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FURTHER RESULTS ON PAIR MEAN CORDIAL GRAPHS

R. PONRAJ[1*], S. PRABHU[1], §

Abstract. Let a graph $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u)+\lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u)+\lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \leq 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called a pair mean cordial graph. In this paper, we examine the pair mean cordial labeling of some graphs including lily graph, torch graph, twig graph, triangular prism, parachute graph and diamond graph.

Keywords: lily graph, torch graph, twig graph, triangular prism, parachute graph, diamond graph.

AMS Subject Classification: 05C78

1. Introduction

In this paper, a simple, finite and undirected graph is represented by the pair $G = (V, E)$, where V and E are the sets of all vertices and edges respectively. The number of vertices and edges in G determine its order and size accordingly. We site [2] for a survey of graph labeling. Most graph labeling techniques trace their origin to one Rosa first proposed in [18] and I. Cahit introduced the idea of cordial labeling in [1]. We refer the book of Harary[3] for the definitions of the fundamental terminology in graph theory. Cordial associated labeling techniques has been studied in [4-11,17,19-23]. The pair mean cordial labeling behavior of various graphs has been studied in [12-16]. In this paper, we examine

[1] Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India.
e-mail: ponrajmaths@gmail.com; ORCID: https://orcid.org/0000-0001-7593-7429.
e-mail: selvaprabhu12@gmail.com; ORCID: https://orcid.org/ 0000-0003-3439-5330.
[*] Corresponding Author.
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the pair mean cordial labeling of some graphs including the lily graph, torch graph, twig graph, triangular prism, parachute graph and diamond graph.

2. Preliminaries

Definition 2.1. *Let a graph $G = (V, E)$ be a (p, q) graph. Define*

$$\rho = \begin{cases} \frac{p}{2} & p \text{ is even} \\ \frac{p-1}{2} & p \text{ is odd,} \end{cases}$$

and $M = \{\pm 1, \pm 2, \cdots \pm \rho\}$ called the set of labels. Consider a mapping $\lambda : V \to M$ by assigning different labels in M to the different elements of V when p is even and different labels in M to $p-1$ elements of V and repeating a label for the remaining one vertex when p is odd. The labeling as defined above is said to be a pair mean cordial labeling if for each edge uv of G, there exists a labeling $\frac{\lambda(u)+\lambda(v)}{2}$ if $\lambda(u) + \lambda(v)$ is even and $\frac{\lambda(u)+\lambda(v)+1}{2}$ if $\lambda(u) + \lambda(v)$ is odd such that $|\bar{\mathbb{S}}_{\lambda_1} - \bar{\mathbb{S}}_{\lambda_1^c}| \le 1$ where $\bar{\mathbb{S}}_{\lambda_1}$ and $\bar{\mathbb{S}}_{\lambda_1^c}$ respectively denote the number of edges labeled with 1 and the number of edges not labeled with 1. A graph G for which there exists a pair mean cordial labeling is called a pair mean cordial graph.

Definition 2.2. *The graph G^2 of an undirected graph G is another graph that has same set of vertices but in which two vertices are adjacent when their distance in G is at most 2.*

Definition 2.3. *A subdivision graph $S(G)$ is obtained from the graph G by subdividing each edge of G with a vertex.*

Definition 2.4. *The corona of two graph G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of H and attaching each i^{th} apex of G to every vertex in i^{th} copy of H.*

Definition 2.5. *The torch graph O_n, $n \ge 3$ is the graph with $V(O_n) = \{u_i : 1 \le i \le n+4\}$ and $E(O_n) = \{u_iu_{n+1}, u_iu_{n+3} : 2 \le i \le n-2\} \cup \{u_1u_i : n \le i \le n+4]\} \cup \{u_{n-1}u_n, u_nu_{n+2}, u_nu_{n+4}, u_{n+1}u_{n+3}\}$. Thus the torch graph O_n has $n+4$ vertices and $2n+3$ edges.*

Definition 2.6. *A lilly graph I_n can be constructed by two star graphs $2K_{1,n}$, $n \ge 2$ joining two path graphs $2P_n$, $n \ge 2$ with sharing a common vertex. That is $I_n = 2K_{1,n} + 2P_n$.*

Definition 2.7. *The twig graph TW_n, $n \ge 4$ is a tree obtained from a path by attaching exactly two pendant edges to each internal vertex of the path P_n.*

Definition 2.8. *The comb $P_n \odot K_1$ is obtained by joining a pendant edge to each vertices of the path P_n. It has $2n$ vertices and $2n-1$ edges*

Definition 2.9. *A diamond graph Br_n, $n \ge 3$ is defined by connection of a single vertex u to all other vertices u_i, $1 \le i \le n$ of triangular ladder graph TL_n.*

Definition 2.10. *The product graph of the path P_n and cycle C_3 is called the triangular prism and it is denoted by $P_n \times C_3$.*

Definition 2.11. *The parachute graph $P_{m,n}$, $m \ge 3$ is the graph with vertex set $V(P_{m,n}) = \{u, u_i, v_j : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E(P_{m,n}) = \{uu_i, u_1v_1, v_nu_m : 1 \le i \le m\} \cup \{u_iu_{i+1}, v_jv_{j+1} : 1 \le i \le m-1, 1 \le j \le n-1\}$. Obviously the parachute graph $P_{m,n}$ has $m+n+1$ vertices and $2m+n$ edges.*

3. Main Results

Theorem 3.1. *The comb graph $P_n \odot K_1$ is pair mean cordial [12].*

Theorem 3.2. *The star graph $K_{1,n}$ is pair mean cordial if and only if $1 \leq n \leq 6$ [12].*

Theorem 3.3. *The bistar graph $B_{m,n}$, $(m \geq 2, n \geq 2)$ is pair mean cordial if and only if $m + n \leq 9$ [12].*

Theorem 3.4. *The square graph of the comb, $(P_n \odot K_1)^2$ is not a pair mean cordial for all $n \geq 2$.*

Proof. Let $V((P_n \odot K_1)^2) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E((P_n \odot K_1)^2) = \{u_i u_{i+1}, u_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i u_{i+2}, v_i u_{i+2} : 1 \leq i \leq n-2\} \cup \{u_i v_i : 1 \leq i \leq n\}$. Clearly the number of vertices and edges of $(P_n \odot K_1)^2$ are $2n$ and $5n - 5$ respectively. Suppose $(P_n \odot K_1)^2$ is pair mean cordial. If the edge uv get the label 1, then the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is $2n - 3$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2n - 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - (2n - 3) = 3n - 2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3n - 2 - (2n + 3) = n + 1 \geq 4 > 1$, a contradiction. □

Theorem 3.5. *The square graph of $B_{n,n}$, $B_{n,n}^2$ is not a pair mean cordial for all $n \geq 1$.*

Proof. Define $V(B_{n,n}^2) = \{u, v, u_i, v_i : 1 \leq i \leq n\}$ and $E(B_{n,n}^2) = \{uv, uu_i, vv_i, vu_i, uv_i : 1 \leq i \leq n\}$. Obviously it has $2n + 2$ vertices and $4n + 1$ edges. We have the following three cases:

Case 1: $n = 1$

Now suppose that $B_{1,1}^2$ is pair mean cordial. If the edge uv get the label 1, then the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is 1. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 5 - 1 = 4$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3 > 1$, a contradiction.

Case 2: $n = 2$

Suppose $B_{2,2}^2$ is pair mean cordial. If the edge uv get the label 1, then the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is 3. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq 6 - 3 = 4$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3 > 1$, a contradiction.

Case 3: $n > 2$

Suppose that $B_{n,n}^2$ is pair mean cordial. If the edge uv get the label 1, then the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is 4. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 4$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 4 = 4n - 3$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 4n - 3 - 4 = 4n - 7 \geq 5 > 1$, a contradiction. □

Theorem 3.6. *The subdivision of edges of the star graph $K_{1,n}$, $S(K_{1,n})$ is pair mean cordial.*

Proof. The vertex set and edge set of $S(K_{1,n})$ respectively are defined by $V(S(K_{1,n})) = \{u, u_i, x_i : 1 \leq i \leq n\}$ and $E(S(K_{1,n})) = \{ux_i, x_i u_i : 1 \leq i \leq n\}$. It has $2n + 1$ vertices and $2n$ edges. Define $\lambda(u) = 1$. Now we assign the labels $-1, -2, \ldots, -n$ respectively to the vertices $u_1, u_2, \ldots, u_n$ and assign the labels $2, 3, \ldots, n$ to the vertices $x_1, x_2, \ldots, x_{n-1}$ respectively. Finally assign the label 1 to the vertex x_n. Hence the edges uv_n and $u_i v_i$, $1 \leq i \leq n-1$ are labeled with 1 and all other edges are labeled by the integers other than 1. Thus $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = n$. □

Example 3.1. *A pair mean cordial labeling of $S(K_{1,5})$ is shown in Figure 1.*

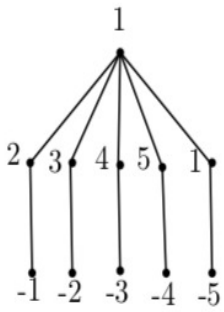

FIGURE 1

Theorem 3.7. *The subdivision of edges of the bistar graph $B_{n,n}$, $S(B_{n,n})$ is pair mean cordial.*

Proof. The vertex set and edge set of $S(B_{n,n})$ respectively are defined by $V(S(B_{n,n})) = \{u, v, w, u_i, x_i, v_i, y_i : 1 \leq i \leq n\}$ and $E(S(B_{n,n})) = \{uw, wv, ux_i, x_iu_i, vy_i, y_iv_i : 1 \leq i \leq n\}$. Then it has $4n + 3$ vertices and $4n + 2$ edges. Define $\lambda(u) = 1$, $\lambda(v) = -2n - 1$ and $\lambda(w) = 1$. Next we assign the labels $-1, -2, \ldots, -n$ respectively to the vertices $u_1, u_2, \ldots, u_n$ and assign the labels $2, 3, \ldots, n+1$ to the vertices $x_1, x_2, \ldots, x_n$ respectively. We now assign the labels $-n-1, -n-2, \ldots, -2n$ respectively to the vertices $v_1, v_2, \ldots, v_n$ and assign the labels $n + 2, n + 3, \ldots, 2n + 1$ to the vertices $y_1, y_2, \ldots, y_n$ respectively. Hence the edges u_ix_i, y_iv_i, $1 \leq i \leq n$ and uv are labeled with 1 and the other edges are not labeled with 1. Thus $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = 2n + 1$. □

Example 3.2. *A pair mean cordial labeling of $S(B_{4,4})$ is shown in Figure 2.*

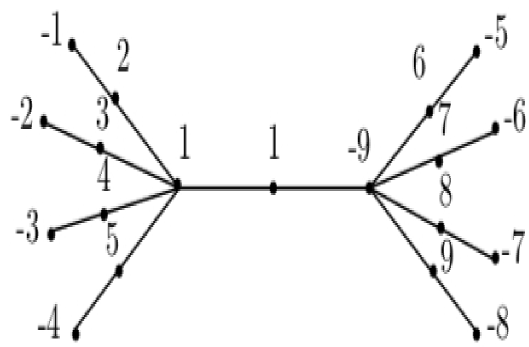

FIGURE 2

Theorem 3.8. *The torch graph O_n is pair mean cordial for all $3 \leq n \leq 7$ and $n = 9$.*

Proof. Let us take the vertex set and edge set from definition 2.5. We have the following two cases:

Case 1: $3 \leq n \leq 7$

Now we define $\lambda(v_{n-1}) = 1$, $\lambda(v_n) = -1$, $\lambda(v_{n+2}) = 3$, $\lambda(v_{n+4}) = 2$ and $\lambda(v_1) = -2$. Next we have to assign labels to the remaining vertices from the following table:

Nature of n	$\lambda(v_{n+3})$	$\lambda(v_{n+1})$	$\lambda(v_2)$	$\lambda(v_3)$	$\lambda(v_4)$	$\lambda(v_5)$
$n=3$	3	-3				
$n=4$	4	-4	-3			
$n=5$	4	-4	-3	-3		
$n=6$	4	5	-3	-4	-5	
$n=7$	4	5	-3	-4	-5	-3

Table 1

Case 2: $n > 7$
Then there are two cases arises:
Subcase 1: n is even
Then $n \geq 8$. Now suppose O_n is pair mean cordial. If the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is 8. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 8$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 8 = 2n - 5$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - 5 - (8) = 2n - 13 \geq 3 > 1$, a contradiction.
Subcase 2: n is odd
If $n = 9$, then we assign the labels $-2, -3, -4, -5, -6, -4, 1, 2, -1$ respectively to the vertices $v_1, v_2, \ldots, v_9$ and assign the labels $6, 4, 5, 3$ to the vertices $v_{10}, v_{11}, v_{12}, v_{13}$ respectively. If $n > 9$, then $n \geq 11$. Suppose O_n is pair mean cordial. If the edge uv get the label 1, the possibilities are $\lambda(u) + \lambda(v) = 1$ or $\lambda(u) + \lambda(v) = 2$. Hence the maximum number of edges with label 1 is 10. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 10$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q - 10 = 2n - 7$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 2n - 7 - (10) = 2n - 17 \geq 5 > 1$, a contradiction.

□

Example 3.3. *A pair mean cordial labeling of O_7 is shown in Figure 3.*

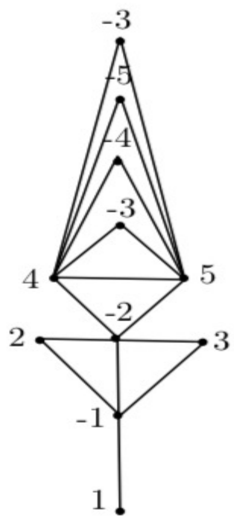

FIGURE 3

Theorem 3.9. *The lilly graph I_n is pair mean cordial for all $n \geq 2$.*

Proof. Let $V(I_n) = \{u_i : 1 \leq i \leq 2n-1\} \cup \{v_i : 1 \leq i \leq 2n\}$ and $E(I_n) = \{u_n v_i : 1 \leq i \leq 2n\} \cup \{u_i u_{i+1} : 1 \leq i \leq 2n-2\}$. Then I_n has $4n-1$ vertices and $4n-2$ edges. Let us assign the labels $2, 3, \dots, n+1$ to the vertices $u_1, u_3, \dots, u_{2n-1}$ respectively. Next we assign the labels $-1, -2, \dots, -n+1$ respectively to the vertices $u_2, u_4, \dots, u_{2n-2}$. We assign the labels $-n, -n-1, \dots, -2n+1$ to the vertices $v_1, v_2, \dots, v_n$ respectively. We assign the labels $n+2, n+3, \dots, 2n-1$ respectively to the vertices $v_{n+1}, v_{n+2}, \dots, v_{2n-2}$. Furthermore assign the label 1 to the vertex v_{2n-1}. Finally if n is odd , assign the label $\frac{-n-1}{2}$ to the vertex v_{2n} and if n is even, assign the label $\frac{n+2}{2}$ to the vertex v_{2n}. Hence the edges $u_i u_{i+1}$, $1 \leq i \leq 2n-2$ and $u_n v_{2n}$ are labeled with 1 and all other edges are labeled by the integers other than 1. Thus $\bar{\mathbb{S}}_{\lambda_1} = 2n-1 = \bar{\mathbb{S}}_{\lambda_1^c}$.

□

Example 3.4. *A pair mean cordial labeling of I_5 is shown in Figure 4.*

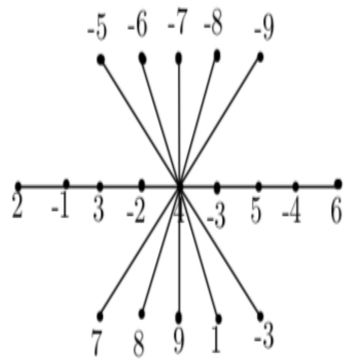

FIGURE 4

Theorem 3.10. *The twig graph TW_n is pair mean cordial for all $n \geq 4$.*

Proof. Define $V(TW_n) = \{u_i, v_j, w_j : 1 \leq i \leq n, 1 \leq j \leq n-2\}$ and $E(TW_n) = \{u_{i+1}v_i, u_{i+1}w_i, u_j u_{j+1} : 1 \leq i \leq n-2, 1 \leq j \leq n-1\}$. Then the twig graph TW_n has $3n-4$ vertices and $3n-5$ edges. We have the following two cases:

Case 1: n is odd

Let $\lambda(u_1) = 1$. Assign the labels $-1, -4, \dots, \frac{-3n+7}{2}$ to the vertices $u_2, u_4, \dots, u_{n-1}$ respectively and assign the labels $-3, -6, \dots, \frac{-3n+9}{2}$ to the vertices $u_3, u_5, \dots, u_{n-2}$ respectively. Then we assign the label $\frac{-3n+5}{2}$ to the vertex u_n. Now we assign the labels $2, 5, \dots, \frac{3n-5}{2}$ to the vertices $v_1, v_3, \dots, v_{n-2}$ respectively and assign the labels $4, 7, \dots, \frac{3n-7}{2}$ respectively to the vertices $v_2, v_4, \dots, v_{n-3}$. Next we assign the labels $3, 6, \dots, \frac{3n-9}{2}$ to the vertices $w_1, w_3, \dots, w_{n-4}$ respectively and assign the labels $-2, -5, \dots, \frac{-3n+11}{2}$ respectively to the vertices $w_2, w_4, \dots, w_{n-3}$. Finally assign the label $\frac{3n-5}{2}$ to the vertex w_{n-2}. Hence the edges $u_{i+1}v_i$, $1 \leq i \leq n-1$ and $u_{2j}w_{2j-1}$, $1 \leq j \leq \frac{n-1}{2}$ are labeled with 1 and the other edges are not labeled with 1.

Case 2: n is even

Let $\lambda(u_1) = 1$. Now we give the labels $-1, -4, \dots, \frac{-3n+4}{2}$ to the vertices $u_2, u_4, \dots, u_n$ respectively and give the labels $-3, -6, \dots, \frac{-3n+6}{2}$ to the vertices $u_3, u_5, \dots, u_{n-1}$ respectively. Next we give the labels $2, 5, \dots, \frac{3n-8}{2}$ to the vertices $v_1, v_3, \dots, v_{n-3}$ respectively and give the labels $4, 7, \dots, \frac{3n-4}{2}$ respectively to the vertices $v_2, v_4, \dots, v_{n-2}$. Furthermore we give the labels $3, 6, \dots, \frac{3n-6}{2}$ to the vertices $w_1, w_3, \dots, w_{n-3}$ respectively and give the

labels $-2, -5, \ldots, \frac{-3n+8}{2}$ respectively to the vertices $w_2, w_4, \ldots, w_{n-2}$. Hence the edges $u_{i+1}v_i$, $1 \le i \le n-2$ and $u_{2j}w_{2j-1}$, $1 \le j \le \frac{n-2}{2}$ are labeled with 1 and the other edges are not labeled with 1.
The following table given that this vertex labeling λ is a pair mean cordial of TW_n for all $n \ge 4$.

□

Nature of n	$\mathbb{S}_{\lambda_1}$	$\mathbb{S}_{\lambda_1^c}$
n is odd	$\frac{3n-5}{2}$	$\frac{3n-5}{2}$
n is even	$\frac{3n-6}{2}$	$\frac{3n-4}{2}$

Table 2

Example 3.5. *A pair mean cordial labeling of TW_9 is shown in Figure 5.*

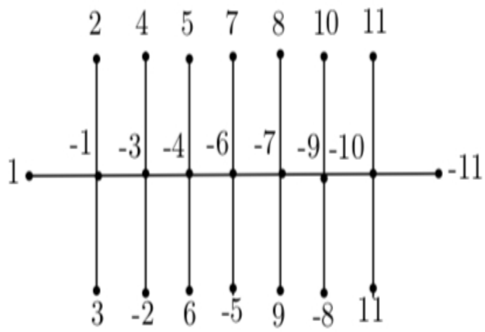

FIGURE 5

Theorem 3.11. *The triangular prism $P_n \times C_3$ is pair mean cordial for all $n \ge 1$ and n is odd.*

Proof. The vertex set and edge set of triangular prism $P_n \times C_3$ respectively are given by $V(P_n \times C_3) = \{u_i, v_i, w_i : 1 \le i \le n\}$ and $E(P_n \times C_3) = \{u_iw_i, v_iw_i, v_iu_i : 1 \le i \le n\} \cup \{u_iu_{i+1}, v_iv_{i+1}, w_iw_{i+1} : 1 \le i \le n-1\}$. Clearly it has $3n$ vertices and $6n-3$ edges. We have the following two cases:
Case 1: n is odd
We assign the labels $-1, -4, \ldots, \frac{-3n+1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_n$ and also assign the labels $3, 6, \ldots, \frac{3n-3}{2}$ to the vertices $v_2, v_4, \ldots, v_{n-1}$ respectively. Then we assign the label 1 to the vertex w_1. Next we assign the labels $4, 7, \ldots, \frac{3n-1}{2}$ respectively to the vertices $w_2, w_4, \ldots, w_{n-1}$ and assign the labels $-3, -6, \ldots, \frac{-3n+3}{2}$ to the vertices $w_3, w_5, \ldots, w_n$ respectively. We now assign the labels $2, 5, \ldots, \frac{3n-5}{2}$ respectively to the vertices $u_1, u_3, \ldots, u_{n-2}$ and assign the labels $-2, -5, \ldots, \frac{-3n+5}{2}$ to the vertices $u_2, u_4, \ldots, u_{n-1}$ respectively. Finally assign the label $\frac{3n-1}{2}$ to the vertex u_n. Hence the edges v_iu_i, $w_{i+1}u_{i+1}$, $1 \le i \le n-1$, $v_{2j-1}v_{2j}$, $w_{2j}w_{2j+1}$, $1 \le j \le \frac{n-1}{2}$ and $u_{n-1}u_n$ are labeled with 1 and all other edges are labeled by the integers other than 1. Thus $\bar{\mathbb{S}}_{\lambda_1} = 3n-2$ and $\bar{\mathbb{S}}_{\lambda_1^c} = 3n-1$.
Case 2: n is even

Suppose that $P_n \times C_3$ is pair mean cordial. If the edge uv get the label 1, then the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges with label 1 is $3n-3$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 3n-3$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(3n-3)=3n$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 3n-(3n-3)=3>1$, a contradiction. □

Example 3.6. *A pair mean cordial labeling of $P_5 \times C_3$ is shown in Figure 6.*

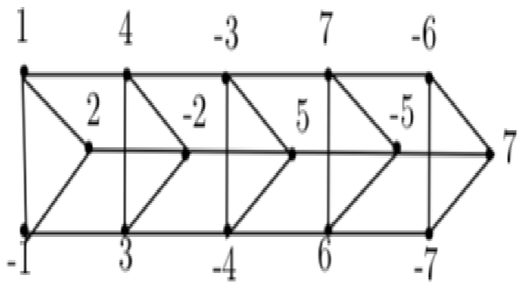

FIGURE 6

Theorem 3.12. *The parachute graph $P_{m,1}$ is pair mean cordial for all odd $m \geq 5$.*

Proof. Let $V(P_{m,1})=\{u,u_i,v_1 : 1 \leq i \leq m\}$ and $E(P_{m,1})=\{uu_i,u_1v_1,v_1u_m : 1 \leq i \leq m\} \cup \{u_iu_{i+1} : 1 \leq i \leq m-1\}$. Then the number of vertices and edges of $P_{m,1}$ respectively are $m+2$ and $2m+1$. The proof is divided into two cases:

Case 1: m is odd

There are two cases arises:

Subcase 1: $m=3$

Let us assume that $P_{3,1}$ is pair mean cordial. Thus if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges with label 1 is 2. That is $\bar{\mathbb{S}}_{\lambda_1} \leq 2$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-2=5$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq 5-2=3>1$, a contradiction.

Subcase 2: $m \geq 5$

First we define $\lambda(u)=3$ and $\lambda(v_1)=1$. Next we assign the labels $2,3,\ldots,\frac{m+1}{2}$ to the vertices $u_1,u_3,\ldots,u_{m-2}$ respectively and assign the labels $-1,-2,\ldots,\frac{-m+1}{2}$ to the vertices $u_2,u_4,\ldots,u_{m-1}$ respectively. Finally assign the label $\frac{-m-1}{2}$ to the vertex u_m. Hence the edges uu_2, uu_4 and u_iu_{i+1}, $1 \leq i \leq m-2$ are labeled with 1 and the other edges are not labeled with 1. Therefore $\bar{\mathbb{S}}_{\lambda_1}=m$ and $\bar{\mathbb{S}}_{\lambda_1^c}=m+1$.

Case 2: m is even

In this case, Suppose $P_{m,1}$ is pair mean cordial. Thus if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges with label 1 is $m-1$. That is $\bar{\mathbb{S}}_{\lambda_1} \leq m-1$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-(m-1)=m+2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m+2-(m-1)=3>1$, a contradiction.

□

Theorem 3.13. *The parachute graph $P_{m,2}$ is pair mean cordial for all even $m \geq 4$.*

Proof. Let $V(P_{m,2})=\{u,u_i,v_1,v_2 : 1 \leq i \leq m\}$ and $E(P_{m,2})=\{uu_i,u_1v_1,v_1v_2, v_2u_m : 1 \leq i \leq m\} \cup \{u_iu_{i+1} : 1 \leq i \leq m-1\}$. Then the number of vertices and edges of

$P_{m,2}$ respectively are $m+3$ and $2m+2$. We have the following two cases:
Case 1: m is odd
In this case, Suppose $P_{m,1}$ is pair mean cordial. Thus if the edge uv get the label 1, the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges with label 1 is m. That is $\bar{\mathbb{S}}_{\lambda_1} \leq m$. Then $\bar{\mathbb{S}}_{\lambda_1^c} \geq q-m = m+2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c} - \bar{\mathbb{S}}_{\lambda_1} \geq m+2-m = 2 > 1$, a contradiction.
Case 2: m is even
Define $\lambda(u)=3$, $\lambda(v_1)=\frac{-m-2}{2}$ and $\lambda(v_2)=1$. Next we assign the labels $2,3,\ldots,\frac{m+2}{2}$ to the vertices $u_1,u_3,\ldots,u_{m-1}$ respectively and assign the labels $-1,-2,\ldots,\frac{-m}{2}$ to the vertices $u_2,u_4,\ldots,u_m$ respectively. Hence the edges uu_2, uu_4 and u_iu_{i+1}, $1\leq i\leq m-1$ are labeled with 1 and all other edges are labeled by the integers other than 1. Therefore $\bar{\mathbb{S}}_{\lambda_1} = \bar{\mathbb{S}}_{\lambda_1^c} = m+1$.

□

Theorem 3.14. *The diamond graph Br_n is not a pair mean cordial for all $n \geq 3$.*

Proof. The vertex set and edge set of diamond graph Br_n respectively are given by $V(Br_n)=\{u,u_i,v_j : 1\leq i\leq n, 1\leq j\leq n-1\}$ and $E(Br_n)=\{uu_i : 1\leq i\leq n\}\cup\{u_iu_{i+1},u_iv_i,v_iu_{i+1} : 1\leq i\leq n-1\}\cup\{v_iv_{i+1} : 1\leq i\leq n-2\}$. Clearly it has $2n$ vertices and $5n-5$ edges. Now suppose that Br_n is pair mean cordial. If the edge uv get the label 1, then the possibilities are $\lambda(u)+\lambda(v)=1$ or $\lambda(u)+\lambda(v)=2$. Hence the maximum number of edges with label 1 is $2n-3$. That is $\bar{\mathbb{S}}_{\lambda_1}\leq 2n-3$. Then $\bar{\mathbb{S}}_{\lambda_1^c}\geq q-(2n-3)=3n-2$. Therefore $\bar{\mathbb{S}}_{\lambda_1^c}-\bar{\mathbb{S}}_{\lambda_1}\geq 3n-2-(2n-3)=n+1\geq 4>1$, a contradiction.

□

Theorem 3.15. *The parachute graph $P_{m,n}$ is pair mean cordial for all $m,n\geq 3$.*

Proof. The vertex set and edge set are defined in definition 2.10. We have the following two cases:
Case 1: m is even
There are two cases arises:
Subcase 1: n is even
Let us define $\lambda(u)=2$. Assign the labels $-1,-2,\ldots,\frac{-m}{2}$ respectively to the vertices $u_1,u_3,\ldots,u_{m-1}$ and also assign the labels $3,4,\ldots,\frac{m+4}{2}$ to the vertices $u_2,u_4,\ldots,u_m$ respectively. Now we assign the labels $\frac{-m-4}{2},\frac{-m-6}{2},\ldots,\frac{-m-n}{2}$ respectively to the vertices $v_1,v_3,\ldots,v_{n-3}$ and assign the labels $\frac{m+6}{2},\frac{m+8}{2},\ldots,\frac{m+n}{2}$ respectively to the vertices $v_2,v_4,\ldots,v_{n-4}$. Next we assign the labels $1,1,\frac{-m-2}{2}$ to the vertices v_{n-2},v_{n-1},v_n respectively. Hence the edges uu_1, u_iu_{i+1}, $1\leq i\leq m-1$, u_mv_n and $v_{2j-1}v_{2j}$, $1\leq j\leq\frac{n-2}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1.
Subcase 2: n is odd
First we assign the labels to the vertices u,u_i as in Subcase 1 of Case 1. Next we assign the labels $\frac{-m-4}{2},\frac{-m-6}{2},\ldots,\frac{-m-n-1}{2}$ respectively to the vertices $v_1,v_3,\ldots,v_{n-2}$ and assign the labels $\frac{m+6}{2},\frac{m+8}{2},\ldots,\frac{m+n+1}{2}$ respectively to the vertices $v_2,v_4,\ldots,v_{n-3}$. Next we assign the labels $1,\frac{-m-2}{2}$ to the vertices v_{n-1},v_n respectively. Hence the edges uu_1, u_iu_{i+1}, $1\leq i\leq m-1$, u_mv_n and $v_{2j-1}v_{2j}$, $1\leq j\leq\frac{n-3}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1.
Case 2: m is odd
There are two cases arises:
Subcase 1: n is even
Now define $\lambda(u)=2$. Assign the labels $-1,-2,\ldots,\frac{-m-1}{2}$ respectively to the vertices

$u_1, u_3, \ldots, u_m$ and also assign the labels $3, 4, \ldots, \frac{m+3}{2}$ to the vertices $u_2, u_4, \ldots, u_{m-1}$ respectively. Then we assign the labels $\frac{-m-5}{2}, \frac{-m-7}{2}, \ldots, \frac{-m-n-1}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-3}$ and assign the labels $\frac{m+7}{2}, \frac{m+9}{2}, \ldots, \frac{m+n+1}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_{n-4}$. Next we assign the labels $1, \frac{-m-3}{2}, \frac{m+5}{2}$ to the vertices v_{n-2}, v_{n-1}, v_n respectively. Hence the edges uu_1, u_iu_{i+1}, $1 \leq i \leq m-1$, u_mv_n, $v_{2j-1}v_{2j}$, $1 \leq j \leq \frac{n-4}{2}$ and $v_{n-1}v_n$ are labeled with 1 and all other edges are labeled by the integers other than 1.

Subcase 2: n is odd

Assign the labels to the vertices u, u_i as in Subcase 1 of Case 2. Next we assign the labels $\frac{-m-3}{2}, \frac{-m-5}{2}, \ldots, \frac{-m-n}{2}$ respectively to the vertices $v_1, v_3, \ldots, v_{n-2}$ and assign the labels $\frac{m+5}{2}, \frac{m+7}{2}, \ldots, \frac{m+n}{2}$ respectively to the vertices $v_2, v_4, \ldots, v_{n-3}$. Next we assign the labels $1, 1$ to the vertices v_{n-1}, v_n respectively. Hence the edges uu_1, u_iu_{i+1}, $1 \leq i \leq m-1$, and $v_{2j-1}v_{2j}$, $1 \leq j \leq \frac{n-1}{2}$ are labeled with 1 and all other edges are labeled by the integers other than 1.

The following table given that this vertex labeling λ is a pair mean cordial of $P_{m,n}$ for all $m, n \geq 3$:

□

Nature of n	$\bar{\mathbb{S}}_{\lambda_1}$	$\bar{\mathbb{S}}_{\lambda_1^c}$
m and n are both even	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
m is even and n is odd	$\frac{2m+n-1}{2}$	$\frac{2m+n+1}{2}$
m is odd and n is even	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
m and n are both odd	$\frac{2m+n-1}{2}$	$\frac{2m+n+1}{2}$

Table 3

Example 3.7. *A pair mean cordial labeling of $P_{6,5}$ is shown in Figure 7.*

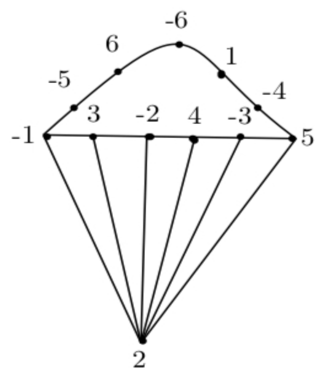

FIGURE 7

4. CONCLUSION

In this paper, we have studied about the pair mean cordial labeling of some graphs including lily graph, torch graph, twig graph, triangular prism, parachute graph and diamond graph. The open problems concern the investigation of the pair mean cordiality of some other special graphs.

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R. Ponraj did his Ph.D in Manonmaniam Sundaranar University, Tirunelveli, India. He has guided 11 Ph.D scholars and published around 175 research papers in reputed journals. He is an author of eight books for undergraduate students. His research interest is in Graph Theory. He is currently an Associate Professor at Sri Paramakalyani College, Alwarkurichi, India.

S. Prabhu did his M.Phil in Madurai Kamaraj University, Madurai, India. He is currently a research scholar (Register number: 21121232091003) in the Department of Mathematics, Sri Paramakalyani College, Alwarkurichi, India (affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli, India). His research interest is in Graph Theory. He has published five papers in journals.