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WARPED PRODUCT OF A QUASI-HEMI SLANT SUBMANIFOLDS WITH TRANS PARA SASAKIAN MANIFOLDS

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Abstract. In the present paper, we define and study quasi hemi-slant submanifolds as a generalization of slant submanifolds, semi-slant submanifolds and hemi-slant submanifolds for a trans para-Sasakian manifold. Further we study warped product submanifolds of a quasi-hemi slant submanifolds with trans para-Sasakian manifolds. We also obtain some results on the existence of such type warped product submanifolds of a quasi-hemi slant submanifolds with trans para-Sasakian manifolds.

Keywords: Warped product, quasi hemi-slant submanifolds, trans para Sasakian manifolds.

AMS Subject Classification: 53C40, 53B25, 53D15.

1. Introduction

Almost Hermitian manifolds of the class W_4 is closely related to locally conformal Kahler manifolds [7]. An almost contact structure on a manifold $\mathcal{M}$ is said to be trans-Sasakian structure [13] if the product manifold $\mathcal{M} \oplus \mathbb{R}$ belongs to the class W_4 . An almost contact metric manifold is trans-Sasakian structures of type (α, β) if it belongs to the class $C_6 \oplus C_5$ [11]. The local nature of the two subclasses, namely the C_5 and the C_6 structures, of trans-Sasakian structures are characterized completely [12]. Moreover, a trans-Sasakian structures of type (α, β) is cosympletic [1] or β Kenmotsu [8] or α Sasakian [8] according to $\alpha = \beta = 0$ or $\alpha = 0$ or $\beta = 0$ respectively. The study of slant submanifolds of almost Hermitian manifolds got momentum after B. Y. Chen [6] paper, as a natural generalization of holomorphic immersions and totally real immersions. Many consequent results on slant submanifolds are collected in his book [5]. Later A. Lotta [10], introduced and studied slant immersions of a Riemannian manifold into almost contact metric manifold. In the course of time this interesting subject have been studied broadly by several geometers

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during last two decades ([6], [15], [16], [17], [18], [19], [22], [23]). In [21] S. Tanno classified the connected almost contact metric manifold whose automorphism group has maximum dimension; there are three classes: (a) homogeneous normal contact Riemannian manifolds with $c > 0$, (b) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (b) global Riemannian product of a line or a circle and a Kaehlerian manifold with constant holomorphic sectional curvature, $C(X, \xi) = 0$. (c) A warped product space $R \times_\delta C^n$, if $C(X, \xi) < 0$. Manifolds of class (a) are characterized by some tensor equations, it has a Sasakian structure and manifolds of class (b) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [9] obtained some tensorial equations to characterize manifolds of class (c). As Kenmotsu manifolds are themselves warped product spaces, it is interesting to study warped product submanifolds in Kenmotsu manifolds. In [2] R. L. Bishop and B. O' Neill introduced the notion of warped product manifolds. In general, these structures are not Sasakian [9]. The study of warped product submanifolds of Kaehler manifolds was introduced by B. Y. Chen [4]. Similar notation have been studied in [14].

2. Preliminaries

Let a $(2n+1)$-dimensional smooth manifold $\mathcal{M}^{(2n+1)}$ is said to be an almost paracontact manifold equipped with almost paracontact structure $(\phi, \xi, \upsilon, <,>)$ consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, a one-form υ and a pseudo-Riemannian metric $<,>$ such that [1]

$$\phi\xi = 0, \quad \phi^2 = I - \upsilon \otimes \xi, \quad \upsilon(\xi) = 1 \tag{1}$$

$$\upsilon \circ \phi = 0, \quad \upsilon(\mathcal{X}) =< \mathcal{X}, \xi >$$

$$< \phi\cdot, \phi\cdot >= - <,> + \upsilon \otimes \upsilon \tag{2}$$

an almost paracontact metric manifold $\bar{\mathcal{M}}$ is called a paracontact metric manifold if there exists a one-form υ such that

$$< \mathcal{X}, \phi\mathcal{Y} >= d\upsilon(\mathcal{X}, \mathcal{Y}) = \frac{1}{2}(\mathcal{X}\upsilon(\mathcal{Y}) - \mathcal{Y}\upsilon(\mathcal{X}) - \upsilon([\mathcal{X}, \mathcal{Y}]) \quad \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{X}M$$

a paracontact metric manifold is called para-Sasakian if it follows,

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = - < \mathcal{X}, \mathcal{Y} > \xi + \eta(\mathcal{Y})\mathcal{X} \tag{3}$$

for all vector fields $\mathcal{X}$ and $\mathcal{Y}$. Further, an almost paracontact metric manifold is called a trans-para-Sasakian manifold if

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = \alpha\{- < \mathcal{X}, \mathcal{Y} > \xi + \upsilon(\mathcal{Y})\mathcal{X}\} + \beta\{< \mathcal{X}, \phi\mathcal{Y} > \xi + \upsilon(\mathcal{Y})\phi\mathcal{X}\} \tag{4}$$

$$\bar{\nabla}_{\mathcal{X}}\xi = -\alpha\phi\mathcal{X} - \beta(\mathcal{X} - \upsilon(\mathcal{X}))\xi, \tag{5}$$

holds for some smooth functions α and β. Now, suppose $\mathcal{M}$ be a submanifold of a contact Lorentzian metric manofold $\bar{\mathcal{M}}$ with the induced metric $<,>$ and ξ be tangent to $\mathcal{M}$. Also suppose ∇ and $\nabla^\perp$ be the induced connections on the tangent bundle $T\mathcal{M}$ and the normal bundle $T^\perp\mathcal{M}$ of $\mathcal{M}$, respectively. Then the Gauss and Weingarteen formulas are given by

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \sigma(\mathcal{X}, \mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{Y} \tag{6}$$

$$\bar{\nabla}_{\mathcal{X}}\lambda = -\Lambda_\lambda\mathcal{X} + \nabla^\perp_{\mathcal{X}}\lambda \tag{7}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and any vector filed λ normal to $\mathcal{M}$, where σ and Λ_λ are the second fundamental form and the shape operator for the immersion of $\mathcal{M}$ into $\bar{\mathcal{M}}$. The second fundamental form σ and shape operator Λ_λ are related by

$$< \sigma(\mathcal{X}, \mathcal{Y}), \lambda >=< \Lambda_\lambda\mathcal{X}, \mathcal{Y} > \tag{8}$$

for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$, we can write

$$\phi\mathcal{X} = T\mathcal{X} + N\mathcal{X} \tag{9}$$

$$\phi\lambda = t\lambda + \mathcal{F}\lambda \tag{10}$$

where $T\mathcal{X}$ and $t\lambda$ are the tangential components of $\phi\mathcal{X}$ and $\phi\lambda$, respectively, where as $N\mathcal{X}$ and $\mathcal{F}\lambda$ are the normal components of $\phi\mathcal{X}$ and $\phi\lambda$, respectively. Thus by using (9) and (10), we can obtain

$$\nabla_{\mathcal{X}}T\mathcal{Y} - T\nabla_{\mathcal{X}}\mathcal{Y} = (\bar{\nabla}_{\mathcal{X}}T)\mathcal{Y}, \nabla^{\perp}_{\mathcal{X}}N\mathcal{Y} - N\nabla_{\mathcal{X}}\mathcal{Y} = (\bar{\nabla}_{\mathcal{X}}N)\mathcal{Y} \tag{11}$$

$$\nabla_{\mathcal{X}}t\lambda - t\nabla^{\perp}_{\mathcal{X}}\lambda = (\bar{\nabla}_{\mathcal{X}}t)\lambda, \nabla^{\perp}_{\mathcal{X}}\mathcal{F}\lambda - \mathcal{F}\nabla^{\perp}_{\mathcal{X}}\lambda = (\bar{\nabla}_{\mathcal{X}}\mathcal{F})\lambda \tag{12}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and vector field λ normal to $\mathcal{M}$. The mean curvature vector σ of $\mathcal{M}$ is given by

$$\mathcal{H} = \frac{1}{m}trace(\sigma) = \frac{1}{m}\sum_{i=1}^{m}\sigma(\varepsilon_i, \varepsilon_i) \tag{13}$$

where m is the dimension of $\mathcal{M}$ and $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_m\}$ is a local orthonormal frame of $\mathcal{M}$. A submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is said to be totally umbilical if

$$< \mathcal{X}, \mathcal{Y} > \mathcal{H} = \sigma(\mathcal{X}, \mathcal{Y}) \tag{14}$$

where σ is the mean curvature vector. A submanifold $\mathcal{M}$ is said to be totally geodesic if $\sigma(\mathcal{X}, \mathcal{Y}) = 0$, For all vector fields $\mathcal{X}, \mathcal{Y}$ tangent to $\mathcal{M}$ and $\mathcal{M}$ is said to be minimal if $\mathcal{H} = 0$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is invariant for $\phi(T_x\mathcal{M}) \subseteq T_x\mathcal{M}$ for every point $x \in \mathcal{M}$ and carring a Riemannian manifold $\mathcal{M}$ isometrically absorbed in an almost contact metric manifold $\bar{\mathcal{M}}$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is anti-invariant for $\phi(T_x\mathcal{M}) \subseteq T_x^{\perp}\mathcal{M}$ for every point $x \in \mathcal{M}$.

If ξ is tangential in $\mathcal{M}$ for a submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ then, the submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is slant [3] for each non zero vector $\mathcal{X}$ tangent to $\mathcal{M}$ at $x \in M$ such that $\mathcal{X}$ is linearly independent to ξ_x, the angle $\theta(\mathcal{X})$ between $\phi\mathcal{X}$ and $T_x\mathcal{M}$ is constant i.e. it does not depend on the choice of the point $x \in M$ and $\mathcal{X} \in T_x\mathcal{M} - \{\xi\}$. In this case, the angle θ is called the slant angle of the submanifold. A slant submanifold $\mathcal{M}$ is proper slant submanifold for neither $\theta = 0$ nor $\theta = \pi/2$. Here $T\mathcal{M} = \mathcal{D}_\theta \oplus \{\xi\}$, where $\mathcal{D}_\theta$ is slant distribution with slant angle θ.

If $\theta = 0$, then the slant submanifolds is said to be an invariant submanifolds and if $\theta = \pi/2$, then slant submanifolds is said to be anti-invariant submanifolds.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is semi-invariant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}^{\perp}$ is anti -invariant i.e. $\phi\mathcal{D}^{\perp} \subset (T^{\perp}\mathcal{M})$.

The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is semi-slant if there exist two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}_\theta$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \{\xi\}$$

where $\mathcal{D}$ is invariant i.e. $\phi\mathcal{D} \subseteq \mathcal{D}$ and $\mathcal{D}_\theta$ is slant with semi slant angle θ. The submanifold $\mathcal{M}$ of an almost contact metric manifold $\bar{\mathcal{M}}$ is hemi-slant [20] if there exist two orthogonal complementary distributions $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ such that

$$T\mathcal{M} = \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \{\xi\}$$

where $\mathcal{D}^\perp$ is anti- invariant i.e. $\phi\mathcal{D}^\perp \subset (T^\perp\mathcal{M})$ and $\mathcal{D}_\theta$ is slant with hemi slant angle θ.

3. Quasi hemi-slant submanifolds of trans para-Sasakian manifolds

The purpose of this section is to study the existence of quasi hemi-slant submanifolds of a trans para-Sasakian manifolds.

We say that $\mathcal{M}$ is quasi hemi-slant submanifold of a trans para-Sasakian manifold $\bar{\mathcal{M}}$ if there exist three orthogonal complementary distributions $\mathcal{D}$, $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ such that
(a) $T\mathcal{M}$ admits the orthogonal direct decomposition

$$T\mathcal{M} = \mathcal{D} \oplus \mathcal{D}_\theta \oplus \mathcal{D}^\perp \oplus \{\xi\}, \quad \xi \in \Gamma(\mathcal{D}_\theta) \tag{15}$$

(b) $\phi\mathcal{D} = \mathcal{D}$
(c) $\phi\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$.
(d) The distribution $\mathcal{D}_\theta$ is a slant with slant constant angle θ, where $\theta =$ slant angle.
In this case, θ is said to be quasi hemi- slant angle of $\mathcal{M}$. If the dimension of distributions $\mathcal{D}, \mathcal{D}_\theta$ and $\mathcal{D}^\perp$ are m_1, m_2 and m_3 respectively, then
(a) $\mathcal{M}$ is a hemi-slant submanifold for $m_1 = 0$.
(b) $\mathcal{M}$ is a semi-invariant submanifold for $m_2 = 0$.
(c) $\mathcal{M}$ is a semi-slant submanifold for $m_3 = 0$.
The quasi hemi-slant submanifold $\mathcal{M}$ is proper if $\mathcal{D} \neq 0$, $\mathcal{D}_\theta \neq 0$, $\mathcal{D}^\perp \neq 0$ and $\theta \neq 0, \pi/2$.
It represents that quasi hemi-slant submanifols is a generalization of invariant, anti-invariant, semi-invarint, slant, hemi-slant, semi-slant submanifolds.

It is clear from the definition that if $\mathcal{D} \neq \{0\}$, $\mathcal{D}_\theta \neq \{0\}$ and $\mathcal{D}^\perp \neq \{0\}$, then $dim\mathcal{D} \geq 2, dim\mathcal{D}_\theta \geq 2$ and $dim\mathcal{D}^\perp \geq 1$. So for proper quasi hemi slant subanifold $\mathcal{M}$, the $dim\mathcal{M} \geq 6$.

Suppose $\mathcal{M}$ be a quasi hemi-slant submanifold of trans para-Sasakian manifold $\bar{\mathcal{M}}$ and the projections on $\mathcal{D}$, $\mathcal{D}_\theta$ and $\mathcal{D}^\perp$ by $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ respectively, then for all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$\mathcal{X} = \mathcal{R}\mathcal{X} + \mathcal{Q}\mathcal{X} + \mathcal{P}\mathcal{X} + v(\mathcal{X})\xi \tag{16}$$

Now put

$$T\mathcal{X} + N\mathcal{X} = \phi\mathcal{X} \tag{17}$$

where $T\mathcal{X}$ and $N\mathcal{X}$ are tangential and normal part of $\phi\mathcal{X}$ on $\mathcal{M}$. From (16) and (17), we derive

$$\phi\mathcal{X} = N\mathcal{R}\mathcal{X} + T\mathcal{R}\mathcal{X} + N\mathcal{Q}\mathcal{X} + T\mathcal{Q}\mathcal{X} + N\mathcal{P}\mathcal{X} + T\mathcal{P}\mathcal{X} \tag{18}$$

As $\phi\mathcal{D} = \mathcal{D}$ and $\phi\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$, we obtain $N\mathcal{P}\mathcal{X} = 0$, and $T\mathcal{R}\mathcal{X} = 0$ and

$$\phi\mathcal{X} = N\mathcal{R}\mathcal{X} + N\mathcal{Q}\mathcal{X} + T\mathcal{Q}\mathcal{X} + T\mathcal{P}\mathcal{X} \tag{19}$$

For all vector field $\mathcal{X}$ tangent to $\mathcal{M}$, we infer

$$T\mathcal{X} = T\mathcal{P}\mathcal{X} + T\mathcal{Q}\mathcal{X}$$

and

$$N\mathcal{X} = N\mathcal{Q}\mathcal{X} + N\mathcal{R}\mathcal{X}$$

Using (19), we deduce the following decompositiona,

$$\phi(T\mathcal{M}) = \mathcal{D} \oplus T\mathcal{D}_\theta \oplus N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp \tag{20}$$

As $N\mathcal{D}_\theta \subseteq T^\perp\mathcal{M}$ and $N\mathcal{D}^\perp \subseteq T^\perp\mathcal{M}$, we obtain

$$T^\perp\mathcal{M} = N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp \oplus \kappa \tag{21}$$

where κ denotes the orthogonal component of $N\mathcal{D}_\theta \oplus N\mathcal{D}^\perp$ in $\Gamma(T^\perp\mathcal{M})$ and invariant with respect to ϕ

For all non-zero vector field λ normal to $\mathcal{M}$, we infer

$$\phi\lambda = t\lambda + f\lambda \tag{22}$$

where $t\lambda$ tangent to $\mathcal{M}$ and $f\lambda$ normal to $\mathcal{M}$.

Proposition 3.1. *For a submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, we infer*

$$\begin{aligned}\nabla_{\mathcal{X}}T\mathcal{Y} &= \Lambda_{N\mathcal{Y}}\mathcal{X} + T\nabla_{\mathcal{X}}\mathcal{Y} + t\sigma(\mathcal{X},\mathcal{Y}) - \alpha < \mathcal{X},\mathcal{Y} > \xi \\ &\quad + \alpha\upsilon(\mathcal{Y})\mathcal{X} + \beta\upsilon(\mathcal{Y})\phi X + \beta < \mathcal{X},\phi\mathcal{Y} > \xi\end{aligned}$$

$$\sigma(\mathcal{X},T\mathcal{Y}) + \nabla^\perp_{\mathcal{X}}N\mathcal{Y} - N\nabla_{\mathcal{X}}\mathcal{Y} - f\sigma(\mathcal{X},\mathcal{Y}) = 0$$

for all vector fields $\mathcal{X},\mathcal{Y}$ tangent to $\mathcal{M}$.

Proposition 3.2. *For a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, we infer*

$$T\mathcal{D} = \mathcal{D}, \quad T\mathcal{D}_\theta = \mathcal{D}_\theta, \quad T\mathcal{D}^\perp = \{0\}, \tag{23}$$

$$tN\mathcal{D}_\theta = \mathcal{D}_\theta, \quad tN\mathcal{D}_\theta = \mathcal{D}^\perp$$

From (17), (22) and $\phi^2 = I - \upsilon \otimes \xi$, we get

Proposition 3.3. *For the endomorphism T and N, t and f of a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$ in the tangent bundle of $\mathcal{M}$, we infer*
(i) $T^2 + tN = I - \upsilon \otimes \xi$ on tangent $\mathcal{M}$
(ii) $NT + fN = 0$ on tangent $\mathcal{M}$
(iii) $Nt + f^2 = I$ on normal $\mathcal{M}$
(iv) $Tt + tf = 0$ on on normal $\mathcal{M}$.

Lemma 3.1. *For a quasi hemi- slant submanifold $\mathcal{M}$ of a trans para-Sasakian $\bar{\mathcal{M}}$, we infer*
(i) $T^2\mathcal{X} = (\cos^2\theta)\mathcal{X}$,
(ii) $< T\mathcal{X}, T\mathcal{Y} >= (\cos^2\theta) < \mathcal{X},\mathcal{Y} >$
(iii) $< N\mathcal{X}, N\mathcal{Y} >= (\sin^2\theta) < \mathcal{X},\mathcal{Y} >$
for all $\mathcal{X},\mathcal{Y} \in \mathcal{D}_\theta$.

Next we state

Proposition 3.4. *For a quasi hemi- slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, we infer*

$$(\bar{\nabla}_{\mathcal{X}}T)\mathcal{Y} = \Lambda_{N\mathcal{Y}}\mathcal{X} + t\sigma(\mathcal{X},\mathcal{Y}) - \alpha < \mathcal{X},\mathcal{Y} > \xi + \alpha\upsilon(\mathcal{Y})\mathcal{X} + \beta < \mathcal{X}, T\mathcal{Y} > \xi + \beta\upsilon(\mathcal{Y})T\mathcal{X} \tag{24}$$

$$(\bar{\nabla}_{\mathcal{X}}N)\mathcal{Y} = \beta\upsilon(\mathcal{Y})N\mathcal{X} + f\sigma(\mathcal{X},\mathcal{Y}) - \sigma(\mathcal{X},T\mathcal{Y}) \tag{25}$$

$$(\bar{\nabla}_{\mathcal{X}}t)\lambda = \Lambda_{f\lambda}\mathcal{X} - T\Lambda_{\lambda}\mathcal{X} \tag{26}$$

and

$$(\bar{\nabla}_{\mathcal{X}}f)\lambda = -\sigma(\mathcal{X},t\lambda) - N\Lambda_{\lambda}\mathcal{X} \tag{27}$$

for all vector fields $\mathcal{X},\mathcal{Y}$ tangent to $\mathcal{M}$ and vector fields λ normal to $\mathcal{M}$.

Proposition 3.5. *For a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, we infer*

$$\nabla_{\mathcal{X}}\xi = -\alpha T\mathcal{X} - \beta\mathcal{X}$$

and

$$\sigma(\mathcal{X},\xi) = -\alpha N\mathcal{X} + \beta\upsilon(\mathcal{X})\xi$$

for all vector fields $\mathcal{X}$ tangent to $\mathcal{M}$.

Lemma 3.2. *For a quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, we infer*

$$\Lambda_{\phi\mathcal{Z}}\mathcal{W} = \Lambda_{\phi\mathcal{W}}\mathcal{Z}$$

for all $\mathcal{Z},\mathcal{W} \in \mathcal{D}^{\perp}$.

Lemma 3.3. *For a quasi hemi- slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, we infer*

$$< [\mathcal{Y},\mathcal{X}],\xi > -2\alpha < T\mathcal{Y},\mathcal{X} >= 0$$

$$< \bar{\nabla}_{\mathcal{Y}}\mathcal{X},\xi > -\alpha < T\mathcal{Y},\mathcal{X} > -\beta < \mathcal{Y},\mathcal{X} > +\beta\upsilon(\mathcal{Y})\upsilon(\mathcal{X}) = 0$$

for all $\mathcal{Y},\mathcal{X} \in \Gamma(\mathcal{D}\oplus\mathcal{D}_{\theta}\oplus\mathcal{D}^{\perp})$.

4. Warped product quasi hemi-slant submanifolds

If $(N_1, <,>_1)$ and $(N_2, <,>_2)$ are two Riemannian manifolds and δ, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times_{\delta} N_2 = (N_1 \times N_2, <,>)$, where

$$<,>=<,>_1 + \delta^2 <,>_2 \tag{28}$$

A warped product manifold $N_1 \times_{\delta} N_2$ is said to be trivial if the warping function δ is constant. We recall the following general formula on a warped product [2] result for later use

$$\nabla_{\mathcal{X}}\mathcal{Z} = \nabla_{Z}\mathcal{X} = (\mathcal{X}\ln\delta)\mathcal{Z}, \tag{29}$$

where $\mathcal{X}$ is tangent to N_1 and $\mathcal{Z}$ is tangent to N_2.
If $\mathcal{M} = N_1 \times_{\delta} N_2$ is a warped product manifold, this means that N_1 is totally geodesic and N_2 is totally umbilical submanifold of $\mathcal{M}$, respectively.

The following corollary shows that the warped product of the type $\mathcal{M} = N_1 \times_{\delta} N_2$ is trivial if $\xi \in N_2$.

Corollary 4.1. *If $\bar{\mathcal{M}}$ is a trans para-Sasakian manifold and N_1 and N_2 be any Riemannian submanifolds of $\bar{\mathcal{M}}$, then there does not exist a warped product submanifold $\mathcal{M} = N_1 \times_\delta N_2$ of $\bar{\mathcal{M}}$ such that ξ is tangential to N_2.*

Lemma 4.1. *If $\mathcal{M} = N_1 \times_\delta N_2$ is a warped product submanifold of trans para-Sasakian manifolds $\bar{\mathcal{M}}$ such that N_1 tangent to ξ , where N_1 and N_2 are any Riemannian submanifolds of $\bar{\mathcal{M}}$, then for any $\mathcal{X}, \mathcal{Y} \in \Gamma(TN_1)$ and $\mathcal{Z}, \mathcal{W} \in \Gamma(TN_2)$, we have*
(i) $\xi \ln \delta = -\alpha T - \beta$,
(ii) $< \sigma(\mathcal{X}, \mathcal{Y}), N\mathcal{Z} >=< \sigma(\mathcal{X}, \mathcal{Z}), N\mathcal{Y} > -\alpha \upsilon(\mathcal{Y}) < \mathcal{X}, \mathcal{Z} >$,
(iii) $< \sigma(\mathcal{X}, \mathcal{Z}), N\mathcal{W} >=< \sigma(\mathcal{X}, \mathcal{W}), N\mathcal{Z} >$,

Lemma 4.2. *If $\mathcal{M} = N_T \times_\delta N_\theta$ is a quasi hemi-slant warped product submanifolds of a trans para-Sasakian manifold $\bar{\mathcal{M}}$, then for any $\mathcal{X}, \mathcal{Y} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$, we have*

$$< t\sigma(\mathcal{X}, \mathcal{Y}), N\mathcal{Z} >= -\alpha \upsilon(\mathcal{Y}) < \mathcal{X}, N\mathcal{Z} > \tag{30}$$

Proof. As N_T is totally geodesic in $\mathcal{M}$ then $(\bar{\nabla}_\mathcal{X} T)\mathcal{Y} \in \Gamma(TN_T)$ and therefore by formula (23):

$$\begin{aligned}(\bar{\nabla}_\mathcal{X} T)\mathcal{Y} &= t\sigma(\mathcal{X}, \mathcal{Y}) + \alpha\{\upsilon(\mathcal{Y})\mathcal{X} - < \mathcal{X}, \mathcal{Y} > \xi\} \\ &\quad + \beta\{< \mathcal{X}, T\mathcal{Y} > \xi + \upsilon(\mathcal{Y}) T\mathcal{X}\}\end{aligned}$$

taking inner product with $\mathcal{Z} \in \Gamma(TN_\theta)$ we get (29). Now we have the following Characterization. □

From Corollary 4.1 the warped product submanifolds of the type $\mathcal{M} = N_1 \times_\delta N_2$ of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$ do not exist if the structure vector field ξ is tangent to N_2. Now, we examine warped product quasi hemi-slant submanifold $\mathcal{M} = N_1 \times_\delta N_2$ of $\bar{\mathcal{M}}$, when $\xi \in TN_1$. Let N_θ and N_T (resp. $N_\perp$) be two slant and invariant (resp. anti-invariant) submanifolds of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$, then their warped product quasi hemi-slant submanifold may given by one of the following forms:
(i) $N_T \times_\delta N_\theta$ *(ii)* $N_\perp \times_\delta N_\theta$,
(iii) $N_\theta \times_\delta N_T$ *(iv)* $N_\theta \times_\delta N_\perp$.

In this paper we are concerned with cases (i) and (ii) . For the warped products of the type (i), we have the following lemma.

Lemma 4.3. *If $\mathcal{M} = N_T \times_\delta N_\theta$ is a warped product quasi hemi-slant submanifold of a trans para-Sasakian manifolds $\bar{\mathcal{M}}$ such that ξ is tangent to N_T where N_T and N_θ are invariant and proper slant submanifolds of $\bar{\mathcal{M}}$, then for any $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$, we have*
(i) $< \sigma(\mathcal{X}, \mathcal{Z}), NT\mathcal{Z} >=< \sigma(\mathcal{X}, T\mathcal{Z}), N\mathcal{Z} >= -\{\mathcal{X} \ln \delta + \upsilon(\mathcal{X})\} \cos^2 \theta ||\mathcal{Z}||^2$,
(ii) $< \sigma(\mathcal{X}, \mathcal{Z}), N\mathcal{Z} >= -(T\mathcal{X} \ln \delta)||\mathcal{Z}||^2$,

Proof. The equality first and second of (i) follows directly by Lemma 4.2 (iii). $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$ we obtain

$$(\bar{\nabla}_\mathcal{X} \phi)\mathcal{Z} = \bar{\nabla}_\mathcal{X} \phi\mathcal{Z} - \phi\bar{\nabla}_\mathcal{X} \mathcal{Z}$$

On using (4) and the fact that ξ is tangent to N_T, then

$$-\alpha < \mathcal{X}, \mathcal{Z} > \xi = \bar{\nabla}_\mathcal{X} \phi Z - \phi\bar{\nabla}_\mathcal{X} \mathcal{Z}$$

Thus, from (6), (7), (9) and (10) we obtain

$$\begin{aligned}\alpha < \mathcal{X}, \mathcal{Z} > \xi &= \nabla_\mathcal{X} T\mathcal{Z} + \sigma(\mathcal{X}, T\mathcal{Z}) - \Lambda_{N\mathcal{Z}}\mathcal{X} + \nabla^\perp_\mathcal{X} N\mathcal{Z} \\ &\quad - T\nabla_\mathcal{X} \mathcal{Z} - N\nabla_\mathcal{X} \mathcal{Z} - t\sigma(\mathcal{X}, \mathcal{Z}) - f\sigma(\mathcal{X}, \mathcal{Z})\end{aligned}$$

Equating the tangential and normal components and using (11), we get

$$(\bar{\nabla}_{\mathcal{X}}T)\mathcal{Z} = \Lambda_{N\mathcal{Z}}\mathcal{X} + t\sigma(\mathcal{X},\mathcal{Z}) + \alpha < \mathcal{X},\mathcal{Z} > \xi \tag{31}$$

and

$$(\bar{\nabla}_{\mathcal{X}}N)\mathcal{Z} = f\sigma(\mathcal{X},\mathcal{Z}) - \sigma(\mathcal{X},T\mathcal{Z}) \tag{32}$$

On the other hand for any $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$, we have

$$(\bar{\nabla}_{\mathcal{Z}}\phi)\mathcal{X} = \bar{\nabla}_{\mathcal{Z}}\phi\mathcal{X} - \phi\bar{\nabla}_{\mathcal{Z}}\mathcal{X}.$$

Using the structure equation of trans para-Sasakian manifolds and the fact that ξ is tangent to N_T , we get

$$\begin{aligned}\alpha < \mathcal{Z},\mathcal{X} > \xi &= \alpha v(\mathcal{X})\mathcal{Z} + \beta < \mathcal{Z},\phi\mathcal{X} > \xi + \beta v(\mathcal{X})\phi\mathcal{Z} - \nabla_{\mathcal{Z}}\phi\mathcal{X}\\ &\quad -\sigma(\mathcal{Z},\phi\mathcal{X}) + T\nabla_{\mathcal{Z}}\mathcal{X} + N\nabla_{\mathcal{Z}}\mathcal{X} + t\sigma(\mathcal{X},\mathcal{Z}) + f\sigma(\mathcal{X},\mathcal{Z})\end{aligned}$$

From orthogonality of distributions, we obtain

$$\begin{aligned}\alpha < \mathcal{Z},\mathcal{X} > \xi &= \alpha v(\mathcal{X})\mathcal{Z} + \beta v(\mathcal{X})T\mathcal{Z} + \beta v(\mathcal{X})N\mathcal{Z} - \nabla_{\mathcal{Z}}T\mathcal{X}\\ &\quad -\sigma(\mathcal{Z},T\mathcal{X}) + T\nabla_{\mathcal{Z}}\mathcal{X} + N\nabla_{\mathcal{Z}}\mathcal{X} + t\sigma(\mathcal{X},\mathcal{Z}) + f\sigma(\mathcal{X},\mathcal{Z})\end{aligned}$$

Equating tangential and normal components, we get

$$(\bar{\nabla}_{\mathcal{Z}}T)\mathcal{X} = t\sigma(\mathcal{X},\mathcal{Z}) - \alpha < (\mathcal{Z},\mathcal{X})\xi + \alpha v(\mathcal{X})\mathcal{Z} + \beta v(\mathcal{X})T\mathcal{Z} \tag{33}$$

and

$$N(\nabla_{\mathcal{Z}}\mathcal{X}) = \sigma(T\mathcal{X},\mathcal{Z}) - f\sigma(\mathcal{X},\mathcal{Z}) + \beta v(\mathcal{X})N\mathcal{Z} \tag{34}$$

Then, from (31) and (33) we have

$$(\bar{\nabla}_{\mathcal{X}}T)\mathcal{Z} + (\bar{\nabla}_{\mathcal{Z}}T)\mathcal{X} = \Lambda_{N\mathcal{Z}}\mathcal{X} + 2t\sigma(\mathcal{X},\mathcal{Z}) + \alpha v(\mathcal{X})\mathcal{Z} + \beta v(\mathcal{X})N\mathcal{Z} \tag{35}$$

Using (12) and (28), we obtain

$$(T\mathcal{X}\ln\delta)\mathcal{Z} - (\mathcal{X}\ln\delta)T\mathcal{Z} = \Lambda_{N\mathcal{Z}}\mathcal{X} + 2t\sigma(\mathcal{X},\mathcal{Z}) + \alpha v(\mathcal{X})\mathcal{Z} + \beta v(\mathcal{X})T\mathcal{Z} \tag{36}$$

Taking product with $T\mathcal{Z}$ and then using (8), we get

$$\begin{aligned}-(\mathcal{X}\ln\delta) < T\mathcal{Z},T\mathcal{Z} > &= < \sigma(\mathcal{X},T\mathcal{Z}),N\mathcal{Z} > +2 < t\sigma(\mathcal{X},\mathcal{Z}),N\mathcal{Z})\\ &\quad +\beta v(\mathcal{X}) < T\mathcal{Z},T\mathcal{Z} >\end{aligned}$$

Then on applying Lemma 3.4 (ii) we obtain

$$-\{(\mathcal{X}\ln\delta) + \beta v(\mathcal{X})\}\cos^2\theta||\mathcal{Z}||^2 = 2 < \phi\sigma(\mathcal{X},\mathcal{Z}),TZ > + < \sigma(\mathcal{X},T\mathcal{Z}),N\mathcal{Z})$$

or

$$-\{(\mathcal{X}\ln\delta) + \beta v(\mathcal{X})\}\cos^2\theta||\mathcal{Z}||^2 = -2 < \sigma(\mathcal{X},\mathcal{Z}),NT\mathcal{Z} > + < \sigma(\mathcal{X},T\mathcal{Z}),N\mathcal{Z})$$

Thus by Lemma 3.1 (iii), we obtain

$$< \sigma(\mathcal{X},\mathcal{Z}),NT\mathcal{Z} > -\{\mathcal{X}\ln\delta + \beta v(\mathcal{X})\}\cos^2\theta||\mathcal{Z}||^2 \tag{37}$$

This is the first and third equality of (i). Now, for part (ii), taking product in (36) with $\mathcal{Z} \in \Gamma(TN_\theta)$ we obtain

$$(T\mathcal{X}\ln\delta)||\mathcal{Z}||^2 =< \sigma(\mathcal{X},\mathcal{Z}),N\mathcal{Z} > +2 < t\sigma(\mathcal{X},\mathcal{Z}),\mathcal{Z})$$

or

$$(T\mathcal{X}\ln\delta)||\mathcal{Z}||^2 =< \sigma(\mathcal{X},\mathcal{Z}),N\mathcal{Z} > -2 < \sigma(\mathcal{X},\mathcal{Z}),N\mathcal{Z})$$

that is,

$$< \sigma(\mathcal{X},\mathcal{Z}),N\mathcal{Z} >= -(T\mathcal{X}\ln\delta)||\mathcal{Z}||^2$$

□

The following theorems provide an explicit mechanism of warped product quasi hemi-slant submanifold $\mathcal{M} = N_T \times_\delta N_\theta$ of trans para-Sasakian manifold.

Theorem 4.1. *If $\mathcal{M} = N_T \times_\delta N_\theta$ is a warped product quasi hemi-slant submanifold of a trans para-Sasakian manifold $\bar{\mathcal{M}}$ such that $\sigma(\mathcal{X}, \mathcal{Z}) \in \mu$, then at least one of the following statements is true:*
(i) $\mathcal{X} \ln \delta = -\beta \upsilon(\mathcal{X})$
(ii) *$\mathcal{M}$ is a CR-warped product,*
(iii) *$\mathcal{M}$ is an invariant submanifold for each $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$.*

Proof. The given statement is $\sigma(\mathcal{X}, \mathcal{Z}) \in \mu$ for each $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$, then by (37) we have

$$-\{\mathcal{X} \ln \delta + \beta \upsilon(\mathcal{X})\} \cos^2 \theta ||\mathcal{Z}||^2 = 0 \tag{38}$$

This means that either $\mathcal{X} \ln \delta + \beta \upsilon(\mathcal{X}) = 0$ or $\theta = \frac{\pi}{2}$ i.e., $\mathcal{M} = N_T \times_\delta N_\perp$ is a CR-warped product submanifold or $N_\theta = 0$. This proves the theorem. □

Theorem 4.2. *If $\mathcal{M} = N_T \times_\delta N_\theta$ is a warped product quasi hemi-slant submanifold of a trans para-Sasakian manifold $\bar{\mathcal{M}}$ such that $\xi \in \Gamma(TN_T)$, then $(\bar{\nabla}_\mathcal{X} N)\mathcal{Z} \neq \mu$ for each $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$, where μ is an invariant normal subbundle of $T\mathcal{M}$.*

Proof. As ξ is tangent to TN_T, then by (2) we have

$$< \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}, \phi \mathcal{Z} >= - < \bar{\nabla}_\mathcal{X} \mathcal{Z}, \mathcal{Z} >$$

For any $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$, using (6) and (28), we obtain

$$< \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}, \phi \mathcal{Z} >= - < \bar{\nabla}_\mathcal{X} \mathcal{Z}, \mathcal{Z} >= -(\mathcal{X} \ln \delta) ||\mathcal{Z}||^2 \tag{39}$$

On the other hand, we have

$$(\bar{\nabla}_\mathcal{X} \phi)\mathcal{Z} = \bar{\nabla}_\mathcal{X} \phi \mathcal{Z} - \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}$$

for any $\mathcal{X} \in \Gamma(TN_T)$ and $\mathcal{Z} \in \Gamma(TN_\theta)$. On using (4) and the fact that $\xi \in \Gamma(TN_T)$, then by orthogonality of two distributions, we have

$$-\alpha < \mathcal{X}, \mathcal{Z} > \xi = \bar{\nabla}_\mathcal{X} \phi \mathcal{Z} - \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}$$

Then by (9), we have

$$-\alpha < \mathcal{X}, \mathcal{Z} > \xi + \phi \bar{\nabla}_\mathcal{X} \mathcal{Z} = \bar{\nabla}_\mathcal{X} T\mathcal{Z} + \bar{\nabla}_\mathcal{X} N\mathcal{Z}$$

On using (6) and (7), we obtain

$$-\alpha < \mathcal{X}, \mathcal{Z} > \xi + \phi \bar{\nabla}_\mathcal{X} \mathcal{Z} = \nabla_\mathcal{X} T\mathcal{Z} + \sigma(\mathcal{X}, T\mathcal{Z}) - \Lambda_{N\mathcal{Z}} \mathcal{X} + \nabla^\perp_\mathcal{X} N\mathcal{Z}$$

Taking product with $\phi \mathcal{Z}$ and using (8), (9), we get

$$< \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}, \phi \mathcal{Z} >=< \nabla_\mathcal{X} T\mathcal{Z}, T\mathcal{Z} > + < \nabla^\perp_\mathcal{X} N\mathcal{Z}, N\mathcal{Z} >$$

Thus by (11) and (28) we obtain

$$< \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}, \phi \mathcal{Z} >= (\mathcal{X} \ln \delta) < T\mathcal{Z}, T\mathcal{Z} > + < (\bar{\nabla}_\mathcal{X} N)\mathcal{Z}, N\mathcal{Z} > + < N \nabla_\mathcal{X} \mathcal{Z}, N\mathcal{Z} >$$

Which, on using Lemma 3.4, implies

$$< \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}, \phi \mathcal{Z} >= (\mathcal{X} \ln \delta) \cos^2 \theta ||\mathcal{Z}||^2 + < (\bar{\nabla}_\mathcal{X} N)\mathcal{Z}, N\mathcal{Z} > + \sin^2 \theta < \nabla_\mathcal{X} \mathcal{Z}, \mathcal{Z} >$$

By (28) and (39), we get

$$-(\mathcal{X} \ln \delta) ||\mathcal{Z}||^2 = (\mathcal{X} \ln \delta) cos^2 \theta ||\mathcal{Z}||^2 + < (\bar{\nabla}_\mathcal{X} N)\mathcal{Z}, N\mathcal{Z} > + (\mathcal{X} \ln \delta) \sin^2 \theta ||\mathcal{Z}||^2$$

Therefore,

$$< (\bar{\nabla}_\mathcal{X} N)\mathcal{Z}, N\mathcal{Z} >= -2(\mathcal{X} \ln \delta) ||\mathcal{Z}||^2 \tag{40}$$

As $\mathcal{Z} \in \Gamma(TN_\perp)$, then $N\mathcal{Z} \in \Gamma(NT\mathcal{M})$ then by orthogonality of normal space, we obtain $(\bar{\nabla}_\mathcal{X} N)\mathcal{Z} \neq \mu$. □

The other case is dealt with by the following theorem.

Theorem 4.3. *If $\mathcal{M} = N_\perp \times_\delta N_\theta$ is a warped product quasi hemi-slant submanifold of a trans para-Sasakian manifold $\bar{\mathcal{M}}$ such that $\xi \in \Gamma(TN_\perp)$, then for each $\mathcal{Z} \in \Gamma(TN_\perp)$, at least one of the following statements is true:*
(i) $\mathcal{Z} \ln \delta = -\beta \upsilon(\mathcal{Z})$,
(ii) $\mathcal{M}$ is an anti-invariant submanifold.

Proof. Let $\mathcal{X} \in \Gamma(TN_\theta)$ and $\mathcal{Z} \in \Gamma(TN_\perp)$, we have

$$(\bar{\nabla}_\mathcal{X} \phi)\mathcal{Z} = \bar{\nabla}_\mathcal{X} \phi Z - \phi \bar{\nabla}_\mathcal{X} \mathcal{Z}$$

Using (4), (6), (7) and (9) we obtain

$$\begin{aligned} \alpha < \mathcal{X}, \mathcal{Z} > \xi &= \alpha \upsilon(\mathcal{Z})\mathcal{X} - \beta < T\mathcal{X}, \mathcal{Z} > \xi + \beta \upsilon(\mathcal{Z})T\mathcal{X} + \beta \upsilon(\mathcal{Z})N\mathcal{X} \\ &\quad + \Lambda_{N\mathcal{Z}}\mathcal{X} - \nabla^\perp_\mathcal{X} N\mathcal{Z} + T\nabla_\mathcal{X}\mathcal{Z} + N\nabla_\mathcal{X}\mathcal{Z} + t\sigma(\mathcal{X}, \mathcal{Z}) + f\sigma(\mathcal{X}, \mathcal{Z}) \end{aligned}$$

From the orthogonality of distributions, we have

$$-\alpha < \mathcal{X}, \mathcal{Z} > \xi + \alpha \upsilon(\mathcal{Z})\mathcal{X} + \beta \upsilon(\mathcal{Z})T\mathcal{X} = -\Lambda_{N\mathcal{Z}}\mathcal{X} - T\nabla_\mathcal{X}\mathcal{Z} - t\sigma(\mathcal{X}, \mathcal{Z})$$

Thus by (28), we have

$$-\alpha < \mathcal{X}, \mathcal{Z} > \xi + \alpha \upsilon(\mathcal{Z})\mathcal{X} + \beta \upsilon(\mathcal{Z})T\mathcal{X} = -\Lambda_{N\mathcal{Z}}\mathcal{X} - (\mathcal{Z} \ln \delta)T\mathcal{X} - t\sigma(\mathcal{X}, \mathcal{Z}) \tag{41}$$

Taking product with $T\mathcal{X}$ in equation (41) and making use of formula (8) and Lemma 3.4, we obtain

$$\begin{aligned} \beta \upsilon(\mathcal{Z}) \cos^2 ||\mathcal{Z}||^2 &= - < \sigma(\mathcal{X}, T\mathcal{X}), N\mathcal{Z} > -(\mathcal{Z} \ln \delta) \cos^2 \theta ||\mathcal{X}||^2 \\ &\quad - < t\sigma(\mathcal{X}, \mathcal{Z}), T\mathcal{X} > \end{aligned}$$

That is,

$$\begin{aligned} \{\beta \upsilon(\mathcal{Z}) + (\mathcal{Z} \ln \delta)\} \cos^2 \theta ||\mathcal{X}||^2 &= - < \sigma(\mathcal{X}, T\mathcal{X}), N\mathcal{Z} > \\ &\quad + < \sigma(\mathcal{X}, \mathcal{Z}), NT\mathcal{X} > \end{aligned} \tag{42}$$

As $\theta \neq \frac{\pi}{2}$, interchanging $\mathcal{X}$ by $T\mathcal{X}$ in (42) and taking account of Lemma 3.4, we deduce that

$$\begin{aligned} \{\beta \upsilon(\mathcal{Z}) + (\mathcal{Z} \ln \delta)\} \cos^4 \theta ||\mathcal{X}||^2 &= \cos^2 \theta < \sigma(T\mathcal{X}, \mathcal{X}), N\mathcal{Z} > \\ &\quad - \cos^2 \theta < \sigma(T\mathcal{X}, \mathcal{Z}), N\mathcal{X} > \end{aligned}$$

i.e.,

$$\{\beta \upsilon(\mathcal{Z}) + (\mathcal{Z} \ln \delta)\} \cos^2 \theta ||\mathcal{X}||^2 = < \sigma(T\mathcal{X}, \mathcal{X}), N\mathcal{Z} > - < \sigma(T\mathcal{X}, \mathcal{Z}), N\mathcal{X} > \tag{43}$$

Adding equations (42) and (43), we get

$$2\{\beta \upsilon(\mathcal{Z}) + (\mathcal{Z} \ln \delta)\} \cos^2 \theta ||\mathcal{X}||^2 = - < \sigma(T\mathcal{X}, \mathcal{Z}), N\mathcal{X} > + < \sigma(\mathcal{X}, \mathcal{Z}), NT\mathcal{X} > \tag{44}$$

The right hand side of the above equation is zero by Lemma 4.2 (iii), then

$$\{\beta \upsilon(\mathcal{Z}) + (\mathcal{Z} \ln \delta)\} \cos^2 \theta ||\mathcal{X}||^2 = 0 \tag{45}$$

Thus, either $\beta \upsilon(\mathcal{Z}) = -(\mathcal{Z} \ln \delta)$ or $\theta = \frac{\pi}{2}$ or $N_\theta = 0$. □

Theorem 4.4. *A quasi hemi-slant submanifold $\mathcal{M}$ of a trans para-Sasakian manifold $\bar{\mathcal{M}}$ with integrable invariant distribution $\mathcal{D}_T\oplus <\xi>$ and integrable slant distribution $\mathcal{D}_\theta$ is locally a quasi hemi-slant warped product if and only if $\nabla_{\mathcal{Z}}T\mathcal{Z}\in\mathcal{D}_\theta$ and there exists a C^∞- function α on $\mathcal{M}$ with $\mathcal{Z}\alpha=0$,*

$$\Lambda_{N\mathcal{Z}}\mathcal{X}=\mathcal{X}\alpha T\mathcal{Z}-T\mathcal{X}\alpha\mathcal{Z}+\beta\upsilon(\mathcal{Z})T\mathcal{X} \tag{46}$$

for all $\mathcal{X}\in\Gamma(\mathcal{D}_T\oplus\{\xi\})$ and $\mathcal{Z}\in\Gamma(\mathcal{D}_\theta)$.

Proof. From (10) and (28) we have

$$\Lambda_{N\mathcal{Z}}\mathcal{X}+t\sigma(\mathcal{X},\mathcal{Z})+\alpha\{\upsilon(\mathcal{Z})\mathcal{X}-<\mathcal{X},\mathcal{Z}>\xi\}=0 \tag{47}$$

Similarly,

$$T\mathcal{X}\ln\delta\mathcal{Z}-\mathcal{X}\ln\delta T\mathcal{Z}=t\sigma(\mathcal{X},\mathcal{Z})+\alpha\{\upsilon(\mathcal{Z})\mathcal{X}-<\mathcal{X},\mathcal{Z}>\xi\}+\beta\upsilon(\mathcal{Z})T\mathcal{X} \tag{48}$$

from (47) and (48), we get

$$\Lambda_{N\mathcal{Z}}\mathcal{X}=\mathcal{X}\ln\delta T\mathcal{Z}-T\mathcal{X}ln\delta\mathcal{Z}+\beta\upsilon(\mathcal{Z})T\mathcal{X} \tag{49}$$

taking inner product with $\mathcal{W}\in\Gamma(TN_\theta)$, we have

$$\begin{aligned}<\Lambda_{N\mathcal{Z}}\mathcal{X},\mathcal{W}> &= \mathcal{X}\ln\delta<T\mathcal{Z},\mathcal{W}>-T\mathcal{X}\ln\delta<\mathcal{Z},\mathcal{W}>\\ &\quad+\beta\upsilon(\mathcal{Z})<T\mathcal{X},\mathcal{W}>\end{aligned} \tag{50}$$

From Lemma 4.3 and (50) we get the desired result.
Conversely, let $\mathcal{M}$ be a quasi hemi-slant submanifold of $\bar{\mathcal{M}}$ satisfying the hypothesis of the theorem, then for any $\mathcal{X},\mathcal{Y}\in\Gamma(\mathcal{D}_T\oplus\{\xi\})$ and $\mathcal{Z}\in\Gamma(\mathcal{D}_\theta)$

$$<t\sigma(\mathcal{X},\mathcal{Y})+\alpha\upsilon(\mathcal{Y})\mathcal{X},N\mathcal{Z}>=0 \tag{51}$$

that means $\sigma(\mathcal{X},\mathcal{Y})\in\mu$. Then from (24)

$$-N\nabla_{\mathcal{X}}\mathcal{Y}=f\sigma(\mathcal{X},\mathcal{Y})-\sigma(\mathcal{X},T\mathcal{Y}) \tag{52}$$

Since $\sigma(\mathcal{X},\mathcal{Y})\in\mu$, then we have $N\nabla_{\mathcal{X}}\mathcal{Y}=0$, that is, $\nabla_{\mathcal{X}}\mathcal{Y}\in\Gamma(\mathcal{D}_T\oplus\{\xi\})$. Hence, each leaf of $\mathcal{D}_T\oplus\{\xi\}$ is totally geodesic in $\mathcal{M}$.
Further, suppose N_θ be a leaf of $\mathcal{D}_\theta$ and σ_θ be second fundamental form of the immersion of N_θ in $\mathcal{M}$, then for any $\mathcal{X}\in\Gamma(\mathcal{D}_T\oplus\{\xi\})$ and $\mathcal{Z}\in\Gamma(\mathcal{D}_\theta)$, we have

$$<\sigma_\theta(\mathcal{Z},\mathcal{Z}),\phi\mathcal{X}>=<\nabla_{\mathcal{Z}}\mathcal{Z},\phi\mathcal{X}> \tag{53}$$

using (6), (7) and (9), the above equation yields

$$<\sigma_\theta(\mathcal{Z},\mathcal{Z}),\phi\mathcal{X}>=<\nabla_{\mathcal{Z}}T\mathcal{Z},\mathcal{X}>+<\Lambda_{N\mathcal{Z}}\mathcal{Z},\mathcal{X}> \tag{54}$$

applying (46), we get

$$<\sigma_\theta(\mathcal{Z},\mathcal{Z}),\phi\mathcal{X}>=-T\mathcal{X}\ln\delta<\mathcal{Z},\mathcal{Z}> \tag{55}$$

Replacing $\mathcal{X}$ by $T\mathcal{X}$, the above equation gives

$$\sigma_\theta(\mathcal{Z},\mathcal{Z})=\nabla\alpha<\mathcal{Z},\mathcal{Z}> \tag{56}$$

From above equation it is easy to derive

$$\sigma_\theta(\mathcal{Z},\mathcal{W})=\nabla\alpha<\mathcal{Z},\mathcal{W}> \tag{57}$$

that is, N_θ is totally umbilical and as $\mathcal{Z}\alpha=0$, for all $\mathcal{Z}\in\Gamma(\mathcal{D}_\theta)$, $\nabla\mu$ is defined on N_T, this mean that mean curvature vector of N_θ is parallel, that is, the leaves of $\mathcal{D}_\theta$ are extrinsic spheres in $\mathcal{M}$. Hence, the tangent bundle of a Riemannian manifold $\mathcal{M}$ splits into an orthogonal sum $T\mathcal{M}=\mathcal{E}_0\oplus\mathcal{E}_1$ of nontrivial vector subbundles such that $\mathcal{E}_1$ is spherical and its orthogonal complement $\mathcal{E}_0$ is autoparallel, then the manifold $\mathcal{M}$

is locally isometric to a warped product $\mathcal{M}_0 \times_\delta \mathcal{M}_1$, we can say $\mathcal{M}$ is locally semi-slant warped product submanifold $N_T \times_\delta N_\theta$, where the warping function $\delta = e^\alpha$. □

5. Conclusion

Thus there exist quasi hemi-slant submanifolds as a generalization of slant submanifolds, semi-slant submanifolds and hemi-slant submanifolds for a trans para-Sasakian manifold. We worked out some important results in the direction of warped product submanifolds of a quasi-hemi slant submanifolds within the framework of trans para-Sasakian manifolds with their geometry. The existence of such warped product of the types $N_T \times_\delta N_\theta$ and $N_\perp \times_\delta N_\theta$ in trans para Sasakian manifolds is shown some interesting results.

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