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STUDY OF SOME EVOLUTION EQUATIONS INVOLVING RIESZ FRACTIONAL DERIVATIVE WITH SINGULAR INITIAL DATA

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Abstract. The objective of this work is to study some evolution problems involving the Riesz fractional derivative with singular initial data which can be distributions. It is a question of proving the existence and uniqueness of the solutions of these problems in the extended Colombeau algebra $\mathcal{G}^e_R$. It is established that the existence and uniqueness generalized solutions hold for both evolution problems associated to the Schrödinger equation and the heat equation involving the corresponding Riesz fractiononal operators derivatives.

Keywords: Colombeau algebra, Generalized functions, Evolution problem, Generalized semigroup, Riesz fractional derivative.

AMS Subject Classification: 83-02, 99A00

1. Introduction

The objective of this research is to study some evolution problems involving the Riesz fractional derivative with singular initial data. More precisely, it is a question of proving the existence and uniqueness of the solutions of these problems in the extended $\mathcal{G}^e$ of Colombeau algebra. In Colombeau-type regularization methods, the basic concept is to represent nonsmooth objects by means of smooth function nets. These nets may or may not converge, but possess mild asymptotics, and regularizing nets are identified by minimizing the discrepancy with the moderateness scale. The equivalence classes of regularization moderates nets with respect a negligible nets are called elements of Colombeau algebra, i.e. sequences of smooth functions satisfying conditions of asymptotically in the regularization parameter ε, the reader can see [6, 8]. In physics, fractional calculus is a useful tool for analyzing nonlocal and memory effects [12]. This is followed by successful applications of anomalous diffusion and evolution problems [17, 18]. In the literature, more precisely in

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recent years many researchers have begin to focus on as an interesting and popular tool to the fractional derivative theory, since it model many phenomena in various fields of engineering, physics, and economics. Often found in viscoelastic, electrochemical, control, porous media, electromagnetic, see [11, 13] and the references therein. The fractional evolution problem $D^\alpha \mu = A\mu$, $m-1 < \alpha \leq m$, $m \in \mathbb{N}_0$ has been studied in [2], they provides necessary and sufficient conditions for solving this problem with A is an unbounded closed operator in Banach space X. In order to reduce the Cauchy problem for a linear inhomogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation, the authors in [21] used the well known Duhamel's principle, in their paper one of the possible generalizations of the classical Duhamel's principle to the time-fractional pseudo-differential equations is established, all this with the usual functions which do not pose problems of singularities, or of product see [4, 10]. But this method remains invalid for systems carrying a product of two distributions, which we have treated here. The authors in [19] provide a method for dealing fractional differential equations including singularities, based on Colombeau's theory of generalized functions, more generally they gives an extension of Colombeau's algebra to fractional derivatives. In [5] the existence and uniqueness of solutions for an abstract Caputo type fractional evolution problem with generalized real numbers in the initial conditions is given, more precisely, mild solutions of our proposed model is constructed by using Laplace transform and a density function.

Also in [1] authors study the existence and uniqueness of generalized mild solutions only for nonlinear Schrödinger equations with singular initial conditions but juste in a simple Colombeau algebras of generalized functions, by the semigroup theory, in this context, there are many articles dealing with this type of problem; the reader can see [3, 5, 7, 9, 16, 20] and the references therein. Our idea is inspired from here, this time with the fractional operator of Riesz derivative. We apply it to solving some PDEs with fractional derivatives in terms of time and space variables, we have studied the existence and uniqueness results. For this, we start to show that a specific representation of the Riesz derivative that allows analytic continuation can be used to evaluate the integral equation, thus resolving the so-called inconsistency issue. This paper is structured as follows. In Section 2, we review Colombeau's theory. Section 3 is particularly interested to regularize the Riesz fractional derivative to use it to study some generalized evolution problems, which includes both heat equation and Schrödinger nonlinear problem as application.

2. Preliminairies

In this section, we will review some basic properties of generalized functions theory in colombeau sense. Let $n \in \mathbb{N}_0$, we note

$$\mathcal{E}(\mathbb{R}^n) = \left(\mathcal{C}^\infty(\mathbb{R}^n)\right)^{(0,1)}.$$

The set of all moderate functions is given as follows

$$\mathcal{E}_M(\mathbb{R}^n) = \Big\{(\mu_\epsilon)_\epsilon \subset \mathcal{E}(\mathbb{R}^n),\ \forall K \subset \mathbb{R}^n, \forall \alpha \in \mathbb{N}_0^n, \exists N \in \mathbb{N}, \sup_{x\in K} |\partial^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\}.$$

The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \Big\{(\mu_\epsilon)_\epsilon \subset \mathcal{E}(\mathbb{R}^n),\ \forall K \subset \mathbb{R}^n,\ \forall \alpha \in \mathbb{N}_0^n, \forall p \in \mathbb{N},\ \sup_{x\in K} |\partial^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{p})\Big\}.$$

The Colombeau algebra is defined as a factor set

$$\mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n).$$

The ring of all generalized real numbers is given by the following set

$$\widetilde{\mathbb{R}} = \mathcal{E}\left(\mathbb{R}\right)/\mathcal{I}\left(\mathbb{R}\right),$$

where

$$\mathcal{E}(\mathbb{R}) = \Big\{(x_\epsilon)_\epsilon \in (\mathbb{R})^{(0,1)},\ \exists m \in \mathbb{N}, |x_\epsilon| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-m})\Big\},$$

and

$$\mathcal{I}(\mathbb{R}) = \Big\{(x_\epsilon)_\epsilon \in (\mathbb{R})^{(0,1)},\ \forall m \in \mathbb{N}, |x_\epsilon| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{m})\Big\}.$$

The ring $\widetilde{\mathbb{R}}$ is formed by factoring moderate sets of real numbers with respect to negligible sets, algebra $\mathcal{E}(\mathbb{R})$ contains the ideal $\mathcal{I}(\mathbb{R})$. The space of distributions with compact support $\mathcal{E}'(\Omega)$ is embedded into $\mathcal{G}(\Omega)$, whre Ω is an open subset of $\mathbb{R}^n$ through convolution

$$i : \begin{cases} \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega) \\ \qquad \omega \to (\omega * (\phi_\epsilon)/\Omega)_{\epsilon\in(0,1)} + \mathcal{N}(\Omega), \end{cases}$$

where

$$\phi_\epsilon(x) = \epsilon^{-n}\phi(\frac{x}{\epsilon}),\ \phi \in \mathcal{C}_0^\infty(\Omega),\ \phi(x) \geq 0,\ \int_\Omega \phi = 1,\ \int_\Omega x^\alpha \phi = 0, \forall\alpha \in \Omega,\ |\alpha| > 0 \tag{1}$$

is obtained by scaling a fixed test function in $\mathcal{S}(\mathbb{R}^n)$ of integral one and with all higher order moments are zero. By the sheaf property, i can be extended in a unique way to an embedding from the space of distributions $\mathcal{D}'(\Omega)$ into Colombeau algebra $\mathcal{G}(\Omega)$. The extended Colombeau algebra of generalized functions $\mathcal{G}^e(\Omega)$ on the open set Ω of $\mathbb{R}^n$ is defined in the sense of the extension of integer derivatives to a fraction those first introduced by M. Stojanovic see [19] for details. Let $\mathcal{E}(\Omega)$ be the algebra of all sequences $(\mu_\epsilon)_{\epsilon>0}$ of real valued functions, $\mu_\epsilon \in \mathcal{C}^\infty(\Omega)$. The algebra of extended moderate functions is given by

$$\mathcal{E}_M^e(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in (\mathcal{E}(\Omega))^I,\ \forall K \subset \mathbb{R}, \forall\alpha \in \mathbb{R}_+, \exists N \in \mathbb{N}, \\ \sup_{x\in K} |D^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\},$$

and the set of negligeable functions is defined by

$$\mathcal{N}^e(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in (\mathcal{E}(\Omega))^I, \forall K \subset \mathbb{R}, \forall\alpha \in \mathbb{R}_+,\ \forall q \in \mathbb{N}, \\ \sup_{x\in K} |D^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{q})\Big\}.$$

The extended Colombeau algebra $\mathcal{G}^e(\Omega)$ is given by the factor algebras

$$\mathcal{G}^e(\Omega) = \mathcal{E}_M^e(\Omega)/\mathcal{N}^e(\Omega).$$

Where, $m-1 < \alpha \leq m,\ m \in \mathbb{N}$ and D^α is the Caputo fractional derivative, which defined (see [14]), for suitably smooth function. For the fractiaonal derivatives and fractional integral, we can see [15] and the references therein. Embedding of the Schwartz distributions space $\mathcal{S}'(\mathbb{R}^n)$ into $\mathcal{G}_\tau^e(\mathbb{R}^n)$ is given by $\omega \to [(\omega * \phi_\epsilon)_{\epsilon\in I}]$, more detail about the space can be found in [19].

3. Main results

We begin this section by embedding the Riesz fractional derivative into extended Colombeau algebra $\mathcal{G}_r^e$ described below. Let $\mu = [(\mu_\epsilon)_\epsilon] \in \mathcal{G}_r^e$, for any $\epsilon \in (0,1)$ and K is a compact of $\mathbb{R}$, we have

$$\begin{aligned} |R^\alpha \mu_\epsilon(t,x)| &\leq \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{+\infty} \frac{|\mu'_\epsilon(t,\xi)|}{|x-\xi|^\alpha} d\xi, \quad \text{for } \alpha \in (0,1) \qquad (2) \\ \sup_{x\in K} |R^\alpha \mu_\epsilon(t,x)| &\leq \frac{1}{\Gamma(1-\alpha)} \sup_{x\in K} \int_{-\infty}^{+\infty} \frac{|\mu'_\epsilon(t,\xi)|}{|x-\xi|^\alpha} d\xi \\ &\leq \frac{1}{\Gamma(1-\alpha)} \sup_{x\in K} |\mu'_\epsilon(t,\xi)| \int_{-\infty}^{+\infty} \frac{1}{|x-\xi|^\alpha} d\xi. \end{aligned}$$

We can write,

$$\int_{-\infty}^{+\infty} \frac{d\xi}{|x-\xi|^\alpha} = \int_{-\infty}^{c} \frac{d\xi}{|x-\xi|^\alpha} + \int_{c}^{+\infty} \frac{d\xi}{|x-\xi|^\alpha},$$

where c is a nonegative constant. Since the second integral in the last equality is not converges at infinity because $\alpha \in (0,1)$, hence the fractional Riesz derivative R^α given in (2) has not a moderate bounds, for this reason we regularize this definition with the convolution method by a well-chosen molifier $\big(|x|^{-\alpha} * \phi_\epsilon(x)\big)$, where ϕ_ϵ is given in (1). Note that we regularize only derivatives with respect to the spacial variable to preserve the evolution-type of equations.

$$\widetilde{R}^\alpha \mu_\epsilon(t,x) = \frac{1}{\Gamma(1-\alpha)} \big(\mu'_\epsilon(t,x) * |x|^{-\alpha} * \phi_\epsilon(x)\big)(t,x) \quad \alpha, \epsilon \in (0,1). \qquad (3)$$

Proposition 3.1. *The Riesz fractional derivative $R^\alpha \mu_\epsilon$ and its regularized $\widetilde{R}^\alpha \mu_\epsilon$ are associated in the Colombeau algebra.*

Proof. We have to prove that

$$|\widetilde{R}^\alpha \mu_\epsilon - R^\alpha \mu_\epsilon| \approx 0.$$

By substracting (2) and (3), we can write

$$\begin{aligned} &\sup_{x\in\mathbb{R}} |\widetilde{R}^\alpha \mu_\epsilon(t,x) - R^\alpha \mu_\epsilon(t,x)| \\ &\qquad = \frac{1}{\Gamma(1-\alpha)} \sup_{x\in\mathbb{R}} \Big((\mu'_\epsilon(t,x) * |x|^{-\alpha} - \mu'_\epsilon(t,x) \Big) \\ &\qquad = \frac{1}{\Gamma(1-\alpha)} \sup_{x\in\mathbb{R}} \Big(|(\mu'_\epsilon(t,x) * |x|^{-\alpha})(t,x)| * |\phi_\epsilon(x) - \delta(x)| \Big). \end{aligned}$$

Since the sequence $(\phi_\epsilon)_\epsilon$ converges to the delta distribution δ in the Schwartz distributions space $\mathcal{D}'$ as $\epsilon \to 0$, we have $\lim_{\epsilon\to 0} |\phi_\epsilon(x) - \delta(x)| = 0$, hence $\widetilde{R}^\alpha \mu_\epsilon \approx R^\alpha \mu_\epsilon$. □

Proposition 3.2. *The regularization of Riesz fractional derivative has a moderate bound.*

Proof. We have,

$$\begin{aligned}
\widetilde{R}^{\alpha}\mu_\epsilon(t,x) &= \frac{1}{\Gamma(1-\alpha)}\Big[\mu'_\epsilon(t,x) * \big(|x|^{-\alpha} * \phi_\epsilon(x)\big)(t,x)\Big]\\
&= \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{+\infty} \mu'_\epsilon(t,x-y)\big(|y|^{-\alpha} * \phi_\epsilon(y)\big)dy\\
&= \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{+\infty} \mu'_\epsilon(t,x-y)\big(\int_{-\infty}^{+\infty} |y-m|^{-\alpha}\phi_\epsilon(m)dm\big)dy\\
&= \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{+\infty} \mu'_\epsilon(t,x-y)\big(\int_{-\infty}^{+\infty} |y-\epsilon l|^{-\alpha}\phi(l)dl\big)dy,
\end{aligned}$$

we get,

$$\begin{aligned}
|\widetilde{R}^{\alpha}\mu_\epsilon(t,x)| &\le \frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{+\infty} \mu'_\epsilon(t,x-y)\big(\sup_{l\in[-M,M]}|\phi(l)|\frac{1}{\epsilon}\int_{y-\epsilon x}^{y+\epsilon M}|\sigma|^{-\alpha}d\sigma\big)dy\\
&\le \frac{1}{\Gamma(1-\alpha)}\sup_{y\in[-M,M]}|\mu'_\epsilon(t,y)|\left(\sup_{l\in[-M,M]}|\phi(l)|\right)\\
&\times \int_0^{2M}|\frac{1}{\epsilon}\int_{y-\epsilon M}^{y+\epsilon M}|\sigma|^{-\alpha}d\sigma|dy\\
&\le \frac{1}{\Gamma(2-\alpha)}\sup_{y\in[-M,M]}|\mu'_\epsilon(t,y)|\left(\sup_{l\in[-M,M]}|\phi(l)|\right)\\
&\times \int_0^{2M}\frac{1}{\epsilon}\big(|y+\epsilon M|^{1-\alpha}-|y-\epsilon M|^{1-\alpha}\big)dy\\
&\le \frac{1}{\Gamma(3-\alpha)}\sup_{y\in[-M,M]}|\mu'_\epsilon(t,y)|\frac{1}{\epsilon^2}\sup_{l\in[-M,M]}|\phi(l)|\\
&\times \Big[|y+\epsilon M|^{2-\alpha}-|y-\epsilon M|^{2-\alpha}\Big]_0^{2M}\\
&\le \frac{1}{\Gamma(3-\alpha)}\frac{1}{\epsilon^2}\sup_{y\in[-M,M]}|\mu'_\epsilon(t,y)|\sup_{l\in[-M,M]}|\phi(l)|\epsilon^{2-\alpha}M^{2-\alpha}\\
&\le \frac{1}{\Gamma(3-\alpha)}\frac{1}{\epsilon^2}\sup_{y\in[-M,M]}|\mu'_\epsilon(t,y)|C_{\alpha,\phi}\epsilon^{-\alpha}M^{2-\alpha}\\
&\le\ C_{\alpha,\phi}M^{2-\alpha}\epsilon^{-\alpha}C\epsilon^{-N_1}.
\end{aligned}$$

Hence,

$$\sup_{x\in\mathbb{R}}|\widetilde{R}^{\alpha}\mu_\epsilon(t,x)| \le C_{\alpha,\phi}K^{2-\alpha}C\epsilon^{-N},\ \text{where } 0<M<K,\ C_{\alpha,\phi}>0,\quad \alpha\in(0,1).$$

The last inequality gives the desired result for the zero order derivative, the proof for higher order derivatives is similar thanks to the Liebniz formula, which ends the proof. $\square$

More general if α is a real number such that $m-1<\alpha\le m$, with m is an integer number the Riesz fractional derivative of order α is defined by

$$R^{\alpha}\mu_\epsilon(t,x) = \frac{1}{\Gamma(m-\alpha)}\int_{-\infty}^{+\infty}\frac{u^{(m)}(t,\xi)}{|x-\xi|^{\alpha+1-m}}d\xi.$$

By the result proving in the two Propositions 3.1 and 3.2, we can defined the new corresponding class of Colombeau algebra $\mathcal{G}^e_r(\Omega)$ by

$$\mathcal{G}^e_R(\Omega) = \mathcal{E}^e_{M,R}(\Omega)/\mathcal{N}^e_R(\Omega),$$

where the set of moderate functions is given by

$$\mathcal{E}^e_{M,R}(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in (\mathcal{E})^I,\ \forall K \subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+,\ \exists N \in \mathbb{N}, \sup_{x\in K} |\widetilde{R}^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\},$$

and the set of negligeable functions is defined by

$$\mathcal{N}^e_R(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in (\mathcal{E}(\Omega))^I, \forall K \subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+,\ \forall q \in \mathbb{N}, \sup_{x\in K} |\widetilde{R}^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{q})\Big\}.$$

In order to study the abstract Cauchy problem associate to this construction,

$$\begin{cases} \mu'(t) = F(t, \mu(t)), \\ \mu(0) = \mu_0, \end{cases} \tag{4}$$

we need to define the corrsponding extension algebra of generalized temered functions by

$$\mathcal{G}^e_{\tau,R}(\mathbb{R}^n) = \mathcal{E}^e_{\tau,R}(\mathbb{R}^n)/\mathcal{N}^e_{\tau,R}(\mathbb{R}^n), \tag{5}$$

with

$$\mathcal{E}^e_{\tau,R}(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in (\mathcal{E})^I, \forall K \subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+,\ \exists N \in \mathbb{N}, \sup_{x\in K} |(1+|x|)^{-N} \widetilde{R}^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{-N})\Big\},$$

and

$$\mathcal{N}^e_{\tau,R}(\Omega) = \Big\{(\mu_\epsilon)_\epsilon \in (\mathcal{E})^I,\ \forall K \subset \mathbb{R}, \forall \alpha \in \mathbb{R}_+,\ \exists N \in \mathbb{N},\ \forall q \geq 0, \sup_{x\in K} |(1+|x|)^{-N} \widetilde{R}^\alpha \mu_\epsilon(x)| = \mathcal{O}_{\epsilon\to 0}(\epsilon^{q})\Big\}.$$

In [19] the authors constructed a generalized solution to the system of equations (4) in the frame of the extended Colombeau algebra of tempered generalized functions $\mathcal{G}_\tau$. It is proved in [19] the existence and uniqueness results for the system (4) for fractional derivative. When $\alpha \in (0,1)$, if $F \in \widetilde{\mathcal{G}}_\tau(\mathbb{R}^{n+1})$, such that $|\nabla_x F| \leq C\,|\log\epsilon|$, problem (4) has a unique solution $\mu \in \widetilde{\mathcal{G}}(\mathbb{R}^n)$. We will prove that the same is true in the extended algebra of generalized functions $\mathcal{G}^e_R(\mathbb{R}^n)$, introduced in this work

Theorem 3.1. *Let $F \in \mathcal{G}^e_{\tau,R}(\mathbb{R}^{n+1})$, assume that $|\nabla_x F| \leq C|\log\epsilon|$. For any given $\mu_0 \in \widetilde{\mathbb{R}}^n$, the problem (4) has a unique solution in the extended Colombeau algebra $\widetilde{\mathcal{G}}^e_R(\mathbb{R})^n$.*

Proof. Consider the regulariwed Riesz fractional derivative $\widetilde{R}^\alpha$, $m-1 < \alpha \leq m\ m \in \mathbb{N}_0$, without loss of generality, we assume that $0 < \alpha < 1$. We apply the regularized fractional Riesz derivative to the representative of the problem (4), we obtain

$$\widetilde{R}^\alpha \mu'_\epsilon(t) = \widetilde{R}^\alpha F_\epsilon(t, \mu_\epsilon(t)),\ \ \mu_\epsilon(0) = \mu_{0,\epsilon},$$

For approximation reasons of F, we will have

$$\widetilde{R}^\alpha \mu'_\epsilon(t) = \widetilde{R}^\alpha F_\epsilon(t, 0) + |\nabla_x F|.\widetilde{R}^\alpha \mu_\epsilon(t) + N_\epsilon(t),$$

where $(N_\epsilon(t))_\epsilon \in \mathcal{G}_R^e(\mathbb{R})^n$. Integrating from 0 to t it yields

$$\int_0^t \widetilde{R}^\alpha \mu'_\epsilon(s)ds = \int_0^t \widetilde{R}^\alpha F_\epsilon(s,0) + |\nabla_x F|.\widetilde{R}^\alpha \mu_\epsilon(s) + N_\epsilon(s)ds,$$

and thus,

$$\widetilde{R}^\alpha \mu_\epsilon(t) = \widetilde{R}^\alpha \mu_{0,\epsilon} + \int_0^t \widetilde{R}^\alpha F_\epsilon(s,0)ds + |\nabla_x F| \int_0^t \widetilde{R}^\alpha \mu_\epsilon(s)ds + \int_0^t N_\epsilon(s)ds,$$

By Gronwall inequality, we have

$$|\widetilde{R}^\alpha \mu_\epsilon(t)| \leq \big(C_T \epsilon^{-N}\big) e^{-T\log\epsilon} \leq C\,\epsilon^{-N}, \tag{6}$$

which shows that $(\mu_\epsilon)_\epsilon$ is moderate. To prove uniqueness, consider two solutions $(\mu_{1,\epsilon})_\epsilon$ and $(\mu_{2,\epsilon})_\epsilon$ to the regularized equation (4), and let their difference be denoted by $\mathcal{H}_\epsilon = \mu_{1,\epsilon} - \mu_{2,\epsilon}$. By subtracting these two equations, we can derive the following estimation

$$|\widetilde{R}^\alpha \mathcal{H}'_\epsilon(t)| \leq |\log\epsilon||\widetilde{R}^\alpha \mathcal{H}_\epsilon(t)| \leq |\log\epsilon| \int_0^t \Big|\frac{\mathcal{H}'_\epsilon(s)}{(t-s)^\alpha}\Big| ds$$

By integration over the interval $[0,t]$ and the Gronwall Lemma, we obtain $|\widetilde{R}^\alpha \mathcal{H}_\epsilon(t)| \leq 0$, then $\Big|\int_0^t \frac{\mathcal{H}'_\epsilon(s)}{(t-s)^\alpha} ds\Big| \leq 0$. Hence,

$$\sup_{s\in[0,T]} |\mathcal{H}'_\epsilon(s)| \approx 0.$$

Then $(\mu'_{1,\epsilon})_\epsilon \approx (\mu'_{2,\epsilon})_\epsilon$, and by integration, we get $(\mu_{1,\epsilon})_\epsilon \approx (\mu_{2,\epsilon})_\epsilon$. Since the initial conditions are the same, then the uniqueness holds. □

Now, we turn out to study the existence and uniqueness solution of the heat equation in $\mathcal{G}_R^e([0,T]\times\mathbb{R}^n)$. Let us consider the following problem

$$\begin{cases} \partial_t \mu(t,x) = \Delta\mu(t,x) + g(\mu(t,x)) & \mu \in \mathcal{G}_R^e([0,T)\times\mathbb{R}^n), \\ \mu(0,x) = \delta(x), \end{cases} \tag{7}$$

where $g(\mu) \in L^\infty_{loc}([0,T],\mathbb{R}^n)$. To study this problem (7) we need the following rgularizations

$$\delta_\epsilon(x) = |\ln\epsilon|^{an}\phi(x|\ln\epsilon|), \quad \|\nabla g_\epsilon(\mu_\epsilon)\|_{L^\infty} \leq |\ln\epsilon|^b.$$

Theorem 3.2. *Let $g \in L^\infty_{loc}\big([0,T],\mathbb{R}^n\big)$, satisfying $\|\nabla g_\epsilon(\mu_\epsilon)\|_{L^\infty} \leq |\ln\epsilon|^b$ the problem* (7) *has a unique solution in the extension $\mathcal{G}_R^e([0,T]\times\mathbb{R}^n)$.*

Proof Let us consider the reqularization problem

$$\begin{cases} \partial_t \mu_\epsilon(t,x) = \Delta\mu_\epsilon(t,x) + g_\epsilon(\mu_\epsilon(t,x)) \\ \mu_\epsilon(0,x) = \delta_\epsilon(x), \end{cases} \tag{8}$$

associated to the problem (7). The integrale form of the equation (8) is given by the Duhamel extension as follows

$$\mu_\epsilon(t,x) = E_{n,\epsilon}(t,x) * \mu_{0,\epsilon}(x) + \int_0^t \int_{\mathbb{R}^n} E_{n,\epsilon}(t-\tau, x-\xi) g_\epsilon(\mu_\epsilon(\tau,x)) d\xi d\tau. \tag{9}$$

Where $E_{n,\epsilon}$ is the heat kernel. The rest of the proof is presented in four steps
Step 1. By taking the norm in the (9), and thank's to Hölder inequality, it follows that

$$\begin{aligned}\| \mu_\epsilon(t,.) \|_{L^\infty} &\leq \| E_{n,\epsilon}(t,x) \|_{L^\infty} \| \mu_{0,\epsilon} \| \\ &+ \int_0^t \Big[\| E_{n,\epsilon}(t-\tau,.) \|_{L^\infty} \| \nabla g_\epsilon(\mu_\epsilon(\theta\mu_\epsilon)) \|_{L^\infty} \| \mu_\epsilon \|_{L^\infty} \Big] d\tau \\ &\leq C|\ln\epsilon|^{an} + \int_0^t c|\ln\epsilon|^b \| \mu_\epsilon(t,.) \|_{L^\infty} \, d\tau. \end{aligned} \tag{10}$$

Now, we use the Gronowall lemma, it yields

$$\begin{aligned}\| \mu_\epsilon(t,.) \|_{L^\infty} &\leq C|l\epsilon|^{a\,n} \exp(cT(|\ln\epsilon|)^b) \\ &\leq C\epsilon^{-N}, \quad \text{where } N>0,\, t\in[0,T],\, \epsilon\in(0,1),\end{aligned}$$

which shows that the family $(\mu_\epsilon)_{\epsilon\in(0,1)}$ is moderate.
Step 2. In this step, we proves the moderateness for the first derivative, we write

$$\partial_x\mu_\epsilon(t,x) = \int_{\mathbb{R}^n} E_{n,\epsilon}(t,x)\partial_\xi\mu_{0,\epsilon}(\xi)d\xi + \int_0^t\int_{\mathbb{R}^n} \partial_x E_{n,\epsilon}(t-\tau,x-\xi)g_\epsilon(\mu_\epsilon(\tau,x))d\xi d\tau.$$

By taking the norm and the abouve regularization, it yields

$$\begin{aligned}\| \partial_x\mu_\epsilon(t,.) \|_{L^\infty} &\leq C|\ln\,\epsilon|^{a\,n-1} + c_1\int_0^t |\ln\,\epsilon|^b \| \mu_\epsilon(\tau,.) \|_{L^\infty}\, d\tau \\ &\leq C_2\,\epsilon^{-N}, \quad \text{where } N>0,\, t\in[0,T),\, \epsilon\in(0,1),\end{aligned}$$

since $(\mu_\epsilon)_\epsilon$ is moderate, we can write

$$\begin{aligned}\| \partial_x\mu_\epsilon(t,.) \|_{L^\infty} &\leq C|\ln\,\epsilon|^{a\,n-1}\Big(c_1T|\ln\,\epsilon|^b\Big) \\ &\leq C\,\epsilon^{-N} \quad \text{where } N>0,\, t\in[0,T),\, \epsilon\in(0,1),\end{aligned}$$

therefore $(\partial_x\mu_\epsilon)_\epsilon$ is moderate.
Step 3. Applying the Riesz fractionnal derivative in the equality (9)it yields

$$\widetilde{R}^\alpha\mu_\epsilon(t,x) = E_{n,\epsilon}(t,x) * \widetilde{R}^\alpha\mu_{0,\epsilon}(x) + \int_0^t\int_{\mathbb{R}^n} E_{n,\epsilon}(t-\tau,x-\xi)g_\epsilon(\widetilde{R}^\alpha\mu_\epsilon(\tau,x))d\xi d\tau.$$

$$\widetilde{R}^\alpha\mu_\epsilon(t,x) = E_{n,\epsilon}(t,x) * \widetilde{R}^\alpha\mu_{0,\epsilon}(x) + \int_0^t\int_{\mathbb{R}^n} E_{n,\epsilon}(t-\tau,x-\xi)\nabla g_\epsilon(\theta\mu_\epsilon)\widetilde{R}^\alpha\mu_\epsilon(\tau,x)d\xi d\tau,$$

with $\theta\in(0,1)$, and by using the moderateness of $(\mu_\epsilon)_\epsilon$, we obtain

$$\begin{aligned}\| \widetilde{R}^\alpha\mu_\epsilon(t,x) \| &\leq \| E_{n,\epsilon}(t,x) \|_{L^1} \| \widetilde{R}^\alpha\mu_{0,\epsilon} \|_{L^1} \\ &+ \int_0^t \| E_{n,\epsilon}(t-\tau,x-.) \|_{L^\infty} \| \nabla g_\epsilon(\theta\mu_\epsilon) \|_{L^\infty} \| \widetilde{R}^\alpha\mu_\epsilon(\tau,.) \|_{L^\infty}\, d\tau, \\ &\leq C\,M_\phi^{4-\alpha-n} + cT\,M_\phi^{4-\alpha-n}(|\ln\,\epsilon|)^b\epsilon^{-N} \\ &\leq C_{\alpha,\phi}\,\epsilon^{-N}, \text{with } N>0 \text{ and } C_{\alpha,\phi} \text{ is a nonegative constant.}\end{aligned}$$

By these three steps above, we can see that regularized of the Riesez fractional derivative of the solution μ_ϵ is an element of the algebra $\mathcal{E}^e_{M,R}([0,T]\times\mathbb{R}^n)$.

step 4. Let us consider $\mathcal{H}_\epsilon(t,x) = \mu_{1,\epsilon}(t,x) - \mu_{2,\epsilon}(t,x)$, where $(\mu_{1,\epsilon})_\epsilon$, $(\mu_{2,\epsilon})_\epsilon$ are two solutions of the equation (8), we can write

$$\begin{cases} \partial_t \mathcal{H}_\epsilon(t,x) = \Delta \mathcal{H}_\epsilon(t,x) + g_\epsilon(\mu_{1,\epsilon}(t,x)) - g_\epsilon(\mu_{2,\epsilon}(t,x)) + N_\epsilon(t,x), \\ \mathcal{H}_\epsilon(0,x) = N_{0,\epsilon}(x) = N_\epsilon(0,x) \in \mathcal{N}_R^e(\mathbb{R}^n), \end{cases} \tag{11}$$

with $(N_\epsilon(t,x))_\epsilon \in \mathcal{N}_R^e([0,T]\times\mathbb{R}^n)$
Now, by applying the Riesz fractional derivative we get

$$\begin{aligned} \| \widetilde{R}^\alpha \mathcal{H}_\epsilon(t,.) \|_{L^\infty} &\leq \| E_{n\epsilon}(t,x-.) \|_{L^1} \| \widetilde{R}^\alpha N_{0\epsilon} \|_{L^\infty} \\ &+ \int_0^t \| E_{n\epsilon}(t,x-.) \|_{L^1} \| \mathcal{H}_\epsilon(\tau,.) \|_{L^\infty} \| \widetilde{R}^\alpha \mathcal{H}_\epsilon(\tau,.) \|_{L^\infty}\, d\tau \\ &+ \int_0^t \| E_{n\epsilon}(t,x-.) \|_{L^1} \| \widetilde{R}^\alpha N_\epsilon(\tau,.) \|_{L^\infty}\, d\tau \\ &\leq C_1\epsilon^q + C_2T(|\ln\epsilon|)^b\epsilon^{-N} + \epsilon^{-N} \\ &\leq \tilde{C}\epsilon^{N_1}, \quad \text{for any } N_1 > 0. \end{aligned}$$

Hence, $\big(\widetilde{R}^\alpha \mathcal{H}_\epsilon(t,x)\big)_\epsilon \in \mathcal{N}_R^e\big([0,T]\times\mathbb{R}^n\big)$.
In the sequel, we will interesting to study existence and uniqueness solution of of the Schrödinger equation in $\mathcal{G}_R^e([0,T]\times\mathbb{R}^n)$. Consider the nonlinear problem associated with Schrödinger equation with potential and initial data are singulars

$$\begin{cases} \frac{1}{i}\partial_t\mu(t,x) - \triangle\mu(t,x) + \nu(x)\mu(t,x) = 0 \\ \nu(x) = \delta(x), \quad \mu(0,x) = \delta(x). \end{cases} \tag{12}$$

We will use later the following Dirac measure regularization

$$\nu_\epsilon(x) = \delta_\epsilon(x) = (\phi_\epsilon(x)) = |\ln\epsilon|^{cn}\phi(x|\ln\epsilon|^c),\ c > 0.$$

with $x \in \mathbb{R}^n$ and $\phi \in \mathcal{S}(\mathbb{R})$, satisfying conditions (1) and for the initial condition, we use the following regularization

$$\mu_{0,\epsilon}(x) = |\ln\epsilon|^{an}\phi(x|\ln\epsilon|^a), \quad a > 0.$$

To give the result about the embedding of Schrödinger equation into $\mathcal{G}_R^e([0,T)\times\mathbb{R}^n]$, we need to

Lemma 3.1. *The regularization of the previous problem is written by*

$$\begin{cases} \frac{1}{i}\partial_t\mu_\epsilon(t,x) - \triangle\mu_\epsilon(t,x) + \nu_\epsilon(x)\mu_\epsilon(t,x) = 0 \\ \nu_\epsilon(x) = \delta_\epsilon(x), \quad \mu_{0,\epsilon}(x) = \delta_\epsilon(x) \end{cases} \tag{13}$$

with ν_ϵ and $\mu_{0,\epsilon}$ are the regularizations of ν and μ_0 respectively. Hence , the problem (12) *has a unique solution in $\mathcal{G}([0,T]\times\mathbb{R}^n)$.*

Proof. The integral solution to the regularized problem (13) is given by

$$\mu_\epsilon(t,x) = \int_{\mathbb{R}^n} S_n(t,x-y)\mu_{0,\epsilon}(y)dy + \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)\nu_\epsilon(y)\mu_\epsilon(\tau,y)dyd\tau \tag{14}$$

with $S_n(t,x)$ the Schrödinger kernel see [16] and the references therein.
The author in [20] proves that the integral solution (14) satisfy the following inequalities

$$\begin{aligned} \| \mu_\epsilon(t,.) \|_{L^\infty(\mathbb{R}^n)} &\leq C|\ln\epsilon|^{an}\exp(CT|\ln\epsilon|^{bn}) \\ &\leq C\epsilon^{-N} \text{ where } C > 0, \text{ and } N > 0 \end{aligned}$$

and for higher order derivatives, we have

$$\| \partial_{x_i}\mu_\epsilon(t,.) \|_{L^\infty(\mathbb{R}^n)} \leq C\big(|\ln\epsilon|^{a(n+1)} + T|\ln\epsilon|^{b(n+1)} \| \mu_\epsilon \|_{L^\infty}\big)\exp(CT|\ln\epsilon|^{bn}).$$

And according to the previous step, there exists $N > 0$ such that

$$||\partial_{x_i}\mu_\epsilon(t,.)||_{L^\infty} \leq C\epsilon^{-N}.$$

We showed also that for the second derivative of x_i, $j \in \{1, ..., n\}$, we obtain

$$\begin{aligned} \| \partial_{x_i}\partial_{x_j}\mu_\epsilon(t,.) \|_{L^\infty} &\leq C\Big(|\ln\epsilon|^{a(n+2)} + |\ln\epsilon|^{b(n+1)} \| \mu_\epsilon \|_{L^\infty} \\ &+ |\ln\epsilon|^{b(n+1)} \| \partial_{y_i}\mu_\epsilon \|_{L^\infty} \\ &+ |\ln\epsilon|^{b(n+1)} \| \partial_{y_j}\mu_\epsilon \|_{L^\infty} \Big)e^{CT|\ln\epsilon|^{bn}} \end{aligned}$$

And thus there exists $N > 0$ such that

$$\| \partial_{x_i}\partial_{x_j}\mu_\epsilon(t,.) \|_{L^\infty(\mathbb{R}^n)} \leq C\epsilon^{-N}$$

which showed the existence of the solution of the nonlinear problem (12) of Schrödinger in the classical Colombeau algebra $\mathcal{G}([0,T]\times\mathbb{R}^n)$. □

Now, we are going to prouve the existence and uniqueness of the solution in the extension $\mathcal{G}^e_R([0,T]\times\mathbb{R}^n)$ of Colombeau algebra.

Theorem 3.3. *Under the assumptions of the previous proposition the problem* (12) *has a unique solution in* $\mathcal{G}^e_R([0,T]\times\mathbb{R}^n)$.

Proof. In the last Lemma, we have proved that entieres derivatives of μ_ϵ are moderates in the classical algebras $\mathcal{G}([0,T]\times\mathbb{R}^n)$, to show theorem, it suffices to prove that the Riesz fractional derivatives $(\widetilde{R}^\alpha\mu_\epsilon)_\epsilon$ with $\alpha \in (m-1, m)$, $m \in \mathbb{N}_0$ are moderates.
Without loss of generality, we have to proves that for all $0 < \alpha < 1$, we have

$$||\widetilde{R}^\alpha\mu_\epsilon(t,.) = \mathcal{O}(\epsilon^{-N}),$$

We apply the regularized Riesz fractional derivative with respect to the spatial variable x to (14)

$$\begin{aligned} \widetilde{R}^\alpha\mu_\epsilon(t,x) &= \int_{\mathbb{R}^n} S_n(t,x-y)\widetilde{R}^\alpha\mu_{0,\epsilon}(y)dy \\ &+ \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)\widetilde{R}^\alpha\nu_\epsilon(y)\mu_\epsilon(\tau,y)dyd\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)\nu_\epsilon(y)\widetilde{R}^\alpha\mu_\epsilon(\tau,y)dyd\tau \end{aligned}$$

$$\begin{aligned} \| \widetilde{R}^\alpha\mu_\epsilon(t,.) \|_{L^\infty} &\leq ||S_n(t,x-.) \|_{L^1} \| \widetilde{R}^\alpha\mu_{0,\epsilon} \|_{L^\infty} \\ &+ \| S_n(t-\tau,x-.) \|_{L^1} \int_0^t \| \widetilde{R}^\alpha\nu_\epsilon \|_{L^\infty} \| \mu_\epsilon(\tau,.) \|_{L^\infty} d\tau \\ &+ \| S_n(t-\tau,x-.) \|_{L^1} \int_0^t \| \nu_\epsilon(y) \|_{L^\infty} \| \widetilde{R}^\alpha\mu_\epsilon(\tau,.) \|_{L^\infty} d\tau \end{aligned}$$

$$\begin{aligned} \| \widetilde{R}^{\alpha}\mu_{\epsilon}(t,.) \|_{L^{\infty}} &\leq C\Big(\| \widetilde{R}^{\alpha}\mu_{0,\epsilon} \|_{L^{\infty}} + T \| \widetilde{R}^{\alpha}\nu_{\epsilon} \|_{L^{\infty}} \|\mu_{\epsilon} \|_{L^{\infty}} \Big) \\ &+ C \| \nu_{\epsilon}(.) \|_{L^{\infty}} \int_0^t \| \widetilde{R}^{\alpha}\mu_{\epsilon}(\tau,.) \|_{L^{\infty}} d\tau. \end{aligned}$$

Again the Gronwall inequality, gives the answer to our problem

$$\begin{aligned} \| \widetilde{R}^{\alpha}\mu_{\epsilon}(t,.) \|_{L^{\infty}} &\leq C\exp\Big(CT \| \nu_{\epsilon} \|_{L^{\infty}} \Big) \\ &\Big(\| \widetilde{R}^{\alpha}\mu_{0,\epsilon} \|_{L^{\infty}(\mathbb{R}^n)} + T \| \widetilde{R}^{\alpha}\nu_{\epsilon} \|_{L^{\infty}(\mathbb{R}^n)} \| \mu_{\epsilon} \|_{L^{\infty}} \Big). \end{aligned}$$

Hence,

$$\begin{aligned} \| \widetilde{R}^{\alpha}\mu_{\epsilon}(t,.) \|_{L^{\infty}} &\leq C\exp\Big(CT \| \nu_{\epsilon} \|_{L^{\infty}} \Big) \\ &\Big(C_{\alpha,T}|\ln\epsilon|^{a(n+1)} + TC_{\alpha,T}|\ln\epsilon|^{b(n+1)} \| \mu_{\epsilon} \|_{L^{\infty}} \Big), \end{aligned}$$

there exists $N > 0$ and $C > 0$ such that

$$\| \widetilde{R}^{\alpha}\mu_{\epsilon}(t,.) \|_{L^{\infty}(\mathbb{R}^n)} \leq C\epsilon^{-N}.$$

And thus,

$$(\mu_{\epsilon})_{\epsilon} \in \mathcal{E}^{e}_{M,R}([0,T]\times\mathbb{R}^n)$$

In order to proves uniqness of the solution, we apply the Riesz fractional derivative $\widetilde{R}^{\alpha}$, $\alpha \in (0,1)$ to the integral solution (14) and if $\mathcal{H}_{\epsilon}$ denotes the difference $\mu_{1,\epsilon} - \mu_{2,\epsilon}$, it yields

$$\begin{aligned} \widetilde{R}^{\alpha}\mathcal{H}_{\epsilon}(t,x) = &\int_{\mathbb{R}^n} S_n(t,x-y)\widetilde{R}^{\alpha}N_{0,\epsilon}(y)dy \\ &+ \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)\widetilde{R}^{\alpha}\nu_{\epsilon}(y)\mathcal{H}_{\epsilon}(\tau,y)dyd\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)\nu_{\epsilon}(y)\widetilde{R}^{\alpha}\mathcal{H}_{\epsilon}(\tau,y)dyd\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} S_n(t-\tau,x-y)\widetilde{R}^{\alpha}N_{\epsilon}(\tau,y)dyd\tau, \end{aligned}$$

$$\begin{aligned} \| \widetilde{R}^{\alpha}\mathcal{H}_{\epsilon}(t,.) \|_{L^{\infty}} &\leq \| S_n(t,x-.) \|_{L^1} \| \widetilde{R}^{\alpha}N_{0,\epsilon} \|_{L^{\infty}} \\ &+ \| S_n(t-\tau,x-.) \|_{L^1} \int_0^t \| \widetilde{R}^{\alpha}\nu_{\epsilon} \|_{L^{\infty}} \| \mathcal{H}_{\epsilon}(\tau,.) \|_{L^{\infty}} d\tau \\ &+ \| S_n(t-\tau,x-.) \|_{L^1} \int_0^t \| \nu_{\epsilon}(.) \|_{L^{\infty}} \| \widetilde{R}^{\alpha}(\mu_{1\epsilon}(\tau,.) \\ &- \mu_{2\epsilon}(\tau,.)) \|_{L^{\infty}} d\tau \\ &+ \| S_n(t-\tau,x-.) \|_{L^1} \int_0^t \| \widetilde{R}^{\alpha}N_{\epsilon}(\tau,.) \|_{L^{\infty}} d\tau \end{aligned}$$

$$\begin{aligned}\| \widetilde{R}^{\alpha}(\mu_{1,\epsilon}(t,.) - \mu_{2,\epsilon}(t,.)) \|_{L^\infty} &\leq C\Big(\| N_{0,\epsilon} \|_{L^\infty} + T \| \widetilde{R}^{\alpha}\nu_\epsilon(.) \|_{L^\infty} \times \\ &\| \mu_{1\epsilon} - \mu_{2\epsilon} \|_{L^\infty} + \| \widetilde{R}^{\alpha} N_\epsilon \|_{L^\infty} \Big) \\ &+C \| \nu_\epsilon \|_{L^\infty} \times \\ &\int_0^t \| \widetilde{R}^{\alpha}(\mu_{1\epsilon}(\tau,.) - \mu_{2\epsilon}(\tau,.)) \|_{L^\infty} \, d\tau.\end{aligned}$$

By Gronwall inequality, we have

$$\begin{aligned}\| \widetilde{R}^{\alpha}(\mu_{1,\epsilon}(t,.) - \mu_{2,\epsilon}(t,.)) \|_{L^\infty} &\leq C\Big(\| N_{0,\epsilon} \|_{L^\infty} + T \| \widetilde{R}^{\alpha}\nu_\epsilon \|_{L^\infty} \times \\ &\| \mu_{1\epsilon} - \mu_{2\epsilon} \|_{L^\infty} + \| \widetilde{R}^{\alpha} N_\epsilon \|_{L^\infty} \Big)\times \\ &\exp\Big(CT \| \nu_\epsilon \|_{L^\infty} \Big).\end{aligned}$$

Thus

$$\| \widetilde{R}^{\alpha}(\mu_{1,\epsilon}(t,.) - \mu_{2,\epsilon}(t,.)) \|_{L^\infty} \leq C\epsilon^q, \qquad \text{for all } q.$$

Hence,

$$\mu - \nu \in \mathcal{N}_R^e([0,T] \times \mathbb{R}^n),$$

which end the proof of theorem. □

4. Conclusions

The primary aim was to establish the existence and uniqueness of solutions for these problems within the framework of the extended $\mathcal{G}_R^e$ of Colombeau algebra. Through rigorous analysis and mathematical reasoning, it has been successfully demonstrated that both the Schrödinger equation and the heat equation, encompassing their respective evolution problems, admit generalized solutions that exhibit existence and uniqueness.

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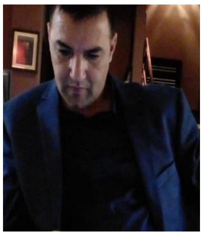

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