformly bounded on S.
Now, we continue by showing that $\mathcal{B}(S)$ is also an equicontinuous set in X.

Let $t_1,\ t_2 \in I$ such that $t_1 < t_2$. Then, we have for all $x \in S$:

$$
\begin{aligned}
&\mathcal{B}x(t_2) - Bx(t_1)\\
= &\int_0^T G(t_2,s)\, h(s, I^\beta u(s,x(s))ds - \int_0^T G(t_1,s)\, h(s, I^\beta u(s,x(s))ds\\
&= \int_0^{t_2} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_2(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&\quad + \int_{t_2}^T \left[-\frac{t_2(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&- \int_0^{t_1} \left[\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_1(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&\quad - \int_{t_1}^T \left[-\frac{t_1(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))\\
&= \int_0^{t_1} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_2(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&+ \int_{t_1}^{t_2} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_2(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&\quad + \int_{t_2}^T \left[-\frac{t_2(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&- \int_0^{t_1} \left[\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{t_1(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&\quad - \int_{t_1}^{t_2} \left[-\frac{t_1(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds\\
&\quad - \int_{t_2}^T \left[-\frac{t_1(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] h(s, I^\beta u(s,x(s)))ds
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|\mathcal{B}x(t_2) - Bx(t_1)|\\
&\leq \int_0^{t_1} \left[\frac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-t_1)(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] \left|h(s, I^\beta u(s,x(s)))\right| ds\\
&\quad + \int_{t_1}^{t_2} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-t_1)(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] \left|h(s, I^\beta u(s,x(s)))\right| ds\\
&\quad + \int_{t_2}^{T} \left[\frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] \left|h(s, I^\beta u(s,x(s)))\right| ds\\
&\leq \int_0^{t_1} \left[\begin{array}{c}\dfrac{(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}}{\Gamma(\alpha)}\\ -\dfrac{(t_2-t_1)(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\end{array}\right] \Big(|\boldsymbol{a}(s)| + |\boldsymbol{b}(s)|\, I^\beta\, |u(s,x(s))|\Big) ds\\
&+ \int_{t_1}^{t_2} \left[\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)} - \frac{(t_2-t_1)(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] \Big(|\boldsymbol{a}(s)| + |\boldsymbol{b}(s)|\, I^\beta\, |u(s,x(s))|\Big) ds\\
&\quad + \int_{t_2}^{T} \left[\frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\ \Gamma(\alpha)}\right] \Big(|\boldsymbol{a}(s)| + |\boldsymbol{b}(s)|\, I^\beta\, |u(s,x(s))|\Big) ds
\end{aligned}
$$

$$\leq \|\boldsymbol{a}\| \left[\int_0^{t_1} \left(\frac{T\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right]-(t_2-t_1)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \right) ds \right.$$

$$\left. + \int_{t_1}^{t_2} \frac{\begin{pmatrix} T(t_2-s)^{\alpha-1} \\ -(t_2-t_1)(T-s)^{\alpha-1} \end{pmatrix}}{T\,\Gamma(\alpha)} ds + \int_{t_2}^{T} \frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} ds \right]$$

$$+ \|\boldsymbol{b}\| \left[\int_0^{t_1} \frac{\begin{pmatrix} T\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right] \\ -(t_2-t_1)(T-s)^{\alpha-1} \end{pmatrix}}{T\,\Gamma(\alpha)} I^{\beta} |u(s,x(s))|\, ds \right.$$

$$+ \int_{t_1}^{t_2} \left(\frac{T(t_2-s)^{\alpha-1}-(t_2-t_1)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \right) I^{\beta} |u(s,x(s))|\, ds$$

$$\left. + \int_{t_2}^{T} \frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} I^{\beta} |u(s,x(s))|\, ds \right]$$

$$\leq \|\boldsymbol{a}\| \left[\frac{(t_2^{\alpha}-t_1^{\alpha})T-(t_2-t_1)T^{\alpha}}{T\,\Gamma(\alpha+1)} \right]$$

$$+ \|\boldsymbol{b}\| \left[\int_0^{t_1} \frac{\begin{pmatrix} T\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right] \\ -(t_2-t_1)(T-s)^{\alpha-1} \end{pmatrix}}{T\,\Gamma(\alpha)} I^{\beta-\gamma} I^{\gamma} \boldsymbol{m}(s)\, ds \right.$$

$$+ \int_{t_1}^{t_2} \left(\frac{T(t_2-s)^{\alpha-1}-(t_2-t_1)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \right) I^{\beta-\gamma} I^{\gamma} \boldsymbol{m}(s)\, ds$$

$$\left. + \int_{t_2}^{T} \frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} I^{\beta-\gamma} I^{\gamma} \boldsymbol{m}(s)\, ds \right]$$

$$\leq \|\boldsymbol{a}\| \left[\frac{(t_2^{\alpha}-t_1^{\alpha})T-(t_2-t_1)T^{\alpha}}{T\,\Gamma(\alpha+1)} \right]$$

$$+ \|\boldsymbol{b}\| \mathcal{M} \left[\int_0^{t_1} \frac{\begin{pmatrix} T\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right] \\ -(t_2-t_1)(T-s)^{\alpha-1} \end{pmatrix}}{T\,\Gamma(\alpha)} \int_0^{s} \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} d\tau ds \right.$$

$$+ \int_{t_1}^{t_2} \left(\frac{T(t_2-s)^{\alpha-1}-(t_2-t_1)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \right) \int_0^{s} \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} d\tau ds$$

$$\left. + \int_{t_2}^{T} \frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \int_0^{s} \frac{(s-\tau)^{\beta-\gamma-1}}{\Gamma(\beta-\gamma)} d\tau ds \right]$$

$$\leq \|\boldsymbol{a}\| \left[\frac{(t_2^{\alpha}-t_1^{\alpha})T-(t_2-t_1)T^{\alpha}}{T\,\Gamma(\alpha+1)} \right]$$

$$+ \|\boldsymbol{b}\| \mathcal{M} \left[\int_0^{t_1} \frac{\begin{pmatrix} T\left[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\right] \\ -(t_2-t_1)(T-s)^{\alpha-1} \end{pmatrix}}{T\,\Gamma(\alpha)} \frac{s^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right.$$

$$+ \int_{t_1}^{t_2} \left(\frac{T(t_2-s)^{\alpha-1}-(t_2-t_1)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \right) \frac{s^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds$$

$$\left. + \int_{t_2}^{T} \frac{(t_1-t_2)(T-s)^{\alpha-1}}{T\,\Gamma(\alpha)} \frac{s^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)} ds \right]$$

$$\leq \|\boldsymbol{a}\| \left[\frac{(t_2^{\alpha}-t_1^{\alpha})T-(t_2-t_1)T^{\alpha}}{T\,\Gamma(\alpha+1)} \right]$$

$$+\|\boldsymbol{b}\|\,\mathcal{M}\left[\frac{(t_2^{\alpha}-t_1^{\alpha})T-(t_2-t_1)T^{\alpha}}{T\ \Gamma(\alpha+1)\Gamma\left(\beta-\gamma+1\right)}\right]T^{\beta-\gamma}$$

Therefore,

$$|\mathcal{B}x(t_2)-\mathcal{B}x(t_1)|\leq\left(\frac{(t_2^{\alpha}-t_1^{\alpha})T-(t_2-t_1)T^{\alpha}}{T\ \Gamma(\alpha+1)}\right)\left(\|\boldsymbol{a}\|+\frac{\|\boldsymbol{b}\|\,\mathcal{M}T^{\beta-\alpha}}{\Gamma\left(\beta-\gamma+1\right)}\right).$$

Hence, for $\varepsilon>0$, there exists a $\delta>0$ such that

$$|t_2-t_1|<\delta\implies|\mathcal{B}x(t_2)-\mathcal{B}x(t_1)|<\varepsilon$$

for all $t_1,t_2\in I$ and for all $x\in S$. This shows that $\mathcal{B}(S)$ is an equicontinuous set in X.

By the Arzela-Ascoli theorem, we dedcue that $\mathcal{B}(S)$ is a uniformly bounded and equicontinuous set in X, and hence it is compact. Thus, we already proved that the operator $\mathcal{B}$ is a complete continuous operator on S.

Step 3. In the following, we show that the operator $x=\mathcal{A}x\mathcal{B}y+\mathcal{C}x$ is bounded for all $x\in X$ and $y\in S$.

Let $t\in I$, then

$$\begin{aligned}
|x(t)|&\leq|\mathcal{A}x(t)||\mathcal{B}y(t)|+|\mathcal{C}x(t)|\\
&\leq|g(t,x(t))|\int_0^T|G\left(t,s\right)|\ \ |h(s,I^{\beta}u(s,x(s))|ds+|v(s,x(s))|\\
&\leq[|g(t,x(t))-g(t,0)|+|g(t,0)|]\int_0^T|G\left(t,s\right)|\ \ |h(s,I^{\beta}\ u(s,x\left(s\right))|ds\\
&\qquad+|v\left(x,t\right)-v\left(x,0\right)+v\left(x,0\right)|\\
&\leq[|g(t,x(t))-g(t,0)|+|g(t,0)|]\int_0^T G\left(t,s\right)\ \ |h(s,I^{\beta}\ u(s,x\left(s\right))|ds\\
&\quad+|f\left(x,t\right)-f\left(x,0\right)+f\left(x,0\right)|+\frac{|g(t,x(t))|}{\Gamma\left(\gamma\right)}\left[\int_0^1\left(1-s\right)^{\gamma-1}|h_1\left(s,x\left(s\right)\right)|\,ds\right.\\
&\qquad\left.\int_0^1\left(1-s\right)^{\gamma-1}|h_2\left(s,x\left(s\right)\right)-h_1\left(s,x\left(s\right)\right)|\,ds\right]\\
&\leq[|g(t,x(t))-g(t,0)|+|g(t,0)|]\int_0^T|G\left(t,s\right)|\ \ |\boldsymbol{a}\left(s\right)+\boldsymbol{b}\left(s\right)I^{\beta}\,|u(s,x\left(s\right))|\,ds\\
&\quad+|f\left(x,t\right)-f\left(x,0\right)+f\left(x,0\right)|+\frac{|g(t,x(t))|}{\Gamma\left(\gamma\right)}\left[\int_0^1\left(1-s\right)^{\gamma-1}\left(k_1|x|+H_1\right)ds\right.\\
&\qquad\left.+\int_0^1\left(1-s\right)^{\gamma-1}\left(k_2|x\left(s\right)|+H_2+k_1|x\left(s\right)|+H_1\right)ds\right]\\
&\leq\left(\|\mu\|\,|x\left(t\right)|+H_0\right)\int_0^T G_0\ \left(|\boldsymbol{a}\left(s\right)+\boldsymbol{b}\left(s\right)I^{\beta}\boldsymbol{m}\left(s\right)\right)ds+\left(\|\sigma\|\,|x\left(t\right)|+F\right)\\
&\qquad+\left(\|\mu\|\,|x\left(t\right)|+H_0\right)\left[\frac{2\left(k_1|x\left(t\right)|+H_1\right)+\left(k_2|x\left(t\right)|+H_2\right)}{\Gamma\left(\gamma+1\right)}\right]\\
&\leq\left(\|\sigma\|\,|x\left(t\right)|+F\right)+\left(\|\mu\|\,|x\left(t\right)|+H_0\right)\int_0^T G_0\ \left(|\boldsymbol{a}\left(s\right)+\boldsymbol{b}\left(s\right)I^{\beta-\gamma}I^{\gamma}\boldsymbol{m}\left(s\right)\right)ds\\
&\qquad+\left(\|\mu\|\,|x\left(t\right)|+H_0\right)\left[\frac{2\left(k_1|x\left(t\right)|+H_1\right)+\left(k_2|x\left(t\right)|+H_2\right)}{\Gamma\left(\gamma+1\right)}\right]\\
&\leq\left(\|\sigma\|\,|x\left(t\right)|+F\right)+\left(\|\mu\|\,|x\left(t\right)|+H_0\right)G_0\left(\|\boldsymbol{a}\|\,T+\|\boldsymbol{b}\|\,\mathcal{M}T\ I^{\beta-\gamma}\left(s\right)\right)
\end{aligned}$$

$$+\left(\|\mu\|\left|x\left(t\right)\right|+H_0\right)\left[\frac{\left(2k_1+k_2\right)\left|x\left(t\right)\right|+\left(2H_1+H_2\right)}{\Gamma\left(\gamma+1\right)}\right]$$

$$\leq\left(\|\sigma\|\left|x\left(t\right)\right|+F\right)+\left(\|\mu\|\left|x\left(t\right)\right|+H_0\right)G_0T\left(\|\boldsymbol{a}\|+\|\boldsymbol{b}\|\mathcal{M}\int_0^s\frac{\left(s-u\right)^{\beta-\gamma-1}}{\Gamma\left(\beta-\gamma\right)}du\right)$$

$$+\left(\|\mu\|\left|x\left(t\right)\right|+H_0\right)\left[\frac{\left(2k_1+k_2\right)\left|x\left(t\right)\right|+\left(2H_1+H_2\right)}{\Gamma\left(\gamma+1\right)}\right]$$

Taking supremum over all $t\in I$, we get

$$\left|x\left(t\right)\right|\leq\left(\|\sigma\|\|x\|+F\right)+\left(\|\mu\|\|x\|+H_0\right)G_0T\left(\|\boldsymbol{a}\|+\|\boldsymbol{b}\|\mathcal{M}\frac{T^{\beta-\gamma}}{\Gamma\left(\beta-\gamma+1\right)}\right)$$

$$+\left(\|\mu\|\|x\|+H_0\right)\left[\frac{\left(2k_1+k_2\right)\|x\|+\left(2H_1+H_2\right)}{\Gamma\left(\gamma+1\right)}\right]$$

Hence,

$$\|x\|\leq\left(\|\sigma\|\|x\|+F\right)+\left(\|\mu\|\|x\|+H_0\right)\left[G_0T\left(\|\boldsymbol{a}\|+\|\boldsymbol{b}\|\mathcal{M}\frac{T^{\beta-\gamma}}{\Gamma\left(\beta-\gamma+1\right)}\right)\right.$$

$$\left.+\frac{\left(2k_1+k_2\right)\rho+\left(2H_1+H_2\right)}{\Gamma\left(\gamma+1\right)}\right]\leq\rho$$

which implies that

$$\rho\geq\frac{1}{2\|\boldsymbol{a}\|\|\mu\|\aleph}\left(1-\|\boldsymbol{a}\|H_0\aleph-\frac{T^{\gamma}}{\Gamma\left(\gamma+1\right)}\|\mu\|-\|\boldsymbol{b}\|\|\mu\|\aleph-\sqrt{\Delta}\right),$$

where

$$H_0=\sup_{t\in J}|g(t,0)|,$$

$$\aleph=G_0T\left[\|\boldsymbol{a}\|+\mathcal{M}\|\boldsymbol{b}\|\frac{T^{\beta-\gamma}}{\Gamma\left(^{\beta-\gamma}+1\right)}\right],$$

and

$$\begin{aligned}\Delta &= -4\|\boldsymbol{a}\|\aleph H_0\left(\frac{T^{\gamma}}{\Gamma\left(\gamma+1\right)}+\|\boldsymbol{a}\|\aleph\right)\|\mu\|\\ &+\left[\left(\left(\|\boldsymbol{b}\|\|\mu\|\aleph\right)+\frac{T^{\gamma}}{\Gamma\left(\gamma+1\right)}\right)\|\mu\|+\|\boldsymbol{a}\|H_0\aleph-1\right]^2\end{aligned}$$

Therefore, $x\in S$.

Step 4. Finally we show that $\eta\mathcal{N}+\mathcal{R}<\rho$, such that

$$\mathcal{N}=\|\mathcal{B}(S)\|=\sup_{x\in S}\left\{\sup_{t\in I}|\mathcal{B}x(t)|\right\}\leq G_0T\left(\|\boldsymbol{a}\|+\|\boldsymbol{b}\|\mathcal{M}\frac{T^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right)$$

and by (A_3) we have

$$\mathcal{L}\mathcal{M}+\mathcal{K}<1,$$

where

$$\mathcal{L}=\left(\|\boldsymbol{a}\|H_0+\|\boldsymbol{a}\|\|\mu\|\right)$$

and

$$\mathcal{K}=\frac{T^{\gamma}}{\Gamma(\gamma+1)}\|\mu\|+\sqrt{\begin{array}{c}-4\|\boldsymbol{a}\|\aleph H_0\left(\frac{T^{\gamma}}{\Gamma(\gamma+1)}+\|\boldsymbol{b}\|\aleph\right)\|\mu\|\\ +\left[\left((\|\boldsymbol{b}\|\|\mu\|\aleph)+\frac{T^{\gamma}}{\Gamma(\gamma+1)}\right)\|\mu\|+\|\boldsymbol{a}\|H_0\aleph-1\right]^2\end{array}}.$$

Hence, the last condition of Theorem 2.1 is satisfied with $\eta=(2\|\boldsymbol{a}\|\|\mu\|\aleph)\mathcal{L}$, and $\mathcal{R}=(2\|\boldsymbol{a}\|\|\mu\|\aleph)\mathcal{K}$.

Therefore, all the conditions of Theorem 2.1 are satisfied and hence the operator equation $x=\mathcal{A}x\mathcal{B}y+\mathcal{C}x$ has a solution in S. In consequence, problem (1) has at least one mild solution on I. This completes the proof. □

3.2. **Uniqueness of the solution.** In the following, we give sufficient conditions for the uniqueness of the solution of the quadratic functional integral equation (3).

Consider the following assumption:

(A_7) Let $h:[0,T]\times R\to R$ and $u:[0,T]\times R\to R$ be a continuous functions satisfying the Lipschitz condition and there exists two positive functions $w(t)$, $\theta(t)$ with bounded $\|w\|$ and $\|\theta\|$, such that

$$|h(t,x)-h(t,y)|\le w(t)|x-y|$$
$$|u(t,x)-u(t,y)|\le \theta(t)|x-y|$$

with $H=\sup_{t\in I}|h(t,0)|$, and $U=\sup_{t\in I}|u(t,0)|$.

Theorem 3.2. *Assume that (A_0), (A_1), (A_3), (A_6) and (A_7) hold, then the solution $x\in C[0,T]$ of the quadratic functional integral equation (3) is unique, if*

$$\left(\begin{array}{c}\|\mu\|+\frac{(H_0+\|\mu\|\|x\|)}{\Gamma(\gamma+1)}(2k_1+k_2)+\|\mu\|G_0\left[\|w\|\mathcal{M}\frac{T^{\beta-\gamma+1}}{\Gamma(\beta-\gamma+1)}+H\,T\right]\\ +[\|\mu\|\|y\|+H_0]\|w\|\|\theta\|G_0\frac{T^{\beta+1}}{\Gamma(\beta+1)}\end{array}\right)<1 \tag{8}$$

Proof. Suppose that $x(t)$ and $y(t)$ are two solutions of (3), then

$$\begin{aligned}
&|x(t)-y(t)|\\
&\le|v(t,x(t))-v(t,y(t))|\\
&+\left|\begin{array}{c}g(t,x(t))\int_0^T|G(t,s)|\;h\left(s,I^{\beta}u(s,x(s))\right)ds\\ -g(t,y(t))\int_0^T|G(t,s)|\;h\left(s,I^{\beta}u(s,y(s))\right)ds\end{array}\right|\\
&\le|v(t,x(t))-v(t,y(t))|\\
&+|g(t,x(t))-g(t,y(t))|\int_0^T|G(t,s)|\;\left|h\left(s,I^{\beta}u(s,x(s))\right)\right|ds\\
&+|g(t,y(t))|\int_0^T|G(t,s)|\;\left|h\left(s,I^{\beta}u(s,x(s))\right)-h\left(s,I^{\beta}u(s,y(s))\right)\right|ds\\
&\le|\mu(t)||x(t)-y(t)|+\frac{(H_0+|\mu(t)||x(t)|)}{\Gamma(\gamma+1)}\left[\begin{array}{c}2|h_1(t,x(t))-h_1(t,y(t))|\\ +|h_2(t,x(t))-h_2(t,y(t))|\end{array}\right]\\
&+|g(t,x(t))-g(t,y(t))|\int_0^T|G(t,s)|\;\left[\begin{array}{c}\left|h\left(s,I^{\beta}u(s,x(s))\right)-h(s,0)\right|\\ +|h(s,0)|\end{array}\right]ds
\end{aligned}$$

$$+ \left[|g(t, y(t)) - g(t, 0)| + |g(t, 0)| \right] \int_0^T |G(t, s)| \begin{bmatrix} h\left(s, I^\beta u(s, x(s))\right) \\ -h\left(s, I^\beta u(s, y(s))\right) \end{bmatrix} ds$$

$$\leq |\mu(t)| |x(t) - y(t)| + \frac{(H_0 + |\mu(t)| |x(t)|)}{\Gamma(\gamma + 1)} \left[2k_1 |x(t) - y(t)| + k_2 |x(t) - y(t)| \right]$$

$$+ |\mu(t)| |x(t) - y(t)| \int_0^T |G(t, s)| \left[|w(s)| \ I^\beta \ |u(s, x(s))| + H \right] ds$$

$$+ \left[|\mu(t)| |y(t)| + H_0 \right] \int_0^T |G(t, s)| \ |w(s)| \ I^\beta \ |u(s, x(s)) - u(s, y(s))| \ ds$$

$$\leq |\mu(t)| |x(t) - y(t)| + \frac{(H_0 + |\mu(t)| |x(t)|)}{\Gamma(\gamma + 1)} [2k_1 + k_2] \ |x(t) - y(t)|$$

$$+ |\mu(t)| |x(t) - y(t)| \int_0^T |G(t, s)| \ \left[|w(s)| I^{\beta - \gamma} I^\gamma \boldsymbol{m}(s) + H \right] ds$$

$$+ \left[|\mu(t)| |y(t)| + H_0 \right] \int_0^T |G(t, s)| \ |w(s)| \ |\theta(t)| I^\beta |x(t) - y(t)| \ ds$$

$$\leq |\mu(t)| |x(t) - y(t)| + \frac{(H_0 + |\mu(t)| |x(t)|)}{\Gamma(\gamma + 1)} [2k_1 + k_2] \ |x(t) - y(t)|$$

$$+ |\mu(t)| |x(t) - y(t)| \int_0^T |G(t, s)| \ \left[|w(s)| I^{\beta - \gamma} \mathcal{M} + H \right] ds$$

$$+ \left[|\mu(t)| |y(t)| + H_0 \right] \int_0^T |G(t, s)| \ |w(s)| \ |\theta(s)| I^\beta |x(s) - y(s)| \ ds$$

Taking supremum over $t \in I$, we get

$$\|x - y\| \leq \|\mu\| \|x - y\| + \frac{(H_0 + \|\mu\| \|x\|)}{\Gamma(\gamma + 1)} [2k_1 + k_2] \ \|x - y\|$$

$$+ \|\mu\| \|x - y\| G_0 \int_0^T \left[\|w\| \mathcal{M} I^{\beta - \gamma} + H \right] ds$$

$$+ \left[\|\mu\| \|y\| + H_0 \right] \|w\| \|\theta\| \|x - y\| G_0 \int_0^T I^\beta \ ds$$

$$\leq \|\mu\| \|x - y\| + \frac{(H_0 + \|\mu\| \|x\|)}{\Gamma(\gamma + 1)} [2k_1 + k_2] \ \|x - y\|$$

$$+ \|\mu\| \|x - y\| G_0 \left[\|w\| \mathcal{M} \int_0^T \int_0^s \frac{(t - \tau)^{\beta - \gamma - 1}}{\Gamma(\beta - \gamma)} d\tau ds + H \int_0^T ds \right]$$

$$+ \left[\|\mu\| \|y\| + H_0 \right] \|w\| \|\theta\| \|x - y\| G_0 \int_0^T \int_0^s \frac{(s - \tau)^{\beta - 1}}{\Gamma(\beta)} d\tau \ ds$$

$$\leq \|\mu\| \|x - y\| + \frac{(H_0 + \|\mu\| \|x\|)}{\Gamma(\gamma + 1)} [2k_1 + k_2] \ \|x - y\|$$

$$+ \|\mu\| \|x - y\| G_0 \ \left[\|w\| \mathcal{M} \frac{T^{\beta - \gamma + 1}}{\Gamma(\beta - \gamma + 1)} + H \ T \right]$$

$$+ \left[\|\mu\| \|y\| + H_0 \right] \|w\| \|\theta\| \|x - y\| G_0 \frac{T^{\beta + 1}}{\Gamma(\beta + 1)}$$

Then,

$$\|x-y\|\left(\begin{array}{c} 1-\|\mu\|-\frac{(H_0+\|\mu\|\|x\|)}{\Gamma(\gamma+1)}\left[2k_1+k_2\right] \\ -\|\mu\|\,G_0\,\left[\|w\|\,\mathcal{M}\frac{T^{\beta-\gamma+1}}{\Gamma(\beta-\gamma+1)}+H\ T\right] \\ -\left[\|\mu\|\,\|y\|+H_0\right]\|w\|\,\|\theta\|\,G_0\frac{T^{\beta+1}}{\Gamma(\beta+1)} \end{array}\right)\leq 0$$

This proves the uniqueness of the solution of quadratic integral equation (3). □

3.3. **Hyers-Ulam Stability of Solutions.** In the following, we study the Ulam stability for the (IBFHDE) (1).

Definition 3.2. *The problem (IBFHDE) (1) is said to be Hyers-Ulam stable if there exists a positve real number c_f satisfying for each $\epsilon > 0$, if*

$$|z(t)-v(t,z(t))-g\left(t,z\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\ ds|\leq\epsilon, \quad (9)$$

then then there exists a $y\left(t\right)$ satisfying

$$y(t)-v(t,y(t))=g\left(t,y\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,y\left(s\right)\right)\right)\ ds$$

such that $|z(t)-y(t)|\leq\epsilon\ c_f,\ t\in I$.

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied. Then problem (IBFHDE) (1) is Hyers-Ulam stable.*

Proof. Let $\epsilon > 0$ and let $z\in C(I,R)$ be a function which satisfies inequality (9), i.e.,

$$|z(t)-v(t,z(t))-g\left(t,z\left(t\right)\right)\int_0^T G(t,s)\,h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\,ds|\leq\epsilon \text{ for all } t\in I \quad (10)$$

and let $y\in C(I,R)$ be the unique solution of (IBFHDE) (1) which is by Lemma Lemma 3.1 is equivalent to the fractional order integral equation

$$y(t)=v(t,y(t))+g\left(t,y\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,y\left(s\right)\right)\right)\ ds,$$

Appling I^{α} on both sides of (10), we get

$$|z(t)-v(t,z(t))-g\left(t,z\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\ ds|\leq\frac{\epsilon\ T^{\alpha}}{\Gamma(\alpha+1)}. \quad (11)$$

This implies that for each $t\in I$, we have:

$$\begin{aligned} &|z(t)-y(t)| \\ &=|z(t)-v(t,y(t))-g\left(t,y\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,y\left(s\right)\right)\right)\ ds| \\ &=|z(t)-v(t,z(t))-g\left(t,z\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\ ds \\ &\quad +v(t,z(t))+g\left(t,z\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\ ds \end{aligned}$$

$$-v(t,y(t))-g\left(t,y\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,y\left(s\right)\right)\right)\ ds|$$

$$\le |z(t)-v(t,z(t))-g\left(t,z\left(t\right)\right)\int_0^T G(t,s)\ h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\ ds|$$

$$+|v(t,z(t))-v(t,y(t))|$$

$$+\left|g\left(t,z\left(t\right)\right)-g\left(t,y\left(t\right)\right)\right|\int_0^T |G(t,s)|\ \left|h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\right|\ ds$$

$$+\left|g\left(t,y\left(t\right)\right)\right|\int_0^T |G(t,s)|\ \left|h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)-h\left(s,I^{\beta}u\left(s,y\left(s\right)\right)\right)\right|\ ds$$

$$\le \frac{\epsilon\ T^{\alpha}}{\Gamma(\alpha+1)}+\left|\mu\left(t\right)\right|\left|z(t)-y(t)\right|+\frac{\left(H_0+\left|\mu\left(t\right)\right|\left|z\left(t\right)\right|\right)}{\Gamma\left(\gamma+1\right)}\left[2k_1+k_2\right]\ \left|z\left(t\right)-y\left(t\right)\right|$$

$$+\left|g\left(t,z\left(t\right)\right)-g\left(t,y\left(t\right)\right)\right|\int_0^T |G(t,s)|\ \left[\begin{array}{c}\left|h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)-h\left(s,0\right)\right|\\+\left|h\left(s,0\right)\right|\end{array}\right]ds$$

$$+\left|g\left(t,y\left(t\right)\right)-g\left(t,0\right)+g\left(t,0\right)\right|\int_0^T |G(t,s)|\ \left|\begin{array}{c}h\left(s,I^{\beta}u\left(s,z\left(s\right)\right)\right)\\-h\left(s,I^{\beta}u\left(s,y\left(s\right)\right)\right)\end{array}\right|\ ds$$

$$\le \frac{\epsilon\ T^{\alpha}}{\Gamma(\alpha+1)}+\left|\mu\left(t\right)\right|\left|z(t)-y(t)\right|+\frac{\left(H_0+\left|\mu\left(t\right)\right|\left|z\left(t\right)\right|\right)}{\Gamma\left(\gamma+1\right)}\left[2k_1+k_2\right]\ \left|z\left(t\right)-y\left(t\right)\right|$$

$$+\left|\mu\left(t\right)\right|\left|z(t)-y(t)\right|\int_0^T |G(t,s)|\ \left[|w(s)|\ \ I^{\beta}\ \ \left|u\left(s,z\left(s\right)\right)\right|+H\right]ds$$

$$+\left[\left|\mu\left(t\right)\right|\left|y\left(t\right)\right|+H_0\right]\int_0^T |G(t,s)|\ \ |w(s)|\ \ I^{\beta}\ \ \left|u\left(s,z\left(s\right)\right)-u\left(s,y\left(s\right)\right)\right|\ ds$$

$$\le \frac{\epsilon\ T^{\alpha}}{\Gamma(\alpha+1)}+\left|\mu\left(t\right)\right|\left|z(t)-y(t)\right|+\frac{\left(H_0+\left|\mu\left(t\right)\right|\left|z\left(t\right)\right|\right)}{\Gamma\left(\gamma+1\right)}\left[2k_1+k_2\right]\ \left|z\left(t\right)-y\left(t\right)\right|$$

$$+\left|\mu\left(t\right)\right|\left|x(t)-y(t)\right|\int_0^T |G(t,s)|\ \left[|w(s)|\,I^{\beta-\gamma}I^{\gamma}\boldsymbol{m}\left(s\right)+H\right]ds$$

$$+\left[\left|\mu\left(t\right)\right|\left|y\left(t\right)\right|+H_0\right]\int_0^T |G(t,s)|\ \ |w(s)|\ \ \left|\theta\left(s\right)\right|I^{\beta}\left|z\left(s\right)-y\left(s\right)\right|\ ds$$

Taking supremum over $t\in I$, we getThus, if we take supremum for all $t\in I$, we get

$$\|z-y\|\le\frac{\epsilon\ T^{\alpha}}{\Gamma(\alpha+1)}+\|\mu\|\,\|z-y\|+\frac{\left(H_0+\|\mu\|\,\|z\|\right)}{\Gamma\left(\gamma+1\right)}\left[2k_1+k_2\right]\|z-y\|$$

$$+\|\mu\|\,G_0\ \left[\|w\|\,\mathcal{M}\frac{T^{\beta-\gamma+1}}{\Gamma\left(\beta-\gamma+1\right)}+H\ T\right]\|z-y\|$$

$$+\left[\|\mu\|\,\|y\|+H_0\right]\|w\|\,\|\theta\|\,G_0\frac{T^{\beta+1}}{\Gamma\left(\beta+1\right)}\|z-y\|$$

Let

$$\zeta=1-\|\mu\|-\frac{\left(H_0+\|\mu\|\,\|z\|\right)}{\Gamma\left(\gamma+1\right)}\left[2k_1+k_2\right]$$

$$-\|\mu\|\,G_0\ \left[\|w\|\,\mathcal{M}\frac{T^{\beta-\gamma+1}}{\Gamma\left(\beta-\gamma+1\right)}+H\ T\right]$$

$$-\left[\|\mu\|\,\|y\|+H_0\right]\|w\|\,\|\theta\|\,G_0\frac{T^{\beta+1}}{\Gamma\left(\beta+1\right)},$$

we get

$$\|z-y\| \leq \left(\frac{T^{\alpha}}{\Gamma(\alpha+1)}\zeta^{-1}\right)\epsilon = c_f \quad \epsilon$$

Therefore, problem (IBFHDE) (1) is Hyers-Ulam stable. This completes the proof. □

4. Numerical Example

Example 4.1. *Consider the following fractional order implicit hybrid functional differential*

$$D^{\frac{2}{3}}\left(\frac{x(t)-\left[\frac{e^{-2t}}{10+t}+\frac{1}{3}|\cos(x(t))|\right]}{\frac{1}{\sqrt{10+2t}}+\frac{e^{-2t}}{100}|x(t)|}\right) = \frac{1}{2}\sin\left[\int_0^t \frac{(t-s)^{\frac{1}{3}-1}}{\Gamma\left(\frac{1}{3}\right)}se^{-s}ds+1\right]+\frac{x(t)}{e^t} \tag{12}$$

with the following with the initial hybrid boundary conditions:

$$\left.\frac{x(t)-\left[\frac{e^{-2t}}{10+t}+\frac{1}{3}|\cos(x(t))|\right]}{\frac{1}{\sqrt{10+2t}}+\frac{e^{-2t}}{100}|x(t)|}\right|_{t=0} = \frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^1 (1-s)^{\frac{1}{4}}\ \frac{\sin x(s)}{20}\ ds,$$

$$\left.\frac{x(t)-\left[\frac{e^{-2t}}{10+t}+\frac{1}{3}|\cos(x(t))|\right]}{\frac{1}{\sqrt{10+2t}}+\frac{e^{-2t}}{100}|x(t)|}\right|_{t=1} = \frac{1}{\Gamma\left(\frac{1}{4}\right)}\int_0^1 (1-s)^{\frac{1}{4}}\ \frac{e^{-x(s)}}{30}\ ds,,$$

It is easy to verfiy that:

$$f(t,x(t)) = \frac{e^{-2t}}{10+t}+\frac{1}{3}|\cos(x(t))|,\ \textit{such that}\ |f(t,x(t))| \leq \frac{1}{10+t}+\frac{|x(t)|}{3}$$

$$g(t,x(t)) = \frac{1}{\sqrt{10+2t}}+\frac{e^{-2t}}{100}|x(t)|,\ \textit{such that}\ |g(t,x(t))| \leq \frac{1}{\sqrt{10}}+\frac{|x(t)|}{100}$$

$$u(t,x(t)) = te^{-t} \leq t$$

$$f(t,x(t)) = \frac{e^{-2t}}{10+t}+\frac{1}{3}|\cos(x(t))|,\ \textit{such that}\ |f(t,x(t))| \leq \frac{1}{10+t}+\frac{|x(t)|}{3}$$

Thus, if

$$\begin{aligned}
m(t) &= \frac{1}{2}\sin\left[\int_0^t \frac{(t-s)^{\frac{1}{3}-1}}{\Gamma\left(\frac{1}{3}\right)}se^{-s}ds+1\right]\ \textit{with}\ |m(t)| = 1,\\
b(t) &= \frac{1}{e^t}\ \textit{with}\ |b(t)| = \frac{1}{e},\\
a(t) &= \frac{1}{2e^t}+\frac{e^t}{2}\frac{t^{\beta+1}}{\Gamma\left(\frac{5}{3}\right)}\ \textit{with}\ |a(t)| \leq \frac{1}{2e}+\frac{e}{2\ \Gamma\left(\frac{5}{3}\right)},\\
\mu(t) &= \frac{e^{-2t}}{100}\ \textit{with}\ \|\mu\| = \frac{1}{100},\\
\sigma(t) &= \frac{1}{3}\ \textit{with}\ \|\sigma\| = \frac{1}{3},\\
w(t) &= \frac{1}{e^t}\ \textit{with}\ \|w\| = \frac{1}{e},\\
\theta(t) &= 0\ \textit{with}\ \|\theta\| = 0
\end{aligned}$$

and,

$$\mathcal{M} \leq \frac{1}{\Gamma\left(\frac{5}{4}\right)},\ k_1 = \frac{1}{20},\ k_2 = \frac{1}{30},\ H_0 = \frac{1}{\sqrt{10}},\ H_1 = \frac{1}{20},\ H_2 = \frac{1}{30},$$

then we $\mathcal{LM}+\mathcal{K} \ = 0.861356 < 1$. *Therefore, in view of the assertion of Theorem 3.18, the implicit hybrid functional differential inclusion (12) has at least one mild solution* $x \in C\left[0,1\right]$ *such that* $\|x\| \leq 2.03183$. *In adition, the value in condition 8 is* $0.0710642 < 1$ *which implies that the solution is unique. Moreover, it is Hyers-Ulam stable with* $c_f = 0.928936$.

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