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TRAVELLING WAVE SOLUTIONS FOR THE TIME-FRACTIONAL EQUATIONS BY THE SINE-GORDON EXPANSION METHOD

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Abstract. The aim of this paper is to explore travelling wave solutions by utilising the novel sine-Gordon expansion method for the time-fractional $(1+1)$-dimensional Hirota Satsuma equation and the time-fractional $(2+1)$ -dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equations. Using the traveling wave transformation, the fractional PDE turns into an ODE. Applying the auxiliary equation from the described method, we get an algebraic polynomial, setting the like power to zero, we get a system of algebraic equations. Solving these equations by using mathematical software program, we acquire the solution sets for the constants. Abundant travelling wave solutions are obtained and expressed in terms of hyperbolic functions. Some graphics of the solutions have also been presented. The proposed method is direct and effective in solving nonlinear evolution equations.

Keywords: Sine-Gordon expansion method, travelling wave solution, Hirota Satsuma equation, Caudrey-Dodd-Gibbon-Kotera-Sawada equation.

AMS Subject Classification: 83-02, 99A00

1. Introduction

In the last decade, fractional differential equations(FDEs) are attracted a great deal of attention and widely used in many scientific research field by researchers. FDEs can be classified as an extension of classical ordinary differential equations which they have integer order. The exact solutions of nonlinear fractional partial differential equations (FPDEs) play a significant role in understanding the nonlinear physical systems which are determinated by these FPDEs. For instance FPDEs can be used to describe various complex phenomena such as in fluid flow, acoustic waves, signal processing, viscoelasticity, systems identification, control theory, etc. The investigation of exact solutions of nonlinear FPDEs come up with many interesting and useful methods which are developed by researchers. These methods are as follows; the trial function method [1], the Jacobi elliptic function expansion [2, 50, 53, 54], the fractional sub-equation method [3, 4, 5], the homogeneous balance method [6, 7, 8], the differential transformation method [9, 10], the

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exponential function method [11, 12], the sub-ODE method [13, 14], Adomian decomposition method [15, 16], the $(G^{'}/G)$-expansion method [17, 19, 51], the $(G^{'}/G, 1/G)$-expansion method [18, 64], the $(G^{'}/G^2)$-expansion method [65], the homotopy analysis method [20], the tanh-function expansion method [21, 57, 59, 61], the tanh-coth method [55, 63], the mapping method [52], the modified simple equation (MSE) method [58], the advanced exponential function method [62] and so on.

In this study, we consider the sine-Gordon expansion method [22, 23, 24, 56] for solving the time fractional (1 + 1)-dimensional Hirota-Satsuma (HS) equation and the time-fractional nonlinear (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equation (CDGKS).

The time-fractional (1 + 1)-dimensional Hirota-Satsuma equation is a nonlinear partial differential equation that arises in the field of mathematical physics. It is an extension of the Hirota-Satsuma equation, which is a well-known integrable equation with applications in various areas, including nonlinear optics, plasma physics, and fluid dynamics. The time-fractional version of the (1 + 1)-dimensional Hirota-Satsuma equation incorporates fractional derivatives in time, which allows for modeling anomalous diffusion and nonlocal effects in physical systems. Fractional derivatives are generalizations of ordinary derivatives to non-integer orders, and they introduce memory and long-range interactions into the equation. The study of the time-fractional (1 + 1)-dimensional Hirota-Satsuma equation involves investigating its properties, such as soliton solutions, integrability, and the effect of fractional derivatives on the dynamics of the system. Understanding this equation and its solutions can provide valuable insights into the behavior of complex physical phenomena and help in the development of new mathematical techniques for modeling and analysis [25].

The time-fractional nonlinear (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equation is a significant generalization of the classical Caudrey-Dodd-Gibbon-Kotera-Sawada equation, incorporating fractional derivatives in time. This equation has attracted attention in the field of nonlinear science and mathematical physics due to its ability to model various physical phenomena with nonlocal and memory effects.

In order to transform FPDEs into integer order differential equations, we use the modified Riemann-Liouville derivative proposed by Jumarie [26], which is defined as in the following

$$D_t^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int\limits_0^t (t-\xi)^{-\alpha-1} \left[f(\xi) - f(0) \right] d\xi \ , & \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int\limits_0^t (t-\xi)^{-\alpha} \left[f(\xi) - f(0) \right] d\xi \ , & 0 < \alpha < 1 \\ \left(f^{(n)}(t) \right)^{(\alpha - n)} \ , & n \leq \alpha < n+1 \ , \ n \geq 1, \end{cases}$$

where α is the order of derivative, $f : \mathbb{R} \to \mathbb{R}$, $f(t)$ is a continuous function, $t \to f(t)$ function and $\Gamma(\alpha)$ is the gamma function given as

$$\Gamma(\alpha) = \lim_{n \to \infty} \frac{n^{\alpha} n!}{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n)}.$$

The Jumarie's modified Riemann-Liouville derivative has the following derivative properties

- $D_t^{\alpha} t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}$,
- $D_t^{\alpha} \left(f(t) g(t) \right) = g(t) D_t^{\alpha} f(t) + f(t) D_t^{\alpha} g(t)$,
- $D_t^{\alpha} f(g(t)) = f_g^{'} \left[g(t) \right] D_t^{\alpha} g(t) = D_g^{\alpha} f \left[g(t) \right] \left(g^{'}(t) \right)^{\alpha}$.

The derivative of $\alpha-$order of a constant is nought. The derivatives are applicable to any differentiable or non differentiable functions.

The rest of this paper is prepared as follows. In Section 2, we introduce the sine-Gordon expansion method to find exact solutions for FPDEs. Section 3 is allocated to find new exact solutions for the time fractional $(1+1)$-dimensional Hirota-Satsuma equation with the help of the sine-Gordon expansion technique. In Section 4, we study the time-fractional nonlinear (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equation by the sine-Gordon expansion technique and find new exact solutions. Some plots of the graphs for the solutions are also given. Finally, some discussions are given in the conclusion.

2. Description of the sine-Gordon expansion method

This method deals with the nonlinear hyperbolic partial differential equation, namely the sine-Gordon equation in the following form [27, 28]

$$u_{xx} - u_{tt} = a^2 \sin(u), \tag{1}$$

where $u = u(x,t)$ and a is a parameter. Introducing the travelling wave transformation

$$u(x,t) = U(\xi)\ ,\ \xi = kx - ct, \tag{2}$$

where k is the wave parameter and c is the velocity of the wave. With the help of this transformation (2), Eq. (1) reduces into the nonlinear ODE

$$U'' = \frac{a^2}{k^2 - c^2}\sin(U). \tag{3}$$

In order to simplify (3), we rewrite in the following form

$$\left[\left(\frac{U}{2}\right)'\right]^2 = \frac{a^2}{k^2 - c^2}\sin^2\left(\frac{U}{2}\right) + C, \tag{4}$$

where C is the integration constant. For simplicity , we take $C = 0$, $\dfrac{U}{2} = w(\xi)$ and $\dfrac{a^2}{k^2 - c^2} = m^2$ in (4) , we get

$$w(\xi)' = m\sin(w(\xi))$$

and if we set $m = 1$, we obtain the simplified form of sine-Gordon equation

$$w' = \sin(w)\,. \tag{5}$$

The solution of (5) gives

$$\sin(w) = \sin(w(\xi)) = \frac{2\ell e^{\xi}}{1 + \ell^2 e^{2\xi}} = \operatorname{sech}(\xi)\,, \qquad \text{for} \quad \ell = 1, \tag{6}$$

$$\cos(w) = \cos(w(\xi)) = \frac{-1 + \ell^2 e^{2\xi}}{1 + \ell^2 e^{2\xi}} = \tanh(\xi)\,, \qquad \text{for} \quad \ell = 1, \tag{7}$$

where ℓ is the integration constant.

Next, suppose that the general form of nonlinear evolution equation is

$$F(u, u_x, u_t, u_{xx}, u_{tt}, u_{tx}...) = 0, \tag{8}$$

where F is a polynomial of $u = u(x,t)$ and its partial derivatives. Using the transformation (2), the equation (8) reduces to the following nonlinear ordinary differential equation (ODE)

$$G(U, U', U'', U''', ...) = 0, \tag{9}$$

where G is a polynomial of U and its derivatives with respect to ξ.

Next, we consider the solution for the equation (9) in the following form

$$U(\xi) = A_0 + \sum_{i=1}^{N} \tanh^{i-1}(\xi)\left[B_i \operatorname{sech}(\xi) + A_i \tanh(\xi)\right], \tag{10}$$

and substitute the expressions in (6) and (7) into (10), we get

$$U(w) = A_0 + \sum_{i=1}^{N} \cos^{i-1}(w)\left[B_i \sin(w) + A_i \cos(w)\right]. \tag{11}$$

The value of N will be found out by the homogeneous balancing, which is considering the terms with the highest order derivatives and the highest order nonlinear term in (9). Substituting the expression (11) into (9) and setting each coefficient of $[\sin^p(w), \cos^q(w)]$ to zero acquire a set of algebraic equations. Solving these algebraic equations, we can find the values for k, c, A_n and B_n. Finally, substituting the values of k, c, A_n and B_n into (10), hence we complete the solution for the nonlinear evolution equation (8).

3. The time-fractional $(1+1)$-dimensional Hirota-Satsuma equation

The Hirota-Satsuma-Ito equations (HSI) are discovered from the Boussinesq equation via a Bäcklund transformation [29] by Hirota and Satsuma and known to represent propagation of unidirectional shallow water waves [30]. We write the HSI equations [31]

$$\begin{aligned} w_t &= u_{xxt} + 3uu_t - 3u_x v_t + \alpha u_x, \\ w_x &= -u_y, \\ v_x &= -u. \end{aligned} \tag{12}$$

If we take $\alpha = -1$ and $y \to -x$, equations (12) reduce to the $(1+1)$-dimensional Hirota-Satsuma equation

$$u_t = u_{xxt} + 3uu_t + 3u_x \partial_x^{-1} u_t - u_x, \tag{13}$$

where $u = u(x,t)$ and ∂_x^{-1} denote integration with respect to x. From (13), the time-fractional $(1+1)$-dimensional Hirota-Satsuma equation (HS) can be written as follows

$$D_t^\alpha u = D_t^\alpha(u_{xx}) + 3uD_t^\alpha u + 3u_x \partial_x^{-1}(D_t^\alpha u) - u_x. \tag{14}$$

Applying the following wave transformation

$$u(x,t) = U(\xi)\ , \qquad \xi = ax + \frac{ct^\alpha}{\Gamma(1+\alpha)},$$

to Eq.(14), converts into to the following nonlinear ODE

$$(a+c)U' = ca^2 U''' + 3c(a+1)UU', \tag{15}$$

where c is the velocity of the wave and a is the wave number . The superscripts in U show the order of derivative with respect to ξ. Next, we can integrate Eq. (15) and neglect the integration constant, hence we get

$$(a+c)U = ca^2 U'' + \frac{3}{2}c(a+1)U^2. \tag{16}$$

To determine the value of N in (10) and (11), we homogeneously balance the terms U'' and U^2 in Eq.(16), and get $N = 2$. We rewrite from (10) and (11) for $N = 2$

$$\begin{aligned} U(\xi) &= A_0 + B_1 \operatorname{sech}(\xi) + A_1 \tanh(\xi) + B_2 \tanh(\xi)\operatorname{sech}(\xi) + A_2 \tanh^2(\xi), && (17) \\ U(w) &= A_0 + B_1 \sin(w) + A_1 \cos(w) + B_2 \cos(w)\sin(w) + A_2 \cos^2(w), && (18) \end{aligned}$$

and the derivative of (18) is

$$\begin{aligned} U''(w) &= 2\sin^2(w)\left(1-3\cos^2(w)\right)A_2+\sin(w)\cos(w)(6\cos^2(w)-5)B_2 \\ &\quad -\sin(w)\left(\sin(2w)A_1-\cos(2w)B_1\right). \end{aligned} \tag{19}$$

On substituting the values (18) and (19) into Eq.(16) and setting the coefficients of $\sin^m(w)$,$\cos^n(w)$ to zero, we obtain the following set of algebraic equations:

$$\begin{aligned}
C_0 &: -6ca^2A_2+\frac{3}{2}c(a+1)(B_2^2-A_2^2)=0,\\
C_1 &: 6ca^2B_2+3c(a+1)A_2B_2=0,\\
C_2 &: 2ca^2B_1+3c(a+1)(A_1B_2+A_2B_1)=0,\\
C_3 &: 2ca^2A_1+3c(a+1)(A_1A_2-B_1B_2)=0,\\
C_4 &: (4ca^2-a-c)A_2+\frac{3}{2}c(a+1)(2A_0A_2+A_1^2+2A_2^2-B_1^2-B_2^2)=0,\\
C_5 &: (a+c-ca^2)B_2-3c(a+1)(A_0B_2+A_1B_1+A_2B_2)=0,\\
C_6 &: (a+c-ca^2)B_1-3c(a+1)(A_0B_1+A_1B_2+A_2B_1)=0,\\
C_7 &: (a+c)A_1-3c(a+1)(A_0+A_2)A_1=0,\\
C_8 &: (a+c)(A_0+A_2)-\frac{3}{2}c(a+1)(A_0^2+2A_0A_2+A_1^2+A_2^2)=0.
\end{aligned}$$

Solving these equations by using mathematical software program, we get the following set of solutions:

Set1

$$\left\{c=\frac{-a}{a^2+1}, A_0=\frac{4a^2}{3(a+1)}, A_1=0, A_2=\frac{-2a^2}{a+1}, B_1=0, B_2=\mp\frac{2Ia^2}{a+1}, I=\sqrt{-1}\right\}$$

Plugging these values into (17), hence the solution for the Eq.(14) is

$$u(x,t)=\frac{2a^2}{a+1}\left(\frac{2}{3}-\tanh^2(ax+ct)\mp I\tanh(ax+ct)\operatorname{sech}(ax+ct)\right).$$

The graphics of this solution are given in figure (1) and (2) for real and imaginary part of $u(x,t)$ respectively.

Set2

$$\left\{c=\frac{a}{a^2-1}, A_0=\frac{2a^2}{a+1}, A_1=0, A_2=\frac{-2a^2}{a+1}, B_1=0, B_2=\mp\frac{2Ia^2}{a+1}, I=\sqrt{-1}\right\}$$

Plugging these values into (17), hence the solution for the Eq.(14) is

$$u(x,t)=\frac{2a^2}{a+1}\operatorname{sech}(ax+ct)\left(\operatorname{sech}(ax+ct)\mp I\tanh(ax+ct)\right).$$

The graphics of this solution are similar to those given in figure (1) and (2) for real and imaginary part of $u(x,t)$ respectively.

Set3

$$\left\{c=\frac{a}{4a^2-1}, A_0=\frac{4a^2}{a+1}, A_1=0, A_2=\frac{-4a^2}{a+1}, B_1=0, B_2=0\right\}$$

Plugging these values into (17), hence the solution for the Eq.(14) is

$$u(x,t)=\frac{4a^2}{a+1}\operatorname{sech}^2(ax+ct).$$

The graphic of this solution is similar to that given in figure (1).

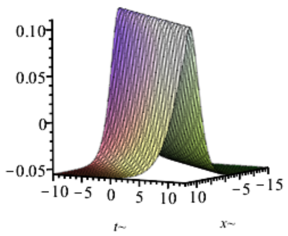

FIGURE 1. Bell-soliton shape of the real part of $u(x,t)$ for the parameters a=$-\frac{1}{4}$,c=$\frac{4}{17}$ and x=-15..14,t=-10..13.

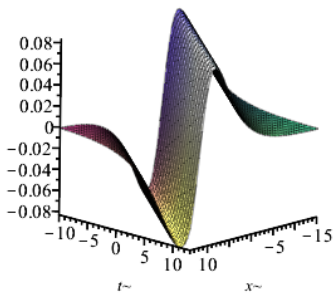

FIGURE 2. Kink-soliton shape of the imaginary part of $u(x,t)$ for the parameters a=$-\frac{1}{4}$,c=$\frac{4}{17}$ and x=-15..14,t=-10..13.

Set4

$$\left\{c=\frac{-a}{4a^2+1}, A_0=\frac{4a^2}{3(a+1)}, A_1=0, A_2=\frac{-4a^2}{a+1}, B_1=0, B_2=0\right\}$$

Plugging these values into (17), hence the solution for the Eq.(14) is

$$u(x,t)=\frac{-4a^2}{a+1}\left(\frac{8}{3}-\operatorname{sech}^2(ax+ct)\right).$$

The graphic of this solution is similar to that given in figure (1).

4. The time-fractional nonlinear (2+1)-dimensional Caudrey-Dodd-Gibbon-Kotera-Sawada equation

Konopelchenko and Dubrovsky proposed the nonlinear (2+1)- Caudrey-Dodd-Gibbon-Kotera-Sawada equation (CDGKS) [32], in the following form

$$36u_t+u_{xxxxx}+15(uu_{xx})_x+45u_xu^2-5u_{xxy}-15uu_y-15u_x\partial_x^{-1}u_y-5\partial_x^{-1}u_{yy}=0, \quad (20)$$

where $u=u(x,y,t)$ and ∂_x^{-1} denote integration with respect to x. The Eq.(20) is an interesting integrable equation that describe large range of nonlinear dispersive physical contexts and has many application in nonlinear sciences. Some of them are; theory of conformal field, 2-dimentional gauge field theory of quantum gravity and the conservative flow of Liouville equation [33, 34, 35]. When $u_y = 0$, Eq.(20) reduces to the following (1+1)-dimensional Sawada- Kotera (SK) equation [36]

$$36u_t + u_{xxxxx} + 15(uu_{xx})_x + 45u_x u^2 = 0. \tag{21}$$

The Eq.(21) is a significant nonlinear evolution equation in the context of physical sciences for describing the motion of long waves in shallow water and has applications in quantum mechanics and in nonlinear optics.The Eq.(21) is an integrable soliton equation, which has multisoliton solutions, Bäcklund transformation and it is a member of higher-order KdV hierarchy [37, 38].

In the last decade, Eq.(20) has been studied by many scientists, and remarkble results have been obtained from the solutions. For example, Geng used the Riccati equation method to derive rational solutions and soliton solutions of the CDGKS and SK equations [39]. Cao et al.worked on the Lax methods to get the equation to integrable ordinary differantial equation and derived the quasi periodic solution [40]. Wang and Xian derived the homoclinic breather-wave solutions, periodic wave solutions and kink solitary wave solutions for the CDGKS equation [41, 42, 43]. The rational solutions and periodic solutions [44] are found by using the tanh method. Applying the Darboux transformation [45] and Hirota bilinear method to the Eq.(20), give rise to quasi-periodic solutions [46] and periodic solitary wave solutions [47], respectively.

The time-fractional CDGKS equation, proposed by Sawada and Kotera [48], and also by Caudrey, Dodd and Gibbon [34, 49], can be written as follows

$$36D_t^{\alpha}u + u_{xxxxx} + 15(uu_{xx})_x + 45u^2u_x - 5u_{xxy} - 15uu_y - 15u_x\partial_x^{-1}u_y - 5\partial_x^{-1}u_{yy} = 0, \tag{22}$$

where $0 < \alpha \leq 1$ is the order of fraction, $u = u(x, y, t)$ and ∂_x^{-1} denote integration with respect to x. Applying the following wave transformation

$$u(x, y, t) = U(\xi)\ , \qquad \xi = ax + by + \frac{ct^{\alpha}}{\Gamma(1+\alpha)},$$

to Eq.(22), which reduces to the following nonlinear ODE

$$(36c - 5b^2)U' - 15b(1 + a)UU' + 45aU^2U' - 5a^2bU''' + 15a^3(UU'')' + a^5U''''' = 0, \tag{23}$$

where a, b, c are constants and the superscripts in U show the order of derivative with respect to ξ. Integrating Eq. (23), we get

$$(36c - 5b^2)U - \frac{15}{2}b(1 + a)U^2 + 15aU^3 - 5a^2bU'' + 15a^3UU'' + a^5U'''' = 0, \tag{24}$$

where the integration constant is neglegted. Considering the homogeneous balance between U'''' and U^3 terms in Eq.(24), we deduce $N = 2$, so we write from (10) and (11) for $N = 2$

$$U(\xi) = A_0 + B_1 \operatorname{sech}(\xi) + A_1 \tanh(\xi) + B_2 \tanh(\xi)\operatorname{sech}(\xi) + A_2 \tanh^2(\xi), \tag{25}$$

$$U(w) = A_0 + B_1 \sin(w) + A_1 \cos(w) + B_2 \cos(w)\sin(w) + A_2 \cos^2(w). \tag{26}$$

The derivatives of (26)

$$
\begin{aligned}
U''(w) &= 2\sin^2(w)\left(1-3\cos^2(w)\right)A_2+\sin(w)\cos(w)(6\cos^2(w)-5)B_2 \\
&\quad -\sin(w)\left(\sin(2w)A_1-\cos(2w)B_1\right), \qquad (27)\\
U''''(w) &= 2\sin^2(w)\left(15\sin^2(2w)-8\right)A_2 \\
&\quad +1/2\sin(2w)\left(120\cos^4(w)-180\cos^2(w)+61\right)B_2 \\
&\quad -4\sin(w)\sin(2w)\left(3\cos^2(w)-2\right)A_1 \\
&\quad +\sin(w)\left(24\cos^4(w)-28\cos^2(w)+5\right)B_1. \qquad (28)
\end{aligned}
$$

Hence we substitute the values (26), (27) and (28) into Eq.(24) and set the coefficients of $\sin^m(w)$,$\cos^n(w)$ to zero, we derive the following set of algebraic equations:

$$
\begin{aligned}
C_0 &: 15a(3B_2^2-A_2^2-8a^4)A_2+90a^3(B_2^2-A_2^2)=0,\\
C_1 &: 15a(3A_2^2-B_2^2+8a^4)B_2+180a^3A_2B_2=0,\\
C_2 &: 24a^5B_1+120a^3(A_1B_2+A_2B_1)+90aA_1A_2B_2+45a(A_2^2-B_2^2)B_1=0,\\
C_3 &: 24a^5A_1+120a^3(A_1A_2-B_1B_2)-90aA_2B_1B_2+45a(A_2^2-B_2^2)A_1=0,\\
C_4 &: 30a^2(4a^3-b)A_2+\frac{15}{2}(a+1)b(B_2^2-A_2^2)+45a[(A_2^2-B_2^2)A_0+(A_1^2-B_1^2)A_2\ +A_2^3]\\
&: +90a(a^2A_0A_2-B_2^2A_2-A_1B_1B_2)+30a^3(A_1^2-B_1^2)+15a^3(10A_2^2-7B_2^2)=0,\\
C_5 &: 30a^2(b-2a^3)B_2+[15b(a+1)-165a^3]A_2B_2-90a(A_0B_2+A_1B_1)A_2\\
&: +15a[3(B_1^2-A_1^2)+B_2^2-6A_2^2]B_2-30a^3(3A_0B_2+2A_1B_1)=0,\\
C_6 &: 10a^2(b-2a^3)B_1-30a(a^2+3A_2)B_1A_0+15(b-9a^3+ab)A_1B_2+15(b-7a^3\\
&: +ab)A_2B_1-90a(A_0+2A_2)A_1B_2+45a(B_2^2-A_1^2)B_1+15a(B_1^2-6A_2^2)B_1=0,\\
C_7 &: 2a^2(5b-4a^3)A_1+15a[3(B_1^2+B_2^2)-(A_1^2+6A_2^2+2a^2A_0)]A_1+15[2a^3\\
&: -b(a+1)+6a(A_0+A_2)]B_1B_2+15[b(a+1)-6a(a^2+A_0)]A_1A_2=0,\\
C_8 &: [5b(4a^2+b)-36c-16a^5)A_2+\frac{15}{2}b(a+1)(A_1^2+2A_2^2-B_1^2-B_2^2)\\
&: +15b[(a+1)-4a^3]A_0A_2+15a^3(B_1^2+B_2^2)-45a[(A_0A_2+A_1^2)A_0+A_2^3\\
&: +(A_2+A_0)(B_1^2+B_2^2)]-30a^3(A_1^2+2A_2^2)+90a(A_1B_1B_2-A_0A_2^2-A_1^2A_2)=0,\\
C_9 &: [a^5+36c-5b(a^2+b)]B_2+15[a^3-b(a+1)](A_0B_2+A_1B_1+A_2B_2)\\
&: +45a(A_2^2+A_1^2+A_0^2)B_2+90a(A_0A_1B_1+A_0A_2B_2+A_1A_2B_1)=0,\\
C_{10} &: [a^5+36c-5b(a^2+b)]B_1+15[a^3-b(a+1)](A_0B_1+A_1B_2+A_2B_1)\\
&: +45a(A_0^2+A_1^2+A_2^2)B_1+90a(A_0A_1B_2+A_0A_2B_1+A_1A_2B_2)=0,\\
C_{11} &: (36c-5b^2)A_1-15(ab+b)(A_0A_1+A_1A_2)+15a(3A_0^2+6A_0A_2+A_1^2+3A_2^2)A_1=0,\\
C_{12} &: [36c-5b^2+45a(A_0A_2+A_1^2)](A_0+A_2)-\frac{15}{2}b(a+1)[(A_2+A_0)^2+A_1^2]\\
&: +15a(A_0^3+A_2^3)=0.
\end{aligned}
$$

Solving these equations by using mathematical software program and set $a=b=1$, we get the following set of solutions:

Set1

$$\left\{c=\frac{1}{4},A_0=1,A_1=0,A_2=-1,B_1=0,B_2=\mp I,I=\sqrt{-1}\right\}$$

Plugging these values into (25), hence the solution for the Eq.(22) is

$$u(x,y,t) = \operatorname{sech}^2\left(x+y+\frac{1}{4}t\right) \pm I\tanh\left(x+y+\frac{1}{4}t\right)\operatorname{sech}\left(x+y+\frac{1}{4}t\right).$$

The graphics of this solution are given in figure (3) and (4) for real and imaginary part of $u(x,y,t)$ respectively.

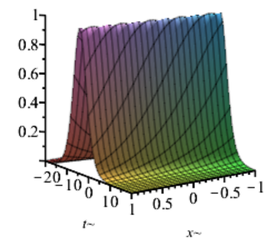

FIGURE 3. Bell-soliton shape of the real part of $u(x,y,t)$ for the parameters a=1, c=$\frac{1}{4}$, and x = -1..1, t = -20..20.

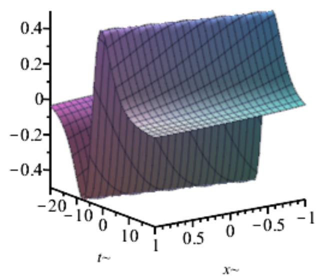

FIGURE 4. Kink-soliton shape of the imaginary part of $u(x,y,t)$ for the parameters a=1, c=$\frac{1}{4}$, and x = -1..1, t = -20..20.

Set2

$$\left\{c=\frac{1}{4}, A_0=2, A_1=0, A_2=-2, B_1=0, B_2=0\right\}$$

Plugging these values into (25), hence the solution for the Eq.(22) is

$$u\left(x,y,t\right) = 2\operatorname{sech}^2\left(x+y+\frac{1}{4}t\right).$$

The graphic of this solution are similar to that given in figure (3).

Set3

$$\left\{c=\frac{1}{4}, A_0=2, A_1=0, A_2=-2, B_1=0, B_2=2I, I=\sqrt{-1}\right\},$$
$$\left\{c=\frac{1}{4}, A_0=2, A_1=0, A_2=-2, B_1=0, B_2=-2I, I=\sqrt{-1}\right\}.$$

Plugging these values into (25), hence the solution for the Eq.(22) is

$$u\left(x,y,t\right)=2\operatorname{sech}^2\left(x+y+\frac{1}{4}t\right)\pm 2I\tanh\left(x+y+\frac{1}{4}t\right)\operatorname{sech}\left(x+y+\frac{1}{4}t\right).$$

The graphics of this solution are similar to those given in figure (3) and (4) for real and imaginary part of $u(x,y,t)$ respectively.

Set4

$$\left\{c=\frac{-67}{96}-\frac{5\sqrt{2505}}{288},A_0=\frac{5}{4}-\frac{\sqrt{2505}}{60},A_1=0,A_2=-2,B_1=0,B_2=0\right\},$$

$$\left\{c=\frac{-67}{96}+\frac{5\sqrt{2505}}{288},A_0=\frac{5}{4}+\frac{\sqrt{2505}}{60},A_1=0,A_2=-2,B_1=0,B_2=0\right\}.$$

Plugging these values into (25), hence the solution for the Eq.(22) is

$$u\left(x,y,t\right)=-\frac{3}{4}\mp\frac{\sqrt{2505}}{60}+2\operatorname{sech}^2\left(x+y+\left(\frac{-67}{96}\mp\frac{5\sqrt{2505}}{288}\right)t\right).$$

The graphic of this solution are similar to that given in figure(3).

Set5

$$\left\{c=\frac{1}{12}-\frac{\sqrt{30}}{36},\ A_0=1-\frac{\sqrt{30}}{15},\ A_1=0,\ A_2=-1,B_1=0,B_2=I,I=\sqrt{-1}\right\},$$

$$\left\{c=\frac{1}{12}+\frac{\sqrt{30}}{36},\ A_0=1+\frac{\sqrt{30}}{15},\ A_1=0,\ A_2=-1,B_1=0,B_2=I,I=\sqrt{-1}\right\},$$

$$\left\{c=\frac{1}{12}-\frac{\sqrt{30}}{36},\ A_0=1-\frac{\sqrt{30}}{15},\ A_1=0,\ A_2=-1,B_1=0,B_2=-I,I=\sqrt{-1}\right\},$$

$$\left\{c=\frac{1}{12}+\frac{\sqrt{30}}{36},\ A_0=1+\frac{\sqrt{30}}{15},\ A_1=0,\ A_2=-1,B_1=0,B_2=-I,I=\sqrt{-1}\right\}.$$

Plugging these values into (25), hence the solution for the Eq.(22) is

$$\begin{aligned} u\left(x,y,t\right) &= \mp\frac{\sqrt{30}}{15}+\operatorname{sech}^2\left(x+y+\left(\frac{1}{12}\mp\frac{\sqrt{30}}{36}\right)t\right) \\ &\quad +I\tanh\left(x+y+\left(\frac{1}{12}\mp\frac{\sqrt{30}}{36}\right)t\right)\operatorname{sech}\left(x+y+\left(\frac{1}{12}\mp\frac{\sqrt{30}}{36}\right)t\right). \end{aligned}$$

The graphics of this solution are similar to those given in figure (3) and (4) for real and imaginary part of $u(x,y,t)$ respectively.

5. Discussion

Travelling wave solutions play a crucial role in understanding the dynamics of nonlinear partial differential equations. For the HS, the travelling wave solutions can often be expressed in terms of hyperbolic functions, such as hyperbolic secant (sech) or hyperbolic tangent (tanh). The exact form of the solution depends on the specific parameters and initial conditions of the equation. The travelling wave solutions for the CDGKS equation can also be expressed in terms of hyperbolic functions. Travelling wave solutions in terms of hyperbolic functions have important implications for the dynamics of these equations. Hyperbolic functions are known for their ability to describe localized structures, such as solitons and solitary waves. These solutions exhibit interesting properties, such as stability, preservation of shape, and interactions with other waves. Studying the travelling wave solutions of the HS and the CDGKS not only provides insights into the behavior of these specific equations but also contributes to the broader understanding of nonlinear wave phenomena in physics. The use of hyperbolic functions in expressing these solutions allows for a concise representation of the wave profiles and facilitates further analysis of their properties. The solutions obtained in this paper are consistent with the solutions obtained in [56].

6. Conclusion

In this paper we have exploited the sine-Gordon expansion method in determining new travelling wave solutions for two time-fractional partial differential equations; the HS and the CDGKS equations. We have found abundant new exact and hyperbolic solutions, which can be regarded as fruitful to further comprehend the dynamics of the nonlinear waves. We used the nonlinear fractional transformation for the nonlinear fractional PDEs to derive its ODEs. This transformation quaranties the reduction from a given fractional PDE to ODE, in which the order is integer. Hence the solutions for the HS and the CDGKS equations are specified by the hyperbolic functions. The presented method is paved the way for exact and hyperbolic solutions of the FPDEs and it is fruitful, efficient and powerful method to work with the systems of FPDEs. In the future, researchers may investigate the specific types of fractional equations for which the Sine-Gordon expansion method is most effective. They can explore the strengths and limitations of the method and identify the conditions under which it yields accurate solutions.

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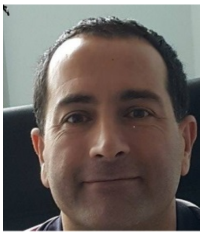

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