TWMS J. App. and Eng. Math. V.15, N.1, 2025, pp. 89-97

$\rho-$STATISTICAL BOUNDEDNESS

M. ET[1], C. CHOUDHURY[2], S. DEBNATH[2*], §

ABSTRACT. In this paper, we introduce and examine the concept of $\rho-$statistical boundedness and give some relations between statistical boundedness and $\rho-$statistical boundedness. We also introduce the notion of $\rho-$statistical upper bound, $\rho-$statistical lower bound, $\rho-$statistical supremum and $\rho-$statistical infimum and investigate their interrelationships.

Keywords: $\rho-$statistical boundedness, $\rho-$statistical convergence, $\rho-$statistical supremum, $\rho-$statistical infimum.

AMS Subject Classification (2020): 40A05, 40C05, 46A45

1. Introduction, Definitions and Preliminaries

Statistical convergence was introduced by Fast [12] and Steinhaus [26] independently in the same year 1951. Though the notion was firstly handled as a summability method by Schoenberg [27]. Salat [21] researched some topological properties of statistical convergence for sequences of real numbers. Fridy [13] defined the concept of statistical Cauchiness and showed that it is equivalent to statistical convergence. He also dealt with some Tauberian theorems. Connor [9] proved that a strongly p-Cesaro summable sequence for $0 < p < \infty$ is statistically convergent and the converse holds for bounded sequences. Recently several generalizations and applications of this concept have been investigated by various authors. For more details, one may refer to [1, 6, 7, 15, 16, 17, 19, 20, 22, 23, 24, 25, 28, 29].

The opinion of statistical convergence depends on the density of subsets of the natural set $\mathbb{N}$. We say that $\delta(E)$ is density of a subset E of $\mathbb{N}$ if the following limit exists such

[1] Department of Mathematics, Fırat University, 23119, Elazıg, Turkey.
e-mail: mikailet68@gmail.com; ORCID: https://orcid.org/0000-0001-8292-7819.
[2] Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India.
e-mail: chiranjibchoudhury123@gmail.com; ORCID: https://orcid.org/0000-0002-5607-9884.
e-mail: shyamalnitamath@gmail.com; ORCID: https://orcid.org/0000-0003-2804-6564.
∗ Corresponding author.
§ Manuscript received: May 12, 2023; accepted: October 6, 2023.
TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.1;
The second author is grateful to the **University Grants Commission, India** for their fellowships funding under the **UGC-SRF** scheme **(F. No. 16-6(DEC. 2018)/2019(NET/CSIR))** during the preparation of this paper.

that

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta\left(E^c\right)=1-\delta\left(E\right)$.

We say that the sequence $x=(x_k)$ is statistically convergent to ℓ if for every $\varepsilon>0$,

$$\delta\left(\left\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\ \right\}\right)=0.$$

Let $K\subset\mathbb{N}$. Then, $\rho-$density of K is defined by

$$\delta_\rho(K)=\lim_{n\to\infty}\frac{1}{\rho_n}\left|\left\{\ k\leq n:k\in K\right\}\right|,$$

provided this limit exists, where and afterwards $\rho=(\rho_n)$ is a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n\dfrac{\rho_n}{n}<\infty$, $\Delta\rho_n=O(1)$, and $\Delta\rho_n=\rho_{n+1}-\rho_n$ for each positive integer n.

If $x=(x_k)$ is a sequence such that x_k holds property $P(k)$ for all k except a set of $\rho-$density zero, then we say that x_k holds $P(k)$ for "almost all k according ρ " and we denote this by "$a.a.k\ (\rho)$".

A sequence $x=(x_k)$ is called $\rho-$statistically convergent [2] to ℓ if

$$\lim_{n\to\infty}\frac{1}{\rho_n}|\{k\leq n:|x_k-\ell|\geq\varepsilon\}|=0$$

for each $\varepsilon>0$. In this case, it is denoted by $x_k\xrightarrow{st_\rho}l$. If $\rho_n=n$ for all $n\in\mathbb{N}$, then $\rho-$statistical convergence is coincides usual statistical convergence [12, 26]. The set of all $\rho-$statisticallly convergent sequences will be denoted by S_ρ.

The concept of statistical boundedness was given by Fridy and Orhan [14] as follows:

The real number sequence x is statistically bounded if there exists a number $M>0$ such that

$$\delta\left(\{k:|x_k|>M\}\right)=0. \tag{1}$$

The set of all statistically bounded sequences will be denoted by

$$S\left(b\right)=\left\{x=(x_k):(1)\text{ holds}\right\}.$$

It can be shown that every bounded sequence is statistically bounded, but the converse is not true, in general. For this consider a sequence $x=(x_k)$ defined by

$$x_k=\begin{cases}k, & \text{if } k \text{ is a square,}\\ 1, & \text{if } k \text{ is not a square.}\end{cases}$$

Clearly, the sequence $x=(x_k)$ is not a bounded sequence, but it is statistically bounded.

Bhardwaj et al. [3, 4, 5] and Et et al. [10, 11] generalized the concept of statistical boundedness.

2. $\rho-$Statistically Bounded Sequences

In this section we introduce the concept of $\rho-$statistical boundedness and give the relation between $\rho-$statistical convergence and $\rho-$statistical boundedness.

Definition 2.1. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers as above. A sequence $x = (x_k)$ is said to be $\rho-$statistically bounded if there exists a $M \geq 0$ such that*

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |x_k| > M\}| = 0, \qquad i.e., \qquad |x_k| \leq M \qquad a.a.k(\rho).$$

The set of all $\rho-$statisticallly bounded sequences will be denoted by $S_\rho(b)$. If $\rho_n = n$ for all $n \in \mathbb{N}$, then $\rho-$statistically boundedness is coincides statistically boundedness.

Theorem 2.1. *$\ell_\infty \subset S_\rho(b)$ and the inclusion is strict, in general.*

Proof. If $x \in \ell_\infty$, then there exists $M > 0$ such that $|x_k| \leq M$ for all $k \in \mathbb{N}$. So, $\{k \leq n : |x_k| > M\} = \emptyset$. This fact implies that $x \in S_\rho(b)$.

For strictness of the inclusion let a sequence $x = (x_k)$ be defined as follows

$$x_k = \begin{cases} \rho_k, & k \text{ is square,} \\ 0, & k \text{ is odd non-square,} \\ 1, & k \text{ is even non-square.} \end{cases} .$$

Then, for any $M > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$\{k : |x_k| > M\} \subset \left\{\rho_1^2, \rho_2^2, \rho_3^2, \cdots, \rho_{k_0}^2, \rho_{k_0+1}^2, \cdots\right\} = A.$$

This follows

$$\begin{aligned} \delta_\rho(\{k : |x_k| > M\}) &\leq \delta_\rho\left(\left\{\rho_1^2, \rho_2^2, \rho_3^2, \cdots, \rho_{k_0}^2, \rho_{k_0+1}^2, \cdots\right\}\right) \\ &= \lim_{n\to\infty} \frac{\sqrt{\rho_n}}{\rho_n} \\ &= 0. \end{aligned}$$

Thus $x \in S_\rho(b)$. However $x \notin \ell_\infty$ and this completes the proof. □

Theorem 2.2. *A $\rho-$statistically convergent sequence is $\rho-$statistically bounded, but converse is not true, in general.*

Proof. Let $x = (x_k) \in S_\rho$. Then there exists an $\ell \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{1}{\rho_n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| = 0$$

holds, for all $\varepsilon > 0$. So it holds $\varepsilon = 1$ in particular. By reverse triangle inequality we see

$$\{k \leq n : |x_k| \geq |\ell| + 1\} \subseteq \{k \leq n : |x_k - \ell| \geq 1\}$$

and

$$\frac{1}{\rho_n} |\{k \leq n : |x_k| \geq |\ell| + 1\}| \leq \frac{1}{\rho_n} |\{k \leq n : |x_k - \ell| \geq 1\}|$$

for all $n \in \mathbb{N}$. Taking limit in the last inequality and choosing $M = |\ell|+1$ we get $x \in S_\rho(b)$.

To show the strictness of the inclusion choose $x = (x_k) = (1, 2, 3, 1, 2, 3, ...)$ and take $\rho_n = n$ for all $n \in \mathbb{N}$. Then, x is not statistically convergent since it has no any dense subsequence which is convergent. However, x is $\rho-$statistically bounded. □

Corollary 2.1. *Every convergent sequence is $\rho-$statistically bounded, but the converse is not true.*

Proof. Proof is clear from Theorem 2.1, because of regularity of $\rho-$statistical convergence. □

Theorem 2.3. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \frac{\rho_n}{n} < \infty$, $\Delta\rho_n = O(1)$, then $S_\rho(b) \subset S(b)$.*

Proof. Let $x = (x_k) \in S_\rho(b)$ be an arbitrary sequence. For a given $M, K > 0$, we have

$$\begin{aligned}\frac{1}{n}\left|\{k \leq n : |x_k| > M\}\right| &= \frac{\rho_n}{n}\frac{1}{\rho_n}\left|\{k \leq n : |x_k| > M\}\right| \\ &\leq \frac{K}{\rho_n}\left|\{k \leq n : |x_k| > M\}\right|.\end{aligned}$$

Since, $\limsup_n \frac{\rho_n}{n} < \infty$, then we get $S_\rho(b) \subset S(b)$. □

Theorem 2.4. *Let $\rho = (\rho_n)$ be a non-decreasing sequence of positive real numbers tending to ∞ such that $\limsup_n \dfrac{\rho_n}{n} < \infty, \Delta\rho_n = O(1)$. If $\rho_n \geq n$ for all $n \in \mathbb{N}$, then $S(b) \subset S_\rho(b)$.*

Proof. If $x \in S(b)$, then for some $M > 0$ we have

$$\begin{aligned}\frac{1}{n}\left|\{k \leq n : |x_k| > M\}\right| &= \frac{\rho_n}{n}\frac{1}{\rho_n}|\{\{k \leq n : |x_k| > M\}\,| \\ &\geq \frac{1}{\rho_n}|\,\{k \leq n : |x_k| > M\}\,|.\end{aligned}$$

This proves the proof. □

Theorem 2.5. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences that satisfy the conditions in Definition 2.1 such that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to\infty} \sup \frac{\rho_n}{\tau_n} < \infty \tag{2}$$

then $S_\rho(b) \subseteq S_\tau(b)$.

Proof. The proof follows from the following inequality

$$\begin{aligned}\frac{1}{\tau_n}\left|\{k \leq n : |x_k| > M\}\right| &= \frac{\rho_n}{\tau_n}\frac{1}{\rho_n}\left|\{k \leq n : |x_k| > M\}\right| \\ &\leq \frac{K}{\rho_n}\left|\{k \leq n : |x_k| > M\}\right|.\end{aligned}$$

□

Theorem 2.6. *Let $\rho = (\rho_n)$ and $\tau = (\tau_n)$ be two sequences that satisfy the conditions in Definition 2.1 such that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to\infty} \inf \frac{\rho_n}{\tau_n} > 0 \tag{3}$$

then $S_\tau(b) \subseteq S_\rho(b)$.

Proof. Suppose that $\rho_n \leq \tau_n$ for all $n \in \mathbb{N}$ and let (3) be satisfied, $K > 0$ we may write

$$\frac{1}{\tau_n}\left|\{k \leq n : |x_k| > M\}\right| \geq \frac{\rho_n}{\tau_n}\frac{1}{\rho_n}\left|\{k \leq n : |x_k| > M\}\right|.$$

Now taking the limit as $n \to \infty$ in the above inequality and using (3), we get $S_\tau(b) \subseteq S_\rho(b)$. □

3. $\rho-$STATISTICAL UPPER BOUND, LOWER BOUND, SUPREMUM, INFIMUM

In 2013, Küçükaslan and Altınok [18] introduced the notion of statistical supremum and statistical infimum using the concept of statistical upper bound and statistical lower bound. In this section, we introduce $\rho-$statistical analogs of the above notion and prove some interesting results.

Definition 3.1. *Let $x = (x_k)$ be a real-valued sequence.*
(i) The real number l is said to be a $\rho-$statistical lower bound of x, if

$$\delta_\rho(\{k \in \mathbb{N} : x_k < l\}) = 0 \, (or \, \delta_\rho(\{k \in \mathbb{N} : x_k \geq l\}) = 1.$$

(ii) The real number u is said to be a $\rho-$statistical upper bound of x, if

$$\delta_\rho(\{k \in \mathbb{N} : x_k > u\}) = 0 \, (or \, \delta_\rho(\{k \in \mathbb{N} : x_k \leq u\}) = 1.$$

The set of all $\rho-$statistical lower and upper bounds of the sequence $x = (x_k)$ is denoted by $L_\rho(x)$ and $U_\rho(x)$, respectively.

Definition 3.2. *Let $x = (x_k)$ be a real-valued sequence.*
(i) The real number i is said to be the $\rho-$statistical infimum of the sequence $x = (x_k)$ if i is the supremum of the set $L_\rho(x)$. In other words,

$$st_\rho - \inf x = \sup L_\rho(x).$$

(ii) The real number s is said to be the $\rho-$statistical supremum of the sequence $x = (x_k)$ if s is the infimum of the set $U_\rho(x)$. In other words,

$$st_\rho - \sup x = \inf U_\rho(x).$$

If we take $\rho_n = n$ for all $n \in \mathbb{N}$, then Definition 3.1 and Definition 3.2 coincides with Definitions given in [18] for natural density.

Theorem 3.1. *(i) If $L(x)$ denotes the set of all usual lower bounds of a sequence $x = (x_k)$, then strictly*

$$L(x) \subset L_\rho(x);$$

(ii) If $U(x)$ denotes the set of all upper lower bounds of a sequence $x = (x_k)$, then strictly

$$U(x) \subset U_\rho(x).$$

Proof. (i) Let $l \in L(x)$. Then, we have $\{k \in \mathbb{N} : x_k \geq l\} = \mathbb{N}$ and consequently, $\delta_\rho(\{k \in \mathbb{N} : x_k \geq l\}) = 1$. Hence, $l \in L_\rho(x)$, proving that $L(x) \subseteq L_\rho(x)$. To prove that the inclusion is strict we construct a counterexample. Let A be a set such that $\delta_\rho(A) = 0$. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & k \notin A \\ (-1)^k k, & otherwise. \end{cases}$$

Then, $1 \in L_\rho(x)$ but $1 \notin L(x)$.
(ii) The proof is similar to that of (i), so omitted. □

Theorem 3.2. *Let $x = (x_k)$ be a real-valued sequence. Then,*
(i) If $l \in L_\rho(x)$, then all real numbers smaller than l are $\rho-$statistical lower bound of x.
(ii) If $u \in U_\rho(x)$, then all real numbers bigger than u are $\rho-$statistical upper bound of x.

Proof. (i) Let $l \in L_\rho(x)$ and $l' < l$. Then, by definition $\delta_\rho(\{k \in \mathbb{N} : x_k \geq l\}) = 1$. Since, $l' < l$, so the inclusion

$$\{k \in \mathbb{N} : x_k \geq l\} \subseteq \{k \in \mathbb{N} : x_k \geq l'\}$$

holds and consequently $\delta_\rho(\{k \in \mathbb{N} : x_k \geq l'\}) = 1$. Hence, $l' \in L_\rho(x)$.
(ii) The proof is similar to that of (i), so omitted. □

Theorem 3.3. *For any real-valued sequence $x = (x_k)$, following inequation*

$$\inf x \leq st_\rho - \inf x \leq st_\rho - \sup x \leq \sup x$$

holds.

Proof. From the definition of usual infimum we have $\delta_\rho(\{k \in \mathbb{N} : x_k \geq \inf x\}) = 1$. Therefore, $\inf x \in L_\rho(x)$ and consequently,

$$\inf x \leq st_\rho - \inf x. \tag{2}$$

In a similar way, one can prove that

$$\sup x \geq st_\rho - \sup x. \tag{3}$$

Now we will show that $st_\rho - \inf x \leq st_\rho - \sup x$. To prove this, it is sufficient to prove that $l \leq u$ for any $l \in L_\rho(x)$ and $u \in U_\rho(x)$.

If possible suppose there exists $l' \in L_\rho(x)$ and $u' \in U_\rho(x)$ such that $l' > u'$. Then, since $l' \in L_\rho(x)$, so by Theorem 3.2, $u' \in L_\rho(x)$, which is a contradiction on the assumption of u'. Hence, we must have $l \leq u$ for any $l \in L_\rho(x)$ and $u \in U_\rho(x)$. In other words,

$$st_\rho - \inf x \leq st_\rho - \sup x. \tag{4}$$

Combining (2), (3), and (4) we obtain the desired result. □

Theorem 3.4. *Let $x = (x_k)$ be a real-valued sequence such that $x_k \to x_0$ as $k \to \infty$. Then, $st_\rho - \inf x = st_\rho - \sup x = x_0$.*

Proof. There are two possible cases:
Case-I: When x_0 is finite.
From the assumption, it is easy to show that for any $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that the following inclusions hold

$$\{k \in \mathbb{N} : x_k \geq x_0 - \varepsilon\} \supseteq \mathbb{N} \setminus \{1, 2, ..., k_0\} \tag{5}$$

and

$$\{k \in \mathbb{N} : x_k \leq x_0 + \varepsilon\} \supseteq \mathbb{N} \setminus \{1, 2, ..., k_0\}. \tag{6}$$

From, (5), (6), and monotonicity properties of $\rho-$density we obtain

$$\delta_\rho(\{k \in \mathbb{N} : x_k \geq x_0 - \varepsilon\}) = \delta_\rho(\{k \in \mathbb{N} : x_k \leq x_0 + \varepsilon\}) = 1.$$

Thus, we have, for any $\varepsilon > 0$, $x_0 - \varepsilon \in L_\rho(x)$ and $x_0 + \varepsilon \in U_\rho(x)$ which means that $L_\rho(x) = (-\infty, x_0)$ and $U_\rho(x) = (x_0, \infty)$. Therefore, by definition, $st_\rho - \inf x = \sup L_\rho(x) = x_0$ and $st_\rho - \sup x = \inf U_\rho(x) = x_0$. Hence, $st_\rho - \inf x = st_\rho - \sup x = x_0$.
Case-II: When x_0 is infinite, i.e., $x_k \to \infty$ as $k \to \infty$.
Then, for given $B > 0$, there exists $k_0 \in \mathbb{N}$ such that the inclusions $\{k \in \mathbb{N} : x_k \geq B\} \supseteq \mathbb{N} \setminus \{1, 2, ..., k_0\}$ and $\{k \in \mathbb{N} : x_k \leq B\} \subseteq \{1, 2, ..., k_0\}$ holds. As a consequence of these inclusions, $\delta_\rho(\{k \in \mathbb{N} : x_k \geq B\}) = 1$ and $\delta_\rho(\{k \in \mathbb{N} : x_k \leq B\}) \neq 1$. Therefore, $B \in L_\rho(x)$ and $B \notin U_\rho(x)$ i.e., $L_\rho(x) = (-\infty, \infty)$ and $U_\rho(x) = \emptyset$. Hence, $st_\rho - \inf x = st_\rho - \sup x = \infty$. □

Theorem 3.5. *For any real-valued sequence $x = (x_k)$, $x_k \xrightarrow{st_\rho} x_0$ if and only if $st_\rho - \inf x = st_\rho - \sup x = x_0$.*

Proof. Firstly we assume that $x_k \xrightarrow{st_\rho} x_0$ holds. Then, by definition of $\rho-$statistical convergence for any $\varepsilon > 0$,

$$\delta_\rho(\{k \in \mathbb{N} : |x_k - x_0| \geq \varepsilon\}) = 0. \tag{7}$$

This implies that, for any $\varepsilon > 0$,

$$\delta_\rho(\{k \in \mathbb{N} : x_k \geq x_0 + \varepsilon\}) = 0 \quad \text{and} \quad \delta_\rho(\{k \in \mathbb{N} : x_k < x_0 + \varepsilon\}) = 1 \tag{8}$$

and

$$\delta_\rho(\{k \in \mathbb{N} : x_k \leq x_0 - \varepsilon\}) = 0 \quad \text{and} \quad \delta_\rho(\{k \in \mathbb{N} : x_k > x_0 - \varepsilon\}) = 1. \tag{9}$$

Now from (8) and (9), we obtain $x_0 + \varepsilon \in U_\rho(x)$ and $x_0 - \varepsilon \in L_\rho(x)$. Eventually, $U_\rho(x) = (x_0, \infty)$ and $L_\rho(x) = (-\infty, x_0)$ holds and we have $st_\rho - \inf x = st_\rho - \sup x = x_0$.

To prove the converse part, let $st_\rho - \inf x = st_\rho - \sup x = x_0$ i.e., $\sup L_\rho(x) = \inf U_\rho(x) = x_0$. Then, by definition of supremum and infimum, there exists at least one $l' \in L_\rho(x)$ and atleast one $l'' \in U_\rho(x)$ such that for any $\varepsilon > 0$, $x_0 - \varepsilon < l'$ and $x_0 + \varepsilon > l''$ holds. Consequently,

$$\{k \in \mathbb{N} : x_k \geq x_0 + \varepsilon\} \subset \{k \in \mathbb{N} : x_k \geq l'\}$$

and

$$\{k \in \mathbb{N} : x_k \leq x_0 + \varepsilon\} \subset \{k \in \mathbb{N} : x_k \leq l''\}.$$

Now since, $l' \in L_\rho(x)$ and $l'' \in U_\rho(x)$, so from the above inclusions we obtain $\delta_\rho(\{k \in \mathbb{N} : x_k \geq x_0 + \varepsilon\}) = 0$ and $\delta_\rho(\{k \in \mathbb{N} : x_k \leq x_0 - \varepsilon\}) = 0$ which altogether implies (7) and this completes the proof. □

Theorem 3.6. *Let $x = (x_k)$ and $y = (y_k)$ be two real-valued sequences such that $\delta_\rho(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Then,*

$$st_\rho - \inf x = st_\rho - \inf y \quad \textit{and} \quad st_\rho - \sup x = st_\rho - \sup y.$$

Proof. We only prove the first part i.e., $st_\rho - \inf x = st_\rho - \inf y$. The proof of the second part can be obtained by applying a similar technique.

Let the given conditions hold and suppose $l \in L_\rho(x)$ be arbitrary. Then, by definition

$$\delta_\rho(\{k \in \mathbb{N} : x_k < l\}) = 0.$$

Consequently,

$$\begin{aligned}\{k \in \mathbb{N} : y_k < l\} &= \{k \in \mathbb{N} : x_k \neq y_k, y_k < l\} \cup \{k \in \mathbb{N} : x_k = y_k, y_k < l\}\\ &\subseteq \{k \in \mathbb{N} : x_k \neq y_k\} \cup \{k \in \mathbb{N} : x_k < l\}.\end{aligned}$$

From the above inclusion, it is clear that $\delta_\rho(\{k \in \mathbb{N} : y_k < l\}) = 0$ which implies $l \in L_\rho(y)$. This proves that

$$L_\rho(x) \subseteq L_\rho(y).$$

Similarly, one can establish $L_\rho(y) \subseteq L_\rho(x)$. Hence, $L_\rho(x) = L_\rho(y)$ holds and eventually $\sup L_\rho(x) = \sup L_\rho(y)$ i.e., $st_\rho - \inf x = st_\rho - \inf y$. □

Remark 3.1. *The converse of the above theorem is not necessarily true. Let $\rho_n = n$, $n \in \mathbb{N}$. Consider the sequences $x = (x_k)$ and $y = (y_k)$ defined by $x_k = 1 - \frac{1}{k}$ and $y_k = 1 + \frac{1}{k}$. Then, it is easy to verify that $st_\rho - \inf x = st_\rho - \inf y = 1$. But $\delta_\rho(\{k \in \mathbb{N} : x_k \neq y_k\}) = 1 \neq 0$.*

4. Conclusion

In this paper, we mainly focus on the concept of $\rho-$statistical boundedness which is a natural generalization of statistical boundedness. In section 2, Theorem 2.3 and Theorem 2.4 are established to show the connection between a $\rho-$statistically bounded sequence and a statistically bounded sequence. Furthermore, Theorem 2.5 and Theorem 2.6 establishes two inclusion relations for the variation on the sequence $\rho = (\rho_n)$. Section 3 basically deals with four new concepts namely $\rho-$statistical upper bound, $\rho-$statistical lower bound, $\rho-$statistical supremum and $\rho-$statistical infimum. Theorem 3.3 establishes an inequation showing how $\rho-$statistical supremum and $\rho-$statistical infimum of a sequence are connected to usual supremum and usual infimum. Theorem 3.5 gives a necessary and sufficient condition for $\rho-$statistical convergence of a real valued sequence $x = (x_k)$.

As a continuation of this research, one may investigate several properties such as solidity, symmetry and monotonicity of the sequence space $S_\rho(b)$. Also, from the application point of view, one may investigate the concept of $\rho-$statistical convergence in neutrosophic normed space, complex uncertain space, credibility space etc.

Acknowledgment

The authors thank the anonymous referees for their constructive suggestions to improve the quality of the paper.

References

[1] Akbas, K. E. and Isik, M., (2020), On asymptotically $\lambda-$statistical equivalent sequences of order α in probability, Filomat, 34(13), pp. 4359–4365.

[2] Aral, N. D., Şengül, H. K. and Et, M., (2021), On $\rho-$statistical convergence, AIP Conference Proceedings 2334, 040001; https://doi.org/10.1063/5.0042243.

[3] Bhardwaj, V. K. and Gupta, S., (2014), On some generalizations of statistical boundedness, J. Inequal. Appl., 2014(12), https://doi.org/10.1186/1029-242X-2014-12.

[4] Bhardwaj, V. K., Dhawan, S. and Gupta, S., (2016), Density by moduli and statistical boundedness, http://dx.doi.org/10.1155/2016/2143018, Article Id 2143018.

[5] Bhardwaj, V. K., Gupta, S., Mohiuddine, S. A. and Kilicman, A., (2014), On lacunary statistical boundedness, J. Inequal. Appl., 2014(311), https://doi.org/10.1186/1029-242X-2014-311.

[6] Cakalli, H., (1995), Lacunary statistical convergence in topological groups, Indian J. Pure Appl. Math., 26(2), pp. 113–119.

[7] Cakalli, H., (2009), A study on statistical convergence, Funct. Anal. Approx. Comput., 1(2), pp. 19–24.

[8] Cakalli, H., (2017), A variation on statistical ward continuity, Bull. Malays. Math. Sci. Soc., 40, pp. 1701-1710.

[9] Connor, J. S., (1988), The statistical and strong $p-$Cesàro convergence of sequences, Analysis, 8, pp. 47-63.

[10] Et, M., Mohiuddine, S. A. and Şengül, H., (2016), On lacunary statistical boundedness of order α, Facta Univ. Ser. Math. Inform., 31(3), pp. 707–716.

[11] Et, M., Bhardwaj, V. K. and Gupta, S., On deferred statistical boundedness of order α, Communications in Statistics-Theory and Methods, https://doi.org/10.1080/03610926.2021.1906434.

[12] Fast, H., (1951), Sur la convergence statistique, Colloq. Math., 2, pp. 241-244.

[13] Fridy, J. A., (1985), On statistical convergence, Analysis, 5(4), pp. 301-313.

[14] Fridy, J. A. and Orhan, C., (1997), Statistical limit superior and limit inferior, Proc. Amer. Math. Soc., 125(12), pp. 3625-3631.

[15] Güngör, M., Et, M. and Altin, Y., (2004), Strongly $(V_\sigma, \lambda, q)-$summable sequences defined by Orlicz functions, Appl. Math. Comput., 157(2), pp. 561–571.

[16] Gürdal, M., Şahiner, A. and Açık, I, (2009), Approximation theory in 2-Banach spaces, Nonlinear Anal., 71(5-6), pp. 1654–1661.

[17] Işik, M. and Akbaş, K. E., (2017), On $\lambda-$statistical convergence of order α in probability, J. Inequal. Spec. Funct., 8(4), pp. 57-64.
[18] Kuçukaslan, M. and Altinok, M., (2013), Statistical supremum-infimum and statistical convergence, Aligarh Bull. Math., 32(1-2), pp. 39–54.
[19] Mursaleen, M., (2000), $\lambda-$statistical convergence, Math. Slovaca, 50(1), pp. 111-115.
[20] Nabiev, A. A., Savaş, E. and Gürdal, M., (2019), Statistically localized sequences in metric spaces, J. Appl. Anal. Comput., 9(2), pp. 739-746.
[21] Salat, T., (1980), On statistically convergent sequences of real numbers, Math. Slovaca, 30(2), pp. 139-150.
[22] Savaş, E. and Et, M., (2015), On $(\Delta_\lambda^m, I)-$statistical convergence of order α, Period. Math. Hungar., 71(2), pp. 135–145.
[23] Savaş, E. and Gürdal, M., (2015), A generalized statistical convergence in intuitionistic fuzzy normed spaces, Sci. Asia, 41(4), pp. 289–294.
[24] Savaş, E. and Gürdal, M., (2015), I-statistical convergence in propabilistic normed spaces, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 77(4), pp. 195–204.
[25] Savaş, E., Kişi, Ö. and Gürdal, M., (2022), On statistical convergence in credibility space, Numer. Funct. Anal. Optim., 43(8), pp. 987–1008.
[26] Steinhaus, H., (1951), Sur la convergence ordinaire et la convergence asymptotique, Colloq. Math., 2, pp. 73-74.
[27] Schoenberg, I. J., (1959), The integrability of certain functions and related summability methods, Amer. Math. Monthly, 66, pp. 361-375.
[28] Tripathy, B. C., (1998), On statistically convergent sequences, Bull. Calcutta Math. Soc., 90, pp. 259–262.
[29] Tripathy, B. C., (1997), On statistically convergent and statistically bounded sequences, Bull. Malaysian Math. Soc., 20, pp. 31-33.

Mikail Et is a full Professor of Mathematics at Firat University, Elazig, Turkey. He received his Ph.D. from Firat University, Elazig, Turkey. His main research interests are Sequence Spaces, Summability Theory, Matrix Transformation, Fuzzy Metric Spaces, Geometric Properties of Banach Spaces. He has published more than 150 research papers in well reputed international journals. Prof. Et is referee of more than 120 scientific journals (most of them are SCI/SCI expended journals). He has also guided 8 Ph.D. and 25 master students so far. He is reviewer for Mathematical Reviews and Zentralblatt Math. He is also member of the editorial board of several mathematical journals

Chiranjib Choudhury has completed his M.Sc in Mathematics from Tripura University (A Central University), Agartala, INDIA. Very recently, he has completed his Ph.D. in Mathematics under the supervision of Dr. Shyamal Debnath at the same university. His research interests include the sequence spaces, summability theory, neutrosophic set theory, etc.

Shyamal Debnath is an Associate Professor of Mathematics at Tripura University (A Central University), Agartala, INDIA. His main research interests are in the field of sequence spaces, summability theory, fuzzy set etc. He has published more than seventy research papers in well reputed international journals. Six students got their Ph.D degree under his supervision and four are pursuing. He is reviewer for Mathematical Reviews and Zentralblatt Math..