TWMS J. App. and Eng. Math. V.15, N.2, 2025, pp. 421-430

FURTHER RESULTS ON THE DOUBLE ROMAN DOMINATION IN GRAPHS

A. OMAR[1*], A. BOUCHOU[2], §

ABSTRACT. A *Roman dominating function* (RDF) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $w(f)$ of a Roman dominating function f is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. A *double Roman dominating function* (DRDF) on a graph G is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 3$ or two vertices v_1 and v_2 for which $f(v_1) = f(v_2) = 2$, and every vertex u for which $f(u) = 1$ is adjacent to at least one vertex v for which $f(v) \geq 2$. The weight $w(f)$ of a double Roman dominating function f is the value $w(f) = \sum_{u \in V} f(u)$. The minimum weight of a double Roman dominating function on a graph G is called the *double Roman domination number* of G, denoted by $\gamma_{dR}(G)$. In this paper,we characterize some classes of graphs G with $\gamma_{dR}(G) \geq 2(n - \Delta(G)) - 1$. Moreover we provide a characterization of extremal graphs of a Nordhaus-Gaddum bound for $\gamma_{dR}(G)$ improving the corresponding results given by L. Volkmann (2023). Finally, we give a characterization of graphs G with $\gamma_{dR}(G) = 2\gamma_R(G) - 1$..

Keywords: Double Roman dominating function, Double Roman domination number, Nordhaus-Gaddum inequalities, Tree.

AMS Subject Classification: 05C69

1. Introduction

All the graphs considered in this paper are simple. Let $G = (V, E)$ denote a graph with vertex set V and edge set E. The order $n = |V|$ of G is the number of its vertices. The complement $\overline{G}$ of $G = (V, E)$ is the graph defined on the vertex set V of G, where an edge belongs to $\overline{G}$ if and only if it does not belong to G. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v of G is $\deg_G(v) = |N(v)|$.

[1] LAMDA-RO Laboratory, Department of Mathematics, University of Blida 1, Algeria.
e-mail: omar.abdelhak@etu.univ-blida.dz; ORCID: https://orcid.org/0000-0001-5655-8968.
[*] Corresponding author.
[2] Department of Mathematics and Computer Science, University of Médéa, Algeria.
e-mail: bouchou.ahmed@yahoo.fr; ORCID: https://orcid.org/0009-0003-8711-4739.
§ Manuscript received: August 09, 2023; accepted: December 14, 2023.

By $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the *maximum degree* and the *minimum degree* of G, respectively. For any set $S \subseteq V$, its open neighborhood is the set $N(S) = \cup_{v\in S} N(v)$, and its closed neighborhood is the set $N[S] = N(S) \cup S$. For any $S \subseteq V$, we denote the subgraph of G induced by S as $G[S]$. We use K_n, P_n and C_n to denote the *complete graph*, the *path* and the *cycle* of order n, respectively. A *tree* is a connected graph with no cycles. A star $K_{1,p}$ for $p \geq 1$, is a tree of order $p+1$ having at least p leaves. For a positive integer t, a *wounded spider* is a star $K_{1,t}$ with at most $t-1$ of its edges subdivided. A graph G of order at least two is called *regular* if its vertices have the same degree and *semiregular* if $\Delta(G) - \delta(G) = 1$. For simplicity, a regular graph each of whose vertices has degree r is called r-regular. For terminology not defined here, we refer the reader to [8].

A subset $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ has a neighbor in S. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

A *Roman dominating function* (RDF) on a graph G is a function $f : V \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight $w(f)$ of a Roman dominating function f is the value $w(f) = \sum_{u\in V} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. For further details on Roman domination and its variations we refer to the reader the book chapters [3, 4] and survey [5].

A *double Roman dominating function* (DRDF) on a graph G is a function $f : V \longrightarrow \{0, 1, 2, 3\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 3$ or two vertices v_1 and v_2 for which $f(v_1) = f(v_2) = 2$, and every vertex u for which $f(u) = 1$ is adjacent to at least one vertex v for which $f(v) \geq 2$. The weight $w(f)$ of a double Roman dominating function f is the value $w(f) = \sum_{u\in V} f(u)$. The minimum weight of a double Roman dominating function on a graph G is called the *double Roman domination number* of G, denoted by $\gamma_{dR}(G)$.

It is clear that any Roman dominating function f on a graph G induces three sets V_0, V_1, V_2 where $V_i = \{v \in V : f(v) = i\}$, $w(f) = \sum_{u\in V} f(u) = |V_1| + 2|V_2|$ and $|V_0| + |V_1| + |V_2| = n$, similarly any double Roman dominating function g on a graph G induces four sets V_0', V_1', V_2', V_3' where $V_i' = \{v \in V : g(v) = i\}$, $w(g) = \sum_{u\in V} g(u) = |V_1'| + 2|V_2'| + 3|V_3'|$ and $|V_0'| + |V_1'| + |V_2'| + |V_3'| = n$.

The double Roman domination number was introduced by Beeler et al. [2], where they obtained relationships of double Roman domination to both domination and Roman domination. Furthermore, they proved that $\gamma_{dR}(G) \leq \frac{5}{4}n$ for any connected graph G with $n \geq 3$ vertices and characterize the graphs attaining this bound. Later Jafari Rad and Rahbani [10] observed that $\gamma_{dR}(G) \leq 2(n - \Delta) + 1$ and presented a characterization of graphs G with $\gamma_{dR}(G) = 2(n - \Delta) + 1$. Further results on double Roman domination in graphs can be found in [1, 7, 11, 12, 13, 14].

In this paper, we first give characterization of some classes of graphs G with $\gamma_{dR}(G) = 2(n - \Delta(G)) + k$, where $k \in \{-1, 0, 1\}$. Moreover we provide a characterization of extremal graphs of a Nordhaus-Gaddum bound for $\gamma_{dR}(G)$ improving the corresponding results given in [10] and [13]. Finally, we give a characterization of graphs G for which the equality $\gamma_{dR}(G) = 2\gamma_R(G) - 1$ holds.

2. Preliminary results

We begin by recalling some important results that will be useful in our investigations.

Proposition 2.1. [2] *In a double Roman dominating function of weight $\gamma_{dR}(G)$, no vertex needs to be assigned the value 1.*

Theorem 2.1. [9] *Let G be a graph of order $n \geq 5$, $\delta(G) \geq 2$ and with no component isomorphic to C_5 or C_7. Then $\gamma_{dR}(G) \leq \frac{11n}{10}$.*

Proposition 2.2. [1] *For any integer $n \geq 1$,*

$$\gamma_{dR}(P_n) = \begin{cases} n & \text{if } n \equiv 0 \pmod 3 \\ n+1 & \text{if } n \equiv 1, 2 \pmod 3. \end{cases}$$

Proposition 2.3. [1] *For any integer $n \geq 3$,*

$$\gamma_{dR}(C_n) = \begin{cases} n & \text{if } n \equiv 0, 2, 3, 4 \pmod 6 \\ n+1 & \text{if } n \equiv 1, 5 \pmod 6. \end{cases}$$

Proposition 2.4. [1] *Let G be a connected graph of order $n \geq 3$. Then*

(1) $\gamma_{dR}(G) = 3$ *if and only if* $\Delta(G) = n - 1$.
(2) $\gamma_{dR}(G) = 4$ *if and only if* $G = \overline{K_2} \vee H$, *where H is a graph with* $\Delta(H) \leq |V(H)| - 2$.
(3) $\gamma_{dR}(G) = 5$ *if and only if* $\Delta(G) = n - 2$ *and* $G \neq \overline{K_2} \vee H$ *for any graph H of order $n - 2$.*

3. Graphs G of order n with $\gamma_{dR}(G) \geq 2(n - \Delta) - 1$

In this section we provide a characterization of some classes of graphs G with $\gamma_{dR}(G) \geq 2(n - \Delta) - 1$, including regular graphs, semiregular graphs and graphs with $\Delta - \delta = 2$.

Using Propositions 2.2 and 2.3, we have the following straightforward observation for nontrivial graphs with $\Delta \leq 2$.

Observation 3.1. *Let G be a graph of order n and p a non-negative integer, with maximum degree $\Delta \leq 2$. Then*

(1) $\gamma_{dR}(G) = 2(n - \Delta) + 1$ *if and only if* $G = pK_1 \cup H$ *where* $H \in \{K_2, P_3, C_3, P_4\}$ *and* $n = p + |V(H)|$.
(2) $\gamma_{dR}(G) = 2(n - \Delta)$ *if and only if* $G = \overline{K_n}$ *or* $G = pK_1 \cup H$, *where* $H \in \{2K_2, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, C_4, C_5, P_5\}$ *and* $n = p + |V(H)|$.
(3) $\gamma_{dR}(G) = 2(n - \Delta) - 1$ *if and only if* $G = pK_1 \cup K_2 \cup H$, *where* $H \in \{C_4, C_5, P_5\}$ *or* $G = pK_1 \cup 2K_2 \cup H$, *where* $H \in \{K_2, P_3, C_3, P_4\}$.

Jafari Rad and Rahbani [10] presented a family of graphs G with $\gamma_{dR}(G) = 2(n - \Delta) + 1$ as follows:

A vertex that belongs to a minimum dominating set of G called a *good vertex*. The set of all good vertices of G is denote by *good*(G), and $G -$ *good*(G) denotes the subgraph of G induced by $V(G) -$ *good*(G). For a graph H, an H-partition is a partition of $V(H)$ into $p + 1$ nonempty sets $A_0, A_1, ..., A_p$ for some integer $p < n$ such that the following hold:

(1) If $p \geq 2$, then for $i \geq 1$ the subgraph of H induced by $V(H) - A_i$ has domination number at least two, or a $\gamma(H[V(H) - A_i])$-set is contained in A_0.
(2) If $1 \leq \gamma(H) \leq 2$, then the following hold:
  - If $\gamma(H) = 1$, then *good*$(H) \subseteq A_0$; and every $\gamma(H -$ *good*$(H))$-set has at most one common vertex with $\bigcup_{i=1}^{p} A_i$ whenever $\gamma(H -$ *good*$(H)) = 2$.
  - If $\gamma(H) = 2$, then $\bigcup_{i=1}^{p} A_i$ contains at most one vertex of a $\gamma(H)$-set, for $i = 1, 2, ..., p$; otherwise a $\gamma(H)$-set is contained in A_i for $i \in \{1, ..., p\}$ and no $\gamma(H)$-set is contained in $\bigcap_{u \in A_0} N(u)$.

Definition 3.1. *Let $A_0, A_1, ..., A_p$ be an H-partition of a graph H. Let $\mathcal{F}$ be the family of graphs G that can be obtained from a graph H by adding $p+1$ new vertices $v_1, v_2, ..., v_p, u$, joining v_i to all of the vertices of A_i for $i = 1, 2, ..., p$, and joining u to all of the vertices of H.*

Theorem 3.1. [10] *If G is graph of order n with maximum degree $\Delta(G)$, then $\gamma_{dR}(G) \leq 2(n - \Delta(G)) + 1$, with equality if and only if $G \in \mathcal{F}$.*

For any vertex $v \in V(G)$, we write $\overline{N}[v] = V(G) - N[v]$. We also denote by t the number of edges joining the vertices of $N(v)$ to the vertices of $\overline{N}[v]$. The corona of a graph G, denoted by $Cor(G)$, is the graph that is obtained by attaching a leaf to each vertex $v \in V$.

Proposition 3.1. *Let G be a graph of order n and p a non-negative integer with maximum degree Δ such that $\Delta - \delta \leq 2$. Then $\gamma_{dR}(G) = 2(n - \Delta) + 1$ if and only if either $G \in \{pK_1 \cup H,\ H \in \{K_2, P_3, C_3, P_4\}\} \cup \{cor(P_3), cor(C_3)\}$, or $\Delta = n - 1$, or $\Delta = n - 2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n - 2$.*

Proof. Let G be a graph of order n with maximum degree Δ and minimum degree δ such that $\Delta - \delta = k \in \{0, 1, 2\}$ and $\gamma_{dR}(G) = 2(n - \Delta) + 1$. If $\Delta \leq 2$, then from Observation 3.1 we obtain $G = pK_1 \cup H$ where $H \in \{K_2, P_3, C_3, P_4\}$ and $n = p + |V(H)|$. Now assume that $\Delta \geq 3$. According to the construction of Family $\mathcal{F}$ described above in Definition 3.1, every vertex in $\overline{N}[v]$ has at least $\Delta - k$ neighbors in $N(v)$, and every vertex in $N(v)$ has at most one neighbor in $\overline{N}[v]$, but at least one vertex which has no neighbor in $\overline{N}[v]$. So we have $(\Delta - k)\left|\overline{N}[v]\right| \leq t \leq |N(v)| - 1$, which provides $(\Delta - k)(n - \Delta - 1) \leq \Delta - 1$, and thus $n \leq \Delta + 2 + \frac{k-1}{\Delta - k}$. Clearly, for $\Delta \geq 2k$, we have $\Delta \geq n - 2$, and by Proposition 2.4, $G \neq \overline{K_2} \vee H$ for any graph H of order $n - 2$. Assume now that $\Delta \leq 2k - 1$. Since $\Delta \geq 3$ and $k \leq 2$, we obtain that $k = 2$ and $\Delta = 3$, and thus $n \in \{4, 5, 6\}$. If $n \in \{4, 5\}$, then $\Delta \geq n - 2$, again by Proposition 2.4, $G \neq \overline{K_2} \vee H$ for any graph H of order $n - 2$. If $n = 6$, then $t = 2$. It is a simple matter to check that $G = cor(P_3)$ or $cor(C_3)$.

The converse is easy to show. □

Next, we present a necessary conditions for connected graphs G of order n and maximum degree Δ, where $2(n - \Delta) - 1 \leq \gamma_{dR}(G) \leq 2(n - \Delta)$.

Lemma 3.1. *Let G be a connected graph of order n with maximum degree Δ. If $\gamma_{dR}(G) = 2(n - \Delta) - p$, where $p \in \{0, 1\}$, then for every vertex v of maximum degree we have:*

(1) *Every vertex of $N(v)$ has at most two neighbors in $\overline{N}[v]$.*
(2) *$\overline{N}[v] \neq \emptyset$ and every component of $G\left[\overline{N}[v]\right]$ has at most two vertices. Moreover*
  i) *If $p = 0$, then $G\left[\overline{N}[v]\right]$ contains at most one edge.*
  ii) *If $p = 1$, then $G\left[\overline{N}[v]\right]$ contains at most two independent edges.*

Proof. Let G be a graph with $\gamma_{dR}(G) = 2(n - \Delta) - p$ where $p \in \{0, 1\}$. Let v be a vertex of maximum degree Δ. If some vertex $u \in N(v)$ has at least three neighbors in $\overline{N}[v]$, then $f = (N(u) \cup N(v) - \{u, v\}, V(G) - (N(u) \cup N(v)), \{u, v\})$ is a DRDF with weight at most $2(n - \Delta) - 2$, a contradiction. Hence (1) follows. If $\overline{N}[v] = \emptyset$, then $\Delta = n - 1$, and so $\gamma_{dR}(G) = 3 = 2(n - \Delta) + 1$, a contradiction. Thus assume that $\overline{N}[v] \neq \emptyset$. Suppose there is a component of $G[\overline{N}[v]]$, say F, has at least three vertices. Let $x \in V(F)$, with $|N_F(x)| \geq 2$. Clearly $f = (N(\{v, x\}), V(G) - N[\{v, x\}], \{v, x\})$ is a DRDF, with weight at most $2(n - \Delta) - 2$, a contradiction. Now suppose that $p = 0$ and $G\left[\overline{N}[v]\right]$ contains two independent edges xy and $x'y'$. Then clearly $g = (N(v) \cup \{y, y'\}, V(G) - (N[v] \cup \{x, x', y, y'\}), \{v, x, x'\})$ is a DRDF, with weight at most

$2(n-\Delta)-1$, a contradiction. Finally suppose that $p=1$ and $G\left[\overline{N}(v)\right]$ contains at least three independent edges xy, $x'y'$ and $x''y''$. Then clearly $g=(N(v)\cup\{y,y',y''\},V(G)-(N[v]\cup\{x,x',x'',y,y',y''\}),\{v,x,x',x''\})$ is a DRDF, with weight at most $2(n-\Delta)-2$, a contradiction. Hence (2) follows. □

Proposition 3.2. *Let G be a graph of order n with maximum degree Δ such that $\Delta-\delta\le 1$. Then $\gamma_{dR}(G)=2(n-\Delta)$ if and only if either $G\in\{\overline{K_n},C_4,C_5,(n-4)K_1\cup 2K_2,K_2\cup P_3,K_2\cup C_3,K_2\cup P_4,P_5\}$, or $\Delta=n-3$ and $\Delta\ge 3$, or $\Delta=n-2$, $\Delta\ge 3$ and $G=\overline{K_2}\vee H$, where H is a graph with $\Delta(H)\le|V(H)|-2$.*

Proof. Let G be a graph of order n with maximum degree Δ such that $\Delta-\delta=k\in\{0,1\}$, and let $v\in V(G)$ be a vertex of maximum degree. Assume that $\gamma_{dR}(G)=2(n-\Delta)$. If $\Delta\le 2$, then from Observation 3.1 we obtain $G\in\left\{\overline{K_n},2K_2,C_4,C_5\right\}$, or $G\in\{(n-4)K_1\cup 2K_2; n\ge 5\}$, or $G\in\{K_2\cup P_3,K_2\cup C_3,K_2\cup P_4,P_5\}$. Now assume that $\Delta\ge 3$. By Lemma 3.1, every vertex in $\overline{N}[v]$ has at least $\Delta-k-1$ neighbors in $N(v)$, and every vertex in $N(v)$ has at most two neighbors in $\overline{N}[v]$, and $\left|\overline{N}[v]\right|\ne 0$. We proceed according to the value of $\left|\overline{N}[v]\right|$.

Case 1. If $\left|\overline{N}[v]\right|\ge 5$, then $2(\Delta-k-1)+3(\Delta-k)\le t\le 2\Delta$, which provides $\Delta\le\left\lfloor\frac{5k+2}{3}\right\rfloor\le 2$, a contradiction.

Case 2. $\left|\overline{N}[v]\right|=4$. Then $\Delta=n-5$, and thus $2(\Delta-k-1)+2(\Delta-k)\le t\le 2\Delta$, which provides $\Delta\le 2k+1$, and thus $k=1$, $\Delta=3$ and $n=8$. By Theorem 2.1, $\gamma_{dR}(G)\le\frac{11n}{10}<2(n-\Delta)$, a contradiction.

Case 3. $\left|\overline{N}[v]\right|=3$. Then $\Delta=n-4$, and thus $2(\Delta-k-1)+(\Delta-k)\le t\le 2\Delta$, which provides $\Delta\le 3k+2$. So $k=1$ and $\Delta\in\{3,4,5\}$. Set $\overline{N}[v]=\{x,y,z\}$, we have three possibilities.

Subcase 3.1. $\Delta=5$. Then $n=9$, which gives $t=10$. Thus $\overline{N}[v]$ has exactly one edge and every vertex in $\overline{N}[v]$ has degree 4. Let $N(v)=\{a,b,c,d,e\}$. Without loss of generality, we assume that $xy\in E(G)$. Since $t=10$, $|N(x)\cap N(v)|=|N(y)\cap N(v)|=3$, and $|N(z)\cap N(v)|=4$. Let $N(z)=\{a,b,c,d\}$. Clearly, x and y have no common neighbor in $\{a,b,c,d\}$, and so x and y have e as a unique common neighbor in $N(v)$. The function $f=(\{x,y,a,b,c,d,v\},\emptyset,\{z,e\})$ is an DRDF on G of weight 6, which contradicts the fact that $\gamma_{dR}(G)=2(n-\Delta)$.

Subcase 3.2. $\Delta=4$. Then $n=8$, which gives $t\in\{7,8\}$. Clearly, $\overline{N}[v]$ is not independent. Without loss of generality, assume that $xy\in E(G)$. Let $N(v)=\{a,b,c,d\}$. Since $|N(z)\cap N(v)|\ge 3$, we may assume that $\{a,b,c\}\subseteq N(z)$. Clearly, xd or $yd\in E(G)$, say $xd\in E(G)$. The function $f=(\{a,b,c,d,y\},\{v,z\},\{x\})$ is an DRDF on G of weight 7, which contradicts the fact that $\gamma_{dR}(G)=2(n-\Delta)$.

Subcase 3.3. $\Delta=3$. Then $n=7$. Note that $\delta\ge 2$. Again by Theorem 2.1, $\gamma_{dR}(G)\le\frac{11n}{10}<2(n-\Delta)$ a contradiction.

Case 4. $\left|\overline{N}[v]\right|=2$. Then $\Delta=n-3$ holds.

Case 5. $\left|\overline{N}[v]\right|=1$. Then $\Delta=n-2$, and thus by Proposition 2.4, $\gamma_{dR}(G)=2(n-\Delta)$ leads $G=\overline{K_2}\vee H$, where H is a graph with $\Delta(H)\le|V(H)|-2$.

The converse is easy to show. □

Proposition 3.3. *Let G be a Δ-regular graph of order $n\ge 2$. Then $\gamma_{dR}(G)=2(n-\Delta)-1$ if and only if $G=3K_2$.*

Proof. Let G be a Δ-regular graph of order $n\ge 2$. Assume that $\gamma_{dR}(G)=2(n-\Delta)-1$. If $\Delta\ge 3$, then by Lemma 3.1, every vertex in $\overline{N}[v]$ has at least $\Delta-1$ neighbors in $N(v)$, and every vertex in $N(v)$ has at most two neighbors in $\overline{N}[v]$. If $\left|\overline{N}[v]\right|\ge 3$, then

$2(\Delta-1)+\Delta \leq t \leq 2\Delta$, which provides $\Delta \leq 2$, a contradiction. Therefore $\left|\overline{N}[v]\right| \leq 2$, and so $\Delta \geq n-3$. By Propositions 3.1 and 3.2, we have $\gamma_{dR}(G) \geq 2(n-\Delta)$, a contradiction. Now assume that $\Delta \leq 2$, then by Observation 3.1, we have $G \in \{3K_2\}$.

The converse is easy to show. □

4. Nordhaus-Gaddum Inequality

Jafari Rad and Rahbani [10], and Volkmann [13] presented Nordhaus-Gaddum type inequalities for the double Roman domination number in terms of the order of the graph G.

Theorem 4.1. [10] *For any graph G of order $n \geq 2$, $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \leq 2n+3$, with equality if and only if $G \in \left\{K_n, \overline{K_n}\right\}$.*

In the following, let $K_n - e$ and $K_n - \{e_1, e_2\}$ represent the complete graph minus an edge and the complete graph minus two independent edges, respectively. Additionally, let $\mathcal{H}_1 = \left\{2K_2, C_4, P_4, C_5, K_n - e, \overline{K_n - e}; n \geq 3\right\}$.

Theorem 4.2. [10] *Let G be a graph of order $n \geq 3$ such that $G \notin \left\{K_n, \overline{K_n}\right\}$. Then $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \leq 2n+2$, with equality if and only if $G \in \mathcal{H}_1$.*

Theorem 4.3. [13] *Let G be a graph of order $n \geq 4$ such that $G \notin \left\{K_n, \overline{K_n}\right\} \cup \mathcal{H}_1$. Then $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \leq 2n+1$, with equality if and only if $G \in \left\{K_n - \{e_1,e_2\}, \overline{K_n - \{e_1,e_2\}}\right\}$ and $n \geq 5$ or $G \in \left\{P_5, 3K_2, \overline{P_5}, \overline{3K_2}\right\}$.*

According to Theorems 4.1, 4.2 and 4.3, if G is a graph such that $G \notin \mathcal{H} = \left\{K_n, \overline{K_n}\right\} \cup \mathcal{H}_1 \cup \mathcal{H}_2$, where $\mathcal{H}_2 = \left\{K_n - \{e_1,e_2\}, \overline{K_n - \{e_1,e_2\}}, P_5, 3K_2, \overline{P_5}, \overline{3K_2}; n \geq 5\right\}$, then we obtain $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \leq 2n$. In the sequel, we provide a characterization of graphs G of order $n \geq 4$ for which $\gamma_R(G)+\gamma_R(\overline{G}) = 2n$. For this purpose, We introduce the following families of graphs :

- $\mathcal{F}_0 = \{4K_2, 2C_3, C_6, C_7\}$.
- $\mathcal{F}_1 = \{(n-6)K_1 \cup 3K_2; n \geq 7, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4\} \cup$ $\{F : F$ is semiregular with $n(F) = 6$ and $\Delta(F) = 3\}$.
- $\mathcal{F}_2 = \{(n-3)K_1 \cup P_3, (n-3)K_1 \cup C_3, (n-4)K_1 \cup P_4; n \geq 4\} \cup$ $\{cor(P_3), cor(C_3), F_1, F_2, F_3\}$, where F_1, F_2 and F_3 are the graphs illustrated in Figure 1.

Theorem 4.4. *Let G be a graph of order $n \geq 4$ such that $G \notin \mathcal{H}$. Then $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \leq 2n$, with equality if and only if G or $\overline{G} \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$.*

Proof. Clearly, the upper bound follows from Theorems 4.1, 4.2 and 4.3, since $G \notin \mathcal{H}$.

Assume now that $\gamma_{dR}(G)+\gamma_{dR}(\overline{G}) = 2n$. By Theorem 3.1, we have

$$\begin{aligned} 2n &= \gamma_{dR}(G)+\gamma_{dR}(\overline{G}) \\ &\leq 2(n-\Delta(G))+1+2\left(n-\Delta\left(\overline{G}\right)\right)+1 \\ &\leq 2n-2(\Delta(G)-\delta(G))+4. \end{aligned}$$

Hence $\Delta(G)-\delta(G) \leq 2$. Therefore G is either regular or semiregular or $\Delta(G)-\delta(G) = 2$. We distinguish three cases.

Case 1. G is regular. Then without loss of generality we consider three possibilities:

Subcase 1.1. $\gamma_{dR}(G) = 2(n-\Delta(G))+1$ and $\gamma_{dR}(\overline{G}) = 2\left(n-\Delta\left(\overline{G}\right)\right)-3$. By Proposition 3.1, we have $G = K_n$, excluded, since $K_n \in \mathcal{H}$.

Subcase 1.2. $\gamma_{dR}(G) = 2(n - \Delta(G))$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 2$. By Proposition 3.2, and since $G \notin \left\{\overline{K_n}, C_4, 2K_2, C_5\right\} \subset \mathcal{H}$, we have $\Delta(G) = n-3$ or $n-2$ with $\Delta(G) \geq 3$. Clearly, if $\Delta(G) = n-3$, then $\overline{G}$ is the disjoint union of p copies of cycles of order n_i, where $p \geq 1$ and $n = \sum_{i=1}^{p} n_i$. Using the fact that $\gamma_{dR}(C_{n_i}) \leq n_i+1$ (see Proposition 2.3), we have $2n-6 = \gamma_{dR}(\overline{G}) = \sum_{i=1}^{p} \gamma_{dR}(C_{n_i}) \leq n+p$, which gives $n \leq p+6$. On the other hand, since $n_i \geq 3$, for $i \in \{1, ..., p\}$, we have $n \geq 3p$, so, $p \leq 3$. Now, it is easy to cheek that if $p = 1$, then $\overline{G} \in \{C_6, C_7\}$, and if $p = 2$ then $\overline{G} \in \{2C_3, C_3 \cup C_4\}$, finally if $p = 3$ then $\overline{G} = 3C_3$. Then $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 2$ leaves $G \in \{2C_3, C_6, C_7\} \subset \mathcal{F}_0$. Now assume that $\Delta(G) = n-2$. Then each component of $\overline{G}$ is a K_2, and thus $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 2$ leaves $\overline{G} = 4K_2$. Hence $\overline{G} \in \mathcal{F}_0$.

Subcase 1.3. $\gamma_{dR}(G) = 2(n - \Delta(G)) - 1$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 1$. By Proposition 3.3, we have $G = 3K_2$, excluded, since $3K_2 \in \mathcal{H}$.

Case 2. G is semi-regular. Then without loss of generality we have two possibilities:

Subcase 2.1. $\gamma_{dR}(G) = 2(n - \Delta(G)) + 1$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 1$. By Proposition 3.1, we have $G = (n-2)K_1 \cup K_2$, $\Delta(G) = n-1$, or $\Delta(G) = n-2$ and $G \neq \overline{K_2} \vee H$ for any graph H of order $n-2$. The graph $(n-2)K_1 \cup K_2$ is excluded, since it is in $\mathcal{H}$. If $\Delta(G) = n-1$, then $\Delta\left(\overline{G}\right) = 1$, and so $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right) - 1$ leaves $\overline{G} = (n-6)K_1 \cup 3K_2$. Hence $\overline{G} \in \mathcal{F}_1$. Now assume that $\Delta(G) = n-2$. Then $\Delta\left(\overline{G}\right) = 2$. By Observation 3.1, we have $\overline{G} = K_2 \cup H$, where $H \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, C_4, C_5, P_5\}$, contradicting the fact that $G \neq \overline{K_2} \vee \overline{H}$.

Subcase 2.2. $\gamma_{dR}(G) = 2(n - \Delta(G))$ and $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right)$. By Proposition 3.2, we have $G \in \{pK_1 \cup 2K_2$ where $p \geq 1, K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4, P_5\}$, or $\Delta(G) = n-3$ and $\Delta(G) \geq 3$, or $\Delta(G) = n-2$, $\Delta(G) \geq 3$ and $G = \overline{K_2} \vee H$, where G is a graph with $\Delta(G) \leq |V(G)| - 2$. The graphs $pK_1 \cup 2K_2$ where $p \geq 1$ and P_5 are excluded, since they are in $\mathcal{H}$. So for $\Delta(G) \leq 2$, $\gamma_{dR}(\overline{G}) = 2\left(n - \Delta\left(\overline{G}\right)\right)$ leaves $G \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4\} \subset \mathcal{F}_1$. Now suppose that $\Delta(G) \geq 3$. If $\Delta(G) = n-2$, then $\Delta(\overline{G}) = 2$, and so $\overline{G} \in \{K_2 \cup P_3, K_2 \cup C_3, K_2 \cup P_4\} \subset \mathcal{F}_1$. Now assume that $\Delta(G) = n-3$. Then $\Delta(\overline{G}) = 3$, which means that $\Delta(\overline{G}) = n-3$, and thus $n = 6$. Therefore G and $\overline{G}$ are semi regular with maximum degree 3. Hence $\overline{G}$ and G are in $\mathcal{F}_1$.

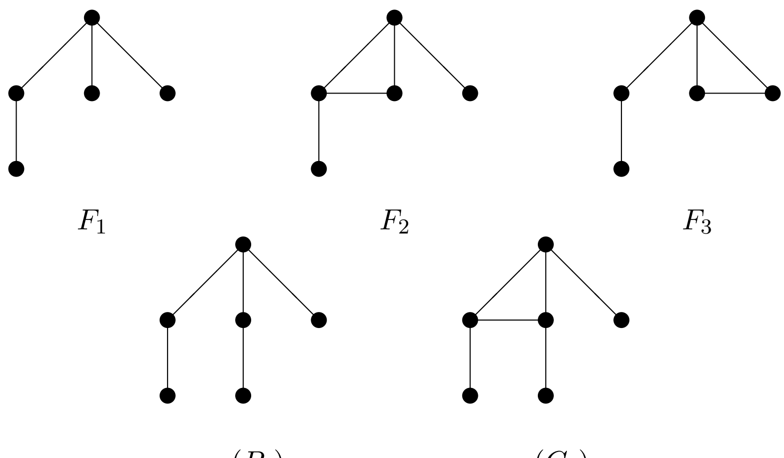

FIGURE 1. Graphs G in $\mathcal{F}_2$ with $\Delta(G) = 3$.

Case 3. $\Delta(G)-\delta(G)=2$. Then we have the only possibility; $\gamma_{dR}(G)=2\left(n-\Delta\left(G\right)\right)+1$ and $\gamma_{dR}(\overline{G})=2\left(n-\Delta\left(\overline{G}\right)\right)+1$. By Proposition 3.1, we have either $M\in\{pK_1\cup H$, where $H\in\{P_3,C_3,P_4\}$ and $p\geq 1\}\cup\{cor(P_3),cor(C_3)\}$, or $\Delta(M)=n-1$, or $\Delta(M)=n-2$ and $M\neq\overline{K_2}\vee H$ for any graph H of order $n-2$, where $M\in\left\{G,\overline{G}\right\}$. Without loss of generality, if $\Delta(G)\leq 2$, then $G\in\{pK_1\cup H$, where $H\in\{P_3,C_3,P_4\}$ and $p\geq 1\}$. Therefore $\overline{G}$ has a vertex with degree $\Delta(\overline{G})=n-1$. Hence $G\in\mathcal{F}_2$. Now suppose that $\Delta(G)\geq 3$. If $\Delta(G)=n-1$, then $\overline{G}$ has an isolated vertex, and so $\overline{G}\in\{pK_1\cup H$, where $H\in\{P_3,C_3,P_4\}$ and $p\geq 1\}$. Hence $\overline{G}\in\mathcal{F}_2$. Assume that $\Delta(G)=n-2$, then $\Delta\left(\overline{G}\right)=3$. By the construction of Family $\mathcal{F}$ described above, we get $n\in\{5,6\}$. It is a simple matter to check that $G\in\{F_1,F_2,F_3,cor(P_3),cor(C_3)\}\subset\mathcal{F}_2$ (see Figure 1).

The converse is easy to see and we omit the details. □

5. Graph with $\gamma_{dR}(G)=2\gamma_R(G)-1$

In this section, we give a characterization of connected graphs with $\gamma_{dR}(G)=2\gamma_R(G)-1$. We begin by recalling some important results that will be useful.

Theorem 5.1. [6] *For any graph G, $\gamma(G)\leq\gamma_R(G)$ with equality if and only if $G=\overline{K_n}$.*

Theorem 5.2. [2] *For any graph G, $\gamma_{dR}(G)\leq 2\gamma_R(G)$ with equality if and only if $G=\overline{K_n}$.*

From Theorem 5.2, if G is a nontrivial connected graph, then $\gamma_{dR}(G)\leq 2\gamma_R(G)-1$. A characterization of the connected graphs G with γ_{dR}-functions of weight $\gamma_{dR}(G)=2\gamma_R(G)-1$, will be shown in the following.

Proposition 5.1. *If G is a connected graph of order n with maximum degree $\Delta(G)$, then $\gamma_{dR}(G)=2\gamma_R(G)-1$ if and only if $\gamma_{dR}(G)=2\left(n-\Delta(G)\right)+1$.*

Proof. Let $f=(V_0,V_1,V_2)$ be an RDF with minimum weight and $\gamma_{dR}(G)=2w\left(f\right)-1$. So $\gamma_{dR}(G)=2\left|V_1\right|+4\left|V_2\right|-1$. It is clear that $g=(V_0,\emptyset,V_1,V_2)$ is a DRDF on G of weight $\gamma_{dR}(G)\leq 2\left|V_1\right|+3\left|V_2\right|$. A simple calculation gives $|V_2|\leq 1$. we have two cases:

Case 1. $V_2=\emptyset$. Then $V_1=V$. However, it is observed that $\gamma_R(G)=n$ if and only if $G=pK_2\cup qK_1$ where $2p+q=n$. Since G is connected, $\gamma_{dR}(G)=2\gamma_R(G)-1$ leaves only $G=K_2$. Hence $\gamma_{dR}(G)=2\left(n-\Delta(G)\right)+1$.

Case 2. $V_2=\{v\}$. Since no edge of G joins V_1 and $\{v\}$, and $\{v\}$ dominates V_0, we have

$$\deg(v)=|V_0|=n-(|V_1|+|V_2|)=n-\gamma_R(G)+1=n-\frac{\gamma_{dR}(G)+1}{2}+1$$

and so $\Delta(G)\geq\frac{2n-\gamma_{dR}(G)+1}{2}$. Hence $\gamma_{dR}(G)\geq 2\left(n-\Delta(G)\right)+1$. Equality holds from the fact that $\gamma_{dR}(G)\leq 2\left(n-\Delta(G)\right)+1$.

Conversely, assume $\gamma_{dR}(G)=2\left(n-\Delta(G)\right)+1$, and let v be a vertex of G with maximum degree $\Delta(G)$. We define $V_0=N(v)$, $V_1=V-N[v]$, and $V_2=\{v\}$, then $f=(V_0,V_1,V_2)$ is an RDF with $w\left(f\right)=n-\Delta(G)+1=\frac{\gamma_{dR}(G)+1}{2}$. Since $\gamma_R(G)\geq\frac{\gamma_{dR}(G)+1}{2}$ for connected graphs, f is an RDF for G with $w\left(f\right)=\gamma_R\left(G\right)$. □

The following result is an immediate consequence of Theorem 3.1 and Propositions 5.1.

Corollary 5.1. *Let G be a connected graph of order n with maximum degree $\Delta(G)$. Then the following statements are equivalent:*

i) $\gamma_{dR}(G)=2\gamma_R(G)-1$.
ii) $\gamma_{dR}(G)=2\left(n-\Delta(G)\right)+1$.
iii) $G\in\mathcal{F}$.

We note that if $\gamma_{dR}(G) = 2\gamma(G)+1$ and $\gamma_R(G) = \gamma(G)+1$, then $\gamma_{dR}(G) = 2\gamma_R(G)-1$. But the converse is not true as shown by the graph in Figure 2.

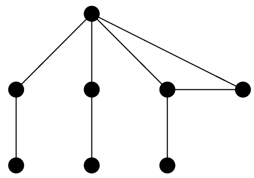

FIGURE 2. Graph with $\gamma(G) = 3$, $\gamma_R(G) = 5$ and $\gamma_{dR}(G) = 9$.

If one of the following equations $\gamma_{dR}(G) = 2\gamma(G)+1$ and $\gamma_R(G) = \gamma(G)+1$ is not hold, then clearly that $\gamma_{dR}(G) \neq 2\gamma_R(G)-1$.

Now in the class of trees, from the construction of Family $\mathcal{F}$, described above, we observe that wounded spiders are the only trees in $\mathcal{F}$, and by other hand wounded spiders are the only trees T for such that $\gamma_{dR}(T) = 2\gamma_R(T)-1$, or $\gamma_R(T) = \gamma(T)+1$, or $\gamma_{dR}(T) = 2\gamma(T)+1$, as shown by Zhang et al. [14], Cockayne et al. [6] and Ahangar et al. [1], respectively. The following result is an immediate consequence of Corollary 5.1.

Corollary 5.2. *Let T be a tree of order n with maximum degree $\Delta(T)$. Then the following statements are equivalent:*

- i) $\gamma_{dR}(T) = 2\gamma_R(T) - 1$.
- ii) $\gamma_{dR}(T) = 2\gamma(T) + 1$.
- iii) $\gamma_R(T) = \gamma(T) + 1$.
- iv) $\gamma_{dR}(T) = 2(n - \Delta(T)) + 1$.
- v) *T is a wounded spider.*

6. Conclusions

In this paper, we provided a characterization of extremal graphs of a Nordhaus-Gaddum bound for $\gamma_{dR}(G)$, improving the corresponding results given in [10] and [13]. Moreover, we gave a characterization of graphs G for which the equality $\gamma_{dR}(G) = 2\gamma_R(G) - 1$ holds, improving the corresponding results given in [14].

Acknowledgement. The authors are grateful to the anonymous referees for their valuable suggestions and useful comments.

References

[1] Ahangar, H. A., Chellali, M. and Sheikholeslami, S. M., (2017), On the double Roman domination in graphs. Discrete Applied Mathematics, 232, pp. 1-7.

[2] Beeler, R. A., Haynes, T. W. and Hedetniemi, S. T., (2016), Double Roman domination. Discrete Applied Mathematics, 211, pp. 23-29.

[3] Chellali, M., Jafari Rad, N., Sheikholeslami, S. M. and Volkmann, L., (2020), Roman domination in graphs, In: Topics in Domination in Graphs, Eds. Haynes T.W., Hedetniemi S.T. and Henning M.A., Springer, pp. 365–409.

[4] Chellali, M., Jafari Rad, N., Sheikholeslami, S. M. and Volkmann, L., (2021), Varieties of Roman domination, In: Structures of Domination in Graphs, Eds. Haynes T.W., Hedetniemi S.T. and Henning M.A., Springer, pp. 273–307.

[5] Chellali, M., Jafari Rad, N., Sheikholeslami, S. M. and Volkmann, L., (2020), Varieties of Roman domination II, AKCE Int. J. Graphs Comb. 17, pp. 966-984.

[6] Cockayne, E. J., Dreyer Jr, P. A., Hedetniemi, S. M. and Hedetniemi, S. T., (2004), Roman domination in graphs. Discrete mathematics, 278(1-3), pp. 11-22.

[7] Hao, G., Chen, X. and Volkmann, L., (2019), Double Roman domination in digraphs. Bulletin of the Malaysian Mathematical Sciences Society, 42, pp. 1907-1920.

[8] Haynes, T. W., Hedetniemi, S. T. and Slater, P. J., (1998), Fundamentals of domination in graphs Marcel Dekker. New York.

[9] Khoeilar, R., Karami, H., Chellali, M. and Sheikholeslami, S. M., (2019), An improved upper bound on the double Roman domination number of graphs with minimum degree at least two. Discrete Applied Mathematics, 270, pp. 159-167.

[10] Jafari Rad, N. and Rahbani, H., (2019), Some progress on the double Roman domination in graphs. Discussiones Mathematicae Graph Theory, 39(1), pp. 41-53.

[11] Poureidi, A., (2023), Double Roman domination in graphs: algorithmic complexity. Communications in Combinatorics and Optimization, 8(3), pp. 491-503.

[12] Rupnik Poklukar, D. and Žerovnik, J., (2023), Double Roman Domination: A Survey. Mathematics, 11(2), 351.

[13] Volkmann, L., (2023), Double Roman and Double Italian Domination. Discussiones Mathematicae Graph Theory 43(3), pp. 721-730

[14] Zhang, X., Li, Z., Jiang, H. and Shao, Z., (2018), Double Roman domination in trees. Information processing letters, 134, pp. 31-34.

Abdelhak Omar received his master's degree in 2018, and is currently pursuing the Ph.D degree in mathematics at Blida 1 University, Algeria. His area of interest includes Roman domination parameters of graphs, Optimization, Combinatorics and Discrete Mathematics.

Ahmed Bouchou received the Ph.D. degree in Mathematics from University of Blida 1, Algeria in 2015. He is currently Associate Professor in Department of Mathematics and Computer Science, University of Médéa, Algeria. He has published more than 14 papers in reputed journals and teaching experience of more than 20 years. His research interest includes domination parameters of graphs, Optimization, Combinatorics and Discrete Mathematics.