TWMS J. App. and Eng. Math. V.15, N.1, 2025, pp. 49-62

EXISTENCE OF PERIODIC SOLUTIONS FOR GENERALIZED NONLINEAR THIRD-ORDER DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this manuscript, we investigate the existence of solution to generalized third-order non-linear delay differential equation with periodic coefficients, which includes many important integral and functional equations that arise in nonlinear analysis and its applications. Our results are obtained by implementing Schauder's fixed point fixed point theorem and rely on a generalization of Ascoli-Arzelá theorem. Moreover, an example is offered to define the primary results.

Keywords: Generalization of Ascoli-Arzelá theorem, fixed point theorem, periodic solutions, third order delay differential equations.

AMS Subject Classification: 34K04, 34A34, 47H10, 34C25.

1. INTRODUCTION

The delay differential equations are frequently encountered as mathematical models of most dynamical processes in mechanics, control theory, physics, chemistry, biology, medicine, economics, atomic energy, information theory (see, for example [9, 11, 14, 15, 16, 18, 19, 23, 24, 27, 28]). In these models, time-delays are related to hidden processes like the stages of the life cycle, the time between infection and the generation of new viruses, the infectious period, the immune period, etc. Hale and Verduyn Lunel [11] pointed out very clearly the importance of the consideration of the delay in the modeling of many phenomena in physics and biology and [11] presented many examples of delay differential equations arise in the modeling of many phenomena. The presence of delay in these equations means that the solutions depend not only on the current time but also on its past values, this feature makes the analysis of delay differential equations more difficult

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§ Manuscript received: June 13, 2023; accepted: November 23, 2023.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.1;

than the differential equations.
Various physical models based on third order differential equations were presented in the recent book [21], for example, the following third order equation of the type

$$x^{'''}(t)-\lambda x^{'}(t)-2x(t)x^{'}(t)=\mu_1(Dx')(t)+\mu_2\sin(t), t\in\mathbb{R} \tag{1}$$

such that $x(t+2\pi)=x(t)$ describes the steady flow of water in a long rectangular tank, oscillating horizontally near a resonant frequency, Cox and Mortell [8] have considered (1) with $(Dx')(t)=x(t)$.
The third-order nonlinear Emden-Fowler delay differential model is written as [10]

$$x^{'''}(t)=h(t,x(t),x(t-\tau),x'(t-\tau(t)),x''(t-\tau(t))), t\in\mathbb{R} \tag{2}$$

Emden-Fowler delay differential model has a variety of applications in various fields.
The third order delay differential equations have been considered by many authors (see [1, 3, 7, 9, 10, 11, 20, 21] and the references therein). For example, Gregus [9] studied the third order linear differential equation, Ardjouni and Djoudi [1] studied the following second order nonlinear duffing equation with delay and variable coefficients:

$$x^{'''}(t)+p(t)x^{''}(t)+q(t)x^{'}(t)+r(t)x(t)=f(t,x(t),x(t-\tau(t)))+\frac{d}{dt}g(t,x(t-\tau(t))), t\in\mathbb{R} \tag{3}$$

Moreover, Nouioua et al. [20] have studied the third order nonlinear delay differential equation with periodic coefficients

$$x^{'''}(t)+p(t)x^{''}(t)+q(t)x^{'}(t)+r(t)x(t)=f(t,x(t),x(t-\tau(t)))+c(t)x'(t-\tau(t)), t\in\mathbb{R}, \tag{4}$$

by using Krasnoselskii's fixed point theorem.
In the current work, we consider the following more general form of nonlinear third delay differential equations

$$x^{'''}(t)+p(t)x^{''}(t)+q(t)x^{'}(t)+r(t)x(t)=f(t,x(t),x(t-\tau(t)),x'(t-\tau(t)),x''(t-\tau(t))), t\in\mathbb{R}. \tag{5}$$

This equation contains as particular cases all the above third order delay differential equation. Specifically, the important models (1) and (2).
We study the existence of the solution to the equation (5) where the derivatives $x', x^{''}$ appear in the nonlinear function, while the derivatives $x', x^{''}$ does not appear in the nonlinear functions of the equations (3) and (4). Our purpose here is to use a generalization of Ascoli-Arzelá theorem given in [4] and Schauder's fixed point theorem to show the existence of a periodic solution for equation (5) under fairly simple conditions.

2. Preliminaries and lemmas

In this section, we provide the following notations and definition.
Let E_1, E_2 be two finite dimensional Banach spaces endowed with the norms $\|.\|_1$, $\|.\|_2$ respectively, and X be a compact subset of E_1. We note by $C^2(X,E_2)$ the space of all functions from X to E_2 with continuous first derivative, this space is endowed with the norm $\|f\|=\sum_{i=0}^{2}\left\|f^{(i)}\right\|_\infty$ such that $\left\|f^{(i)}\right\|_\infty=\sup_{x\in X}\{\|f^{(i)}\|_2\}$ such that $f^{(i)}$ is ith derivative of the function f.
For our purpose, we need the following definition in $C^2(X,E_2)$.

Definition 2.1. [4] *The family $F \subset C^2(X, E_2)$ is called equicontinuous if for every $\epsilon > 0$ there is $\delta > 0$ such that $\left\|f^{(i)}(x) - f^{(i)}(y)\right\|_2 < \varepsilon$ for $i \in \{0, 1, 2\}$ and for all $x, y \in X$ satisfying $\|x - y\|_1 < \delta$.*
The family $F \subset C^2(X, E_2)$ is called equibounded if there is a constant M such that $\|f(x)\|_2 \leq M$ for all $f \in F$ and all $x \in X$.

The following result gives a generalization of Ascoli-Arzelá theorem in $C^2(X, E_2)$.

Theorem 2.1. [4] *Let F be a subset of $C^2(X, E_2)$. Then F is relatively compact if and only if F is equicontinuous and equibounded.*

Let T be a positive constant, we consider the spaces

$$P_T^0 = \{\psi \in C(\mathbb{R}), \psi(t + T) = \psi(t), \forall t \in \mathbb{R}\}.$$

$$P_T^1 = \{\psi \in C^1(\mathbb{R}), \psi(t + T) = \psi(t), \forall t \in \mathbb{R}\}.$$

$$P_T^2 = \{\psi \in C^2(\mathbb{R}), \psi(t + T) = \psi(t), \forall t \in \mathbb{R}\}.$$

It is clear that P_T^i $(i \in \{0, 1, 2\})$ is a Banach space endowed with the norm

$$\|x\| = \sum_{i=0}^{2} \sup_{t \in [0,T]} |x^{(i)}|.$$

Equation (5) will be studied under the following assumptions:

(H_1) $f \in C(\mathbb{R}^5, \mathbb{R})$ such that

$$f(t + T, x, y, z, s) = f(t, x, y, z, s), \quad \forall (t, x, y, z, s) \in \mathbb{R}^5.$$

(H_2) There exist $k_i > 0$, $i \in \{1, ..., 4\}$ and $\phi \in C(\mathbb{R}, \mathbb{R}^+)$ bounded such that

$$|f(t, u_1, u_2, u_3, u_4)| \leq \phi(t) + k_1|u_1| + k_2|u_2| + k_3|u_3| + k_4|u_4|, \quad \forall (t, u_1, u_2, u_3, u_4) \in \mathbb{R}^5.$$

(H_3) There exist differentiable positive T-periodic functions a_1 and a_2 and a positive real constant ρ such that

$$\begin{cases} a_1(t) + \rho = p(t) \\ a_1'(t) + a_2(t) + \rho a_1(t) = q(t), \\ a_2'(t) + \rho a_2(t) = r(t), \end{cases}$$

(H_4) $p, q, r, \tau : \mathbb{R} \longrightarrow \mathbb{R}^+$ are all continuous $T-$periodic functions, such that $\int_0^T p(s) > \rho T$, $\int_0^T q(s) > 0$.

Now, we consider the equation

$$x'''(t) + p(t)x'' + q(t)x' + r(t)x(t) = h(t),$$

where h is a continuous T periodic function. It is easy to check, (see [1, 3]), that by virtue of (H_3) and (H_4), the above equation can be transformed into the following system

$$\begin{cases} y'(t) + \rho y(t) = h(t), \\ x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t), \end{cases}$$

To obtain the main result, we need the following lemmas and Corollaries.

Lemma 2.1. [20] *Suppose that $y, h \in P_T$. Then y is a solution of equation*

$$y'(t) + \rho y(t) = h(t),$$

if and only if

$$y(t) = \int_t^{t+T} G_1(t,s)h(s)ds,$$

where,

$$G_1(t,s) = \frac{\exp(\rho(s-t))}{\exp(\rho T) - 1}.$$

Corollary 2.1. [20] *The Green's function G_1 satisfies the following properties:*

$$G_1(t+T, s+T) = G_1(t,s), G_1(t,t+T) = G_1(t,t)\exp(\rho T),$$
$$G_1(t+T, s) = G_1(t,s)\exp(-\rho T), G_1(t,t+T) = G_1(t,s)\exp(\rho T)$$
$$\frac{\partial G_1(t,s)}{\partial s} = \rho G_1(t,s), \frac{\partial G_1(t,s)}{\partial t} = -\rho G_1(t,s).$$

and $m_1 \leq G_1(t,s) \leq M_1$,

where $m_1 = \dfrac{1}{\exp(\rho T) - 1}, M_1 = \dfrac{\exp(\rho T)}{\exp(\rho T) - 1}.$

Lemma 2.2. [22] *Suppose that (H_3) and (H_4) are satisfied and*

$$\frac{R_1 \exp\left[\left(\int_0^T a_1(v)dv\right) - 1\right]}{Q_1 T} \leq 1,$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp(\int_0^T a_1(v)dv)}{\exp(\int_0^T a_1(v)dv) - 1} a_2(s)ds \right|,$$

$$Q_1 = \left(1 + \exp\left(\int_0^T a_1(v)dv\right)\right)^2 R_1^2.$$

Then there are continuous $T-$periodic functions a and b such that $b(t) > 0, \int_0^T a(v)dv > 0$, and $a(t) + b(t) = a_1(t), b'(t) + a(t)b(t) = a_2(t)$ for $t \in \mathbb{R}$.

Lemma 2.3. [22] *Suppose that the conditions of Lemma 2.2 are satisfied and $y \in P_T$. Then the equation*

$$x''(t) + a_1(t)x'(t) + a_2(t)x(t) = y(t),$$

has a T periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G_2(t,s)y(s)ds,$$

where,

$$G_2(t,s) = \frac{\int_t^s \exp\left(\int_t^u b(v)dv + \int_u^s a(v)dv\right)du + \int_s^{t+T} \exp\left(\int_t^u b(v)dv + \int_u^{s+T} a(v)dv\right)du}{\left(\exp\left(\int_0^T a(u)du\right) - 1\right)\left(\exp\left(\int_0^T b(u)du\right) - 1\right)}.$$

Corollary 2.2. [22] *The Green's function G_2 satisfies the following proprieties*

$$G_2(t,t+T) = G_2(t,t), G_2(t+T,s+T) = G_2(t,s),$$
$$\frac{\partial G_2(t,s)}{\partial t} = -b(t)G_2(t,s) + F(t,s)$$
$$\frac{\partial G_2(t,s)}{\partial s} = a(s)G_2(t,s) - E(t,s).$$

where $F(t,s) = \dfrac{\exp\left(\int_t^s a(v)dv\right)}{\exp\left(\int_0^T a(v)dv\right) - 1}$, $E(t,s) = \dfrac{\exp\left(\int_t^s b(v)dv\right)}{\exp\left(\int_0^T b(v)dv\right) - 1}$.

Lemma 2.4. [17] *Let* $A = \int_0^T p(u)du, B = T^2 \exp\left(\frac{1}{T}\int_0^T \ln(q(u))du\right)$. *If* $A^2 \geq 4B$, *then we have*

$$\min\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \geq \frac{1}{2}\left(A - \sqrt{A^2-4B}\right) := l$$
$$\max\left\{\int_0^T a(u)du, \int_0^T b(u)du\right\} \leq \frac{1}{2}\left(A + \sqrt{A^2-4B}\right) := L$$

Corollary 2.3. [22] *The functions G_2, E and F satisfy*

$$m_2 =: \frac{T}{(e^L-1)^2} \leq G_2(t,s) \leq \frac{T\exp\left(\int_0^T p(u)du\right)}{(e^l-1)^2} := M_2$$
$$E(t,s) \leq \frac{e^L}{e^l-1}, F(t,s) \leq e^L.$$

Lemma 2.5. [3] *Suppose that the conditions of Lemma 2.2 are satisfied and $h \in P_T$. Then the equation*

$$x'''(t) + p(t)x'' + q(t)x' + r(t)x(t) = h(t),$$

has a T-periodic solution. Moreover, the periodic solution can be expressed by

$$x(t) = \int_t^{t+T} G(t,s)h(s)ds,$$

where

$$G(t,s) = \int_t^{t+T} G_2(t,\sigma)G_1(\sigma,s)d\sigma$$

Corollary 2.4. [3] *The Green's function G satisfies the following properties:*

$$G(t+T,s+T) = G(t,s), G(t,t+T) = G(t,t)\exp(\rho T), \frac{\partial G(t,s)}{\partial s} = \rho G(t,s)$$
$$\frac{\partial G(t,s)}{\partial t} = (\exp(-\rho T)-1)G_1(t,t)G_2(t,s) - b(t)G(t,s) + \int_t^{t+T} F(t,\sigma)G_1(\sigma,s)d\sigma,$$

and

$$m \leq G(t,s) \leq M,$$

where

$$m = \frac{T^2}{(\exp(l)-1)^2(\exp(\rho T)-1)}, M = \frac{T^2 \exp(\rho T + \int\limits_0^T a(v)dv)}{(\exp(l)-1)^2(\exp(\rho T)-1)}.$$

In the sequel, we need the following Lebesgue dominated convergence theorem.

Theorem 2.2. [6] *Let Ω be a measurable set of $\mathbb{R}$ and (f_k) be a sequence in $L^1(\Omega,\mathbb{R})$ space such that: $f_k(x) \longrightarrow f(x)$ a.e and there exists a function g in $L^1(\Omega,\mathbb{R})$ such that $|f_k(s)| \leq g(s)$, then $f \in L^1(\Omega,\mathbb{R})$ and*

$$\int_\Omega |f_k - f| \; ds \longrightarrow 0, when \;, k \longrightarrow \infty.$$

We end this section by stating the Schauder's fixed point theorem.

Theorem 2.3. [25] *Let C be a nonempty bounded, closed and convex subset of a Banach space E and T is a continuous operator from C into itself. If $T(C)$ is relatively compact, then T has a fixed point.*

3. Main result

It is easy to check, under the above assumptions, that x is a solution of (5) in P_T^2 if and only if x is the solution of the following integral equation in P_T^2 (see [3])

$$x(t) = \int\limits_t^{t+T} G(t,s) f\big(s, x(s), x(s-\tau(s)), x'(s-\tau(s)), x''(s-\tau(s))\big) ds. \tag{6}$$

Under the hypothesis $(H_1 - H_4)$ and the previous lemmas and corollaries, we will make use of Schauder fixed point theorem to prove the following main result.

Theorem 3.1. *If the hypotheses $(H_1 - H_4)$ hold, and under the following condition:*

$$k = k_6(MT + 2\gamma) < 1 \qquad (*)$$

Then, the neutral integro-differential equation (6) has a positive periodic solution in $C^2(\mathbb{R})$.

Proof. Solving Equation (6) is equivalent to finding a fixed point of the operator A such that A are defined by the following expression

$$Ax(t) = \int\limits_t^{t+T} G(t,s) \Big[f\big(s, x(s), x(s-\tau(s)), x'(s-\tau(s)), x''(s-\tau(s))\big) \Big] ds.$$

It is clear that the operator A is well defined from P_T^2 into itself, moreover

$$(Ax)'(t) = \int\limits_t^{t+T} \frac{\partial G(t,s)}{\partial t} \Big[f(s, x(s), x(s-\tau(s)), x'(s-\tau(s)), x''(s-\tau(s)) \Big] ds.$$

$$(Ax)''(t) = \int\limits_t^{t+T} \frac{\partial^2 G(t,s)}{\partial t^2} \Big[f(s, x(s-\tau(s)), x'(s-\tau(s)), x''(s-\tau(s)) \Big] ds.$$

From Corollary 2.4, we obtain

$$\frac{\partial^2 G(t,s)}{\partial t^2} = (\exp(-\rho T) - 1)\left[\underbrace{\left(\frac{\partial G_1(t,t)}{\partial t} + \frac{\partial G_1(t,t)}{\partial s}\right)}_{=0} G_2(t,t) + G_1(t,t)\frac{\partial G_2(t,s)}{\partial t}\right]$$

$$-b'(t)G(t,s) - \frac{\partial G(t,s)}{\partial t}b(t) + F(t,t+T)G_2(t+T,s) - F(t,t)G_1(t,s) + \int_t^{t+T} F(t,\sigma)G_1(\sigma,s)d\sigma$$

$$\leq (\exp(-\rho T) - 1)\, G_1(t,t)\frac{\partial G_2(t,s)}{\partial t} - b'(t)G(t,s) - \frac{\partial G(t,s)}{\partial t}b(t)$$

$$+F(t,t+T)G_2(t+T,s) - F(t,t)G_1(t,s) + \int_t^{t+T} F(t,\sigma)G_1(\sigma,s)d\sigma$$

and from Corollary 2.2, we obtain

$$\frac{\partial F(t,s)}{\partial t} = -a(t)F(t,s).$$

The proof is split into three steps.

Step I. There exists $\alpha > 0$ such that A transforms $C = \{x \in P_T^2, \|x\| \leq \alpha\}$ into itself . It is clear that C is nonempty, bounded, convex and closed.

To simplify notations, we introduce the constants

$$\lambda = \max_{t\in[0,T]} |\phi(t)|, k_5 = \max\{k_1, k_2\}, k_6 = \max\{k_3, k_4, 2k_5\}, \gamma = \max\{M_3, M_4\}.$$

Moreover, for all $x \in C$ and $t \in [0,T]$, we have

$$\begin{aligned}
|Ax(t)| &= \left| \int_t^{t+T} G(t,s)[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)) + x''(s-\tau(s)))]ds \right| \\
&\leq \int_t^{t+T} |G(t,s)| \left[|\phi(s)| + k_1|x(s)| + \sum_{i=0}^{2} k_{i+2}|x^{(i)}(s-\tau(s))| \right] ds \qquad (7) \\
&\leq MT(\|\phi\|_\infty + (k_1+k_2)\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty) \\
&\leq MT(\|\phi\|_\infty + 2k_5\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty) \\
&\leq MT(\lambda + k_6\|x\|),
\end{aligned}$$

and

$$\begin{aligned}
|(Ax)'(t)| &= \left| \int_t^{t+T} \frac{\partial G(t,s)}{\partial t}[f(s,x(s),x(s-\tau(s)),x'(s-\tau(s)) + x''(s-\tau(s)))]ds \right| \\
&\leq \int_t^{t+T} |\frac{\partial G(t,s)}{\partial t}| \left[\|\phi\|_\infty + k_1|x(s)| + \sum_{i=0}^{2} k_{i+2}|x^{(i)}(s-\tau(s))| \right] ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_t^{t+T} |\frac{\partial G(t,s)}{\partial t}|[\|\phi\|_\infty + (k_1+k_2)\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty \\
&\leq \underbrace{(1-\exp(-\rho T))M_1M_2 + \|b\|_\infty M + TMe^L]}_{=M_3} \times [\|\phi\|_\infty + 2k_5\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq M_3 \times (\lambda + k_6\|x\|)
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
|(Ax)''(t)| = &\left| \int_t^{t+T} \frac{\partial^2 G(t,s)}{\partial t^2}[f(s, x(s-\tau(s)), x'(s-\tau(s)), x''(s-\tau(s))]ds \right|. \\
&\leq \int_t^{t+T} |\frac{\partial^2 G(t,s)}{\partial t^2}| \left[|\phi(s)| + k_1|x(s)| + \sum_{i=0}^{2} k_{i+2}|x^{(i)}(s-\tau(s))| \right] ds \\
&\leq \int_t^{t+T} |\frac{\partial^2 G(t,s)}{\partial t^2}|[\|\phi\|_\infty (k_1+k_2)\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq \Big|(1-\exp(-\rho T))[(\frac{\partial G(t,t)}{\partial t} + \frac{\partial G(t,s)}{\partial s})G_2(t,s) + G_1(t,t)\frac{\partial G_2(t,s)}{\partial t} \\
&- (b)'(t)G(t,s) - (b)(t)\frac{\partial G(t,s)}{\partial t} + F(t,t+T)G_2(t+T,s) - F(t,t)G_1(t,s) \\
&+ \int_t^{t+T} \frac{\partial F(t,\sigma)}{\partial t} G(\sigma,s)d\sigma]\Big| [\|\phi\|_\infty (k_1+k_2)\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq \Big|(1-\exp(-\rho T))G_1(t,t)\frac{\partial G_2(t,s)}{\partial t} - (b)'(t)G(t,s) - (b)(t)\frac{\partial G(t,s)}{\partial t} \\
&+ F(t,t+T)G_2(t+T,s) - F(t,t)G_1(t,s) + \int_t^{t+T} \frac{\partial F(t,\sigma)}{\partial t} G(\sigma,s)d\sigma)\Big| \\
&\times [\|\phi\|_\infty (k_1+k_2)\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq [2M_1M_2(1-\exp(-\rho T))\|b\|_\infty + e^L(\|b\|_\infty TM + M_1 + M_2 + 1) + \|b\|_\infty^2 M \\
&+ TM_1\|\frac{\partial F}{\partial t}\|_\infty] \times [\|\phi\|_\infty + 2k_5\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq [2M_1M_2(1-\exp(-\rho T))\|b\|_\infty + e^L(\|b\|_\infty TM + M_1 + M_2 + 1) + \|b\|_\infty^2 M \\
&+ TM_1\|a\|_\infty e^L] \times [\|\phi\|_\infty + 2k_5\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq M_4 \times [\|\phi\|_\infty + 2k_5\|x\|_\infty + k_3\|x'\|_\infty + k_4\|x''\|_\infty] \\
&\leq M_4 \times (\lambda + k_6\|x\|)
\end{aligned} \tag{9}
$$

such that $M_4 = 2M_1M_2(1-\exp(-\rho T))\|b\|_\infty + e^L(\|b\|_\infty TM + M_1 + M_2 + 1) + \|b\|_\infty^2 M + TM_1\|a\|_\infty e^L$.

Hence, by (7), (8), (9), we obtain

$$\begin{aligned}\|A(x)\| &\leq (MT + M_3 + M_4) \times (\|\phi\|_\infty + k_6\|x\|)\\ &\leq (MT + 2\gamma) \times (\lambda + k_6\|x\|)\end{aligned}$$

We deduce that, A transforms C into itself if

$$\begin{aligned}\|A(x)\| &\leq (MT + 2\gamma) \times (\lambda + k_6\|x\|)\\ &\leq (MT + 2\gamma) \times (\lambda + k_6\alpha)\\ &\leq \alpha.\end{aligned}$$

which implies, under the condition $(*)$, that

$$\frac{\lambda \times (MT + 2\gamma)}{1 - k} \leq \alpha$$

Then, A transforms C into itself for

$$\alpha = \frac{\lambda \times (MT + 2\gamma)}{1 - k}$$

Step 2. The operator A is continuous. Let $(x_n)_n \in C$ be a convergence sequence to $x \in C$, which implies that $(x_n^{(i)})$ converges to $x^{(i)}$ $(i = 0, 1, 2)$ in the space $C([0, T], [-\alpha, \alpha])$. Since f are uniformly continuous on the compact set $[0, T] \times [-\alpha, \alpha]^5$, then the sequence , $f\big(s, x_n(s), x(s - \tau(s)), x_n'(s - \tau(s)), x_n''(s - \tau(s))\big)$ converges to $f\big(s, x(s), x(s - \tau(s)), x'(s - \tau(s)), x''(s - \tau(s))\big)$ in $C([0, T], \mathbb{R})$. It follows that,

$$\begin{aligned}\Big\|Ax_n - Ax\Big\| \leq M\Big(&\|f\big(s, x_n(s), x(s - \tau(s)), x_n'(s - \tau(s)), x_n''(s - \tau(s))\big)\\ &- f\big(s, x(s), x(s - \tau(s)), x'(s - \tau(s)), x''(s - \tau(s))\big\|_\infty\Big).\end{aligned}$$

Which implies that (Ax_n) converges to Ax and the operator A is continuous.
Step 3. $A(C)$ is relatively compact, it is clear that $A(C)$ is equibounded.
Now, to show that $A(C)$ is equicontinuous, take t_1 and t_2 in $I = [0, T]$.
Let $H(s) = f\big(s, x(s), x(s - \tau(s)), x'(s - \tau(s)), x''(s - \tau(s))\big)$, by the assumption (H_2), we have

$$\|H\|_\infty \leq \lambda + k_6\alpha.$$

It follows that

$$\begin{aligned}|Ax(t_1) - Ax(t_2)| &= \left|\int_{t_1}^{t_1+T} G(t_1, s)H(s)ds - \int_{t_2}^{t_2+T} G(t_2, s)H(s)ds\right|\\ &\leq \left|\int_{t_1}^{t_1+T} G(t_1, s)H(s)ds - \int_{t_1}^{t_1+T} G(t_2, s)H(s)ds\right|\\ &+ \left|\int_{t_1}^{t_1+T} G(t_2, s)H(s)ds - \int_{t_2}^{t_2+T} G(t_2, s)H(s)ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{t_1}^{t_1+T} |G(t_1,s)-G(t_2,s)||H(s)|ds + |\int_{t_1}^{t_2} G(t_2,s)H(s)ds \\
&+ \int_{t_2}^{t_2+T} G(t_2,s)H(s)ds + \int_{t_2+T}^{t_1+T} G(t_2,s)H(s)ds - \int_{t_2}^{t_2+T} G(t_2,s)H(s)ds| \\
&\leq \int_{t_1}^{t_1+T} |G(t_1,s)-G(t_2,s)||H(s)|ds + \int_{t_1}^{t_2} |G(t_2,s)H(s)|ds \\
&+ \int_{t_1+T}^{t_2+T} |G(t_2,s)H(s)|ds \\
&\leq \|H\| \int_{t_1}^{t_1+T} |(G(t_1,s)-G(t_2,s)|ds \\
&+ \|H\| \int_{t_1}^{t_2} |G(t_2,s)|ds + \|H\| \int_{t_2+T}^{t_1+T} |G(t_2,s)|ds \\
&\leq \|H\| \int_{t_1}^{t_1+T} |G(t_1,s)-G(t_2,s)|ds + 2M\|H\||t_1-t_2| \\
&\leq (\lambda + k_6\alpha) \int_{t_1}^{t_1+T} |G(t_1,s)-G(t_2,s)|ds \\
&+ 2M\,(\lambda + k_6\alpha)\,|t_1-t_2|.
\end{aligned}
\tag{10}
$$

Using a similar argument as above, we prove that

$$
\begin{aligned}
|A'x(t_1)-A'x(t_2)| &\leq \|H\| \int_{t_1}^{t_1+T} \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| ds \\
&+ \|H\| \int_{t_1}^{t_2} \left|\frac{\partial G(t_2,s)}{\partial t}\right| ds + \|H\| \int_{t_2+T}^{t_1+T} \left|\frac{\partial G(t_2,s)}{\partial t}\right| ds \\
&\leq \|H\| \int_{t_1}^{t_1+T} \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| ds \\
&+ 2M_3\|H\||t_1-t_2| \\
&\leq (\lambda + k_6\alpha) \int_{t_1}^{t_1+T} \left|\frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t}\right| ds \\
&+ 2M_3\,(\lambda + k_6\alpha)\,|t_1-t_2|.
\end{aligned}
\tag{11}
$$

Also, we have

$$\begin{aligned}
|A''x(t_1) - A''x(t_2)| &\le \|H\| \int_{t_1}^{t_1+T} \left| \frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2} \right| ds \\
&+ \|H\| \int_{t_1}^{t_2} \left| \frac{\partial^2 G(t_2,s)}{\partial t^2} \right| ds + \|H\| \int_{t_2+T}^{t_1+T} \left| \frac{\partial^2 G(t_2,s)}{\partial t^2} \right| ds \\
&\le \|H\| \int_{t_1}^{t_1+T} \left| \frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2} \right| ds \qquad (12)\\
&+ 2M_4 \|H\| |t_1 - t_2| \\
&\le (\lambda + k_6\alpha) \int_{t_1}^{t_1+T} \left| \frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2} \right| ds \\
&+ 2M_4 (\lambda + k_6\alpha) |t_1 - t_2|.
\end{aligned}$$

Now, let $\varepsilon > 0$, since the functions $G(t,s)$, $\frac{\partial G(t,s)}{\partial t}$ and $\frac{\partial^2 G(t,s)}{\partial t^2}$ are uniformly continuous on the compact set $[0,T] \times [0,2T]$, then there exists $\delta_1 > 0$ such that, if $|t_2 - t_1| \le \delta_1$, we have for all $s \in [0,2T]$

$$|G(t_2,s) - G(t_1,s)| \le \frac{\varepsilon}{2T(\lambda + k_6\alpha)},$$

$$\left| \frac{\partial G(t_1,s)}{\partial t} - \frac{\partial G(t_2,s)}{\partial t} \right| \le \frac{\varepsilon}{2T(\lambda + k_6\alpha)}, \forall\, s \in [0,2T].$$

and

$$\left| \frac{\partial^2 G(t_1,s)}{\partial t^2} - \frac{\partial^2 G(t_2,s)}{\partial t^2} \right| \le \frac{\varepsilon}{2T(\lambda + k_6\alpha)}, \forall\, s \in [0,2T].$$

Then from (10), (11) and (12), if $|t_2 - t_1| \le \delta = \min(\delta_1, \delta_2, \delta_3, \delta_4)$, where

$$\begin{cases} \delta_2 = \frac{\varepsilon}{4M(\lambda + k_6\alpha)}, \\ \delta_3 = \frac{\varepsilon}{4M_3(\lambda + k_6\alpha)}, \\ \delta_4 = \frac{\varepsilon}{4M_4(\lambda + k_6\alpha)}. \end{cases}$$

We deduce, for $i = 0,1,2$, that

$$|Ax^{(i)}(t_2) - Ax^{(i)}(t_1)| \le \varepsilon.$$

Consequently, the set $A(C)$ is equicontinuous.
Hence, by Theorem (2.1), $A(C)$ is relatively compact.
The proof of Theorem 3.1 then follows from Schauder's fixed point theorem. □

4. Application

Consider the differential equation:

$$x'''(t) + 3x''(t) + \frac{5}{4}x'(t) + \frac{1}{2}x(t) = \cos(6t) + \ln\left(1 + \left(kx(t) + k\sum_{i=0}^{2} x^{(i)}(s - \tau(s))\right)^2\right) \quad (13)$$

Hence, by using the notations of Theorem 3.1, we have
$p(t) = 3, q(t) = \frac{5}{4}, r(t) = \frac{1}{2}$,$T = 2\pi$, $\tau(t) = \sin t + 2$, $\delta = 1$, $\phi(t) = \cos(6t)$ and

$$f(t, u_1, u_2, u_3, u_4) = \cos(6t) + \ln(1 + (ku_1 + ku_2 + ku_3 + ku_4)^2).$$

Doing straightforward computations, it is easy to obtain
$a(t) = b(t) = \frac{1}{2}, a_1(t) = 1, a_2(t) = \frac{1}{4}, \lambda = 1, \rho = 2.$

$$E(t, s) = F(t, s) = \frac{e^{\frac{1}{2}(s-t)}}{(e^{\pi} - 1)}, L = 5\pi, l = \pi, M = \frac{4\pi^2 e^{5\pi}}{(e^{\pi} - 1)^2(e^{4\pi} - 1)},$$

$$M_1 = \frac{e^{4\pi}}{(e^{4\pi} - 1)}, M_2 = \frac{2\pi e^{6\pi}}{(e^{\pi} - 1)^2}, M_3 = \frac{2(e^{4\pi} - 1)\pi e^{6\pi} + 2\pi^2 e^{5\pi}}{(e^{4\pi} - 1)(e^{\pi} - 1)^2}$$

$$M_4 = \frac{\pi^2 e^{5\pi}(1 + 4\pi e^{5\pi}) + e^{5\pi}(e^{\pi} - 1)^2(2e^{4\pi} + \pi e^{4\pi} - 1) + 2\pi e^{6\pi}(e^{4\pi} - 1)(2e^{5\pi} - 1)}{(e^{4\pi} - 1)(e^{\pi} - 1)^2}$$

and $k_1 = k_2 = k_3 = k_4 = k_5 = |k|, k_6 = 2|k|, \gamma = M_4$.
therefore, the inequality in Theorem 3.1 takes the form

$$4|k|\frac{\pi^2 e^{5\pi}(1 + 4\pi e^{5\pi}) + e^{5\pi}(e^{\pi} - 1)^2(2e^{4\pi} + \pi e^{4\pi} - 1) + 2\pi e^{6\pi}(e^{4\pi} - 1)(2e^{5\pi} - 1)}{(e^{4\pi} - 1)(e^{\pi} - 1)^2} + 4|k|\frac{4\pi^3 e^{5\pi}}{(e^{4\pi} - 1)(e^{\pi} - 1)^2} < 1. \tag{14}$$

Then by Theorem 3.1, we conclude, from the inequality (14), that the third order nonlinear delay differential equation (13) has a solution $u \in C^3(I, \mathbb{R})$ if

$$|k| < 9.57 \times 10^{-15}.$$

5. Conclusions

In this paper, we have considered a general form of the third-order nonlinear delay differential equation, where the derivatives x' and x'' appear in the nonlinear function. The existence of a periodic solution has been investigated, under fairly simple conditions, by using a new generalization of Ascoli-Arzelà theorem and Schauder's fixed point theorem. The advantage of using the new generalization of Ascoli-Arzelá theorem given in [4] is that it allows us to study the existence of the solution to the third order delay differential equations where the derivatives x', x'' appear in the nonlinear function, while the derivatives x', x'' does not appear in the nonlinear functions in the previous studies of the third order delay differential equations (see for example [1, 2, 3, 12, 17, 20, 22]) which they have used only the Ascoli-Arzelá theorem in the space of continuous functions. It is important to point out that the general form of the third-order nonlinear delay differential equation (5) contains as particular cases many important integral and functional equations in the literature. Specifically, the particular functional delay $\tau(t) = t - x(t)$ or the nonlinear function contains the term $x(x(t))$ which called iterative differential equation (see for example [5, 13, 26]). Finally, an example is provided to illustrate our main result.

Acknowledgement. The authors are very grateful to the anonymous referees for their valuable comments and suggestions

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