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NON-INTERSECTION POWER GRAPHS AND CO-PRIME GRAPHS OF FINITE GROUPS

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Abstract. In this paper, we define the non-intersection power graph of a finite group G as a graph whose vertex set is G, and edge set consists of unordered pairs $\{u, v\}$ of vertices such that $\langle u \rangle \cap \langle v \rangle = \{e\}$. We find some structural properties, planarity and independence number of non-intersection power graphs of finite groups. We classify all groups whose non-intersection power graph and co-prime graph are identical. Also we calculate some topological indices such as Wiener index, Harary index and Zagreb index of co-prime graphs of some groups.

Keywords: Intersection graphs, Power graphs, Co-prime graph, Wiener index, Harary index, Zagreb index.

AMS Subject Classification: 05C25

1. Introduction

Graphs associated with groups are powerful tools for understanding the structural properties and relationships within a group. These graphs provide a visual representation of group elements and their interactions, shedding light on the group's symmetries, connectivity, and other important characteristics. Defining graphs with vertex set the set of all elements of a finite group have been an interesting research topic for the last few years. Commuting graphs, power graphs, co-prime graphs and enhanced power graphs are some examples of such graphs. Adjacency of the vertices in these graphs differ accordingly as the name indicates. In the power graph, two vertices are adjacent if one is a power of the other, whereas in enhanced power graph two vertices are adjacent if they belong to the same cyclic group. Two vertices in the commuting graph are adjacent if they commute. Since the graphs defined on groups have many useful applications and are related to automata theory, the researchers are enthusiastic to define new graphs on groups and find their graph parameters in group aspects.

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For a semigroup, the directed power graph was first defined by Kelarev and Quinn in 2000 [3], and then in 2009 Chakrabarty et al. presented the undirected power graph of a group [2]. Since then, almost every graph parameters such as connectedness, number of edges, planarity, independence number, chromatic number, traversability, clique number and automorphism are studied by various researchers. The enhanced power graph of a group was introduced by Aalipour et al.[11]. For a finite group G, the power graph of G is a spanning subgraph of the enhanced power graph. The authors in [11] characterized the groups whose power graphs and enhanced power graphs coincide.

For a finite group G, in 2014, Ma et al. [1] presented the co-prime graph of G, as an undirected simple graph whose vertex set is G and edge set consists of set of all unordered pairs $\{u, v\}$ such that $gcd(o(u), o(v)) = 1$, where $o(u)$ and $o(v)$ denotes the orders of u and v respectively. They investigated some graph properties such as diameter, planarity, clique number and automorphisms of co-prime graphs. In [14], the authors calculated the number of edges in co-prime graphs of dihedral groups as well as cyclic groups. The authors in [15] proved that for any finite group, the chromatic number and the clique number of the coprime graph are equal. Additionally, all finite groups with complete r-partite or planar co-prime graphs are categorised. Co-prime graphs of cyclic groups are examined by the authors of [16]. In [17], the co-prime graphs of generalised quaternion groups are investigated. The Laplacian eigen values of dihedral groups and cyclic groups are studied in [18]. In [19], the authors classified groups whose coprime graphs are toroidal and projective. In [20], the calculation of the independence number of coprime graphs for dihedral groups is presented, along with the characterization of groups for which the coprime graph is perfect.

The intersection power graph of a finite group G was defined by Sudip Bera in 2018 [6]. The author defined the graph with vertex set G in which e is adjacent to every other vertices, where e is the identity in G and two non-identity elements u and v are adjacent if and only if $\langle u\rangle \cap \langle v\rangle \neq \{e\}$. The author proved that the intersection power graph is complete if and only if either G is a generalized quaternion group or cyclic p-group. Also, the author studied dominatability as well as automorphisms of intersection power graphs of some groups. In [13] the authors characterize the independence number of the intersection power graph of a group, finite groups with dominatable intersection power graphs and the finite groups whose intersection power graphs equal to their power graphs, enhanced power graphs, commuting graphs, and order supergraphs. In [21] the authors determined some finite groups whose intersection power graphs having book thickness at most two. Certain finite groups whose intersection power graphs can be embedded on the torus or projective plane are classified in [22]. A study on proper intersection power graphs containing perfect code is conducted in [23].

Motivated by the definition of intersection power graphs, in this paper we define non-intersection power graph of finite group G. We study some structural properties of non-intersection power graphs such as completeness, clique number and planarity, and also characterize the finite groups whose non-intersection power graphs and co-prime graphs coincide.

Topological indices of graphs are numerical measures used to describe the topological structure or properties of graphs. They play a crucial role in graph theory and have many applications in various scientific disciplines such as chemistry, biology, computer science, and network analysis. There are several types of topological indices, each of them focus on different aspects of the graph's structure. Some examples include Wiener index, Zagreb index and Harary index.

We refer to [12] for basic definitions and concepts in graph theory. For an undirected simple graph Γ, we denote the vertex set by $V(\Gamma)$ and edge set by $E(\Gamma)$. We denote $u \sim v$ if $\{u, v\}$ is an edge in Γ. The degree of a vertex u denoted by $deg(u)$. The distance between two vertices u and v, denoted by $d(u, v)$, is the smallest length of any $u - v$ path in Γ. Union of two graphs $\Gamma_1(V_1, E_1)$ and $\Gamma_2(V_2, E_2)$ is a graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, which is denoted by $\Gamma_1 \cup \Gamma_2$. The join $\Gamma_1 \vee \Gamma_2$ of two vertex disjoint graphs G_1 and G_2 is the graph union $\Gamma_1 \cup \Gamma_2$ together with all the edges joining V_1 and V_2.

Throughout this paper, we consider only finite groups. The order of the group G is denoted by $|G|$ which is the cardinality of the group. The order of an element u in G is denoted by $o(u)$. We adopt the notation e to represent the identity element in G. The generalized quaternion groups are the groups given by the representation $Q_{4n} = \langle u, v : u^n = v^2, u^{2n} = e, v^{-1}uv = u^{-1} \rangle$. Note that Q_{4n} is a p-group ($p = 2$) when $n = 2^k$ for some integer k.

2. Definition and some Properties

We begin with the definition of non-intersection power graphs of finite groups.

Definition 2.1. *The non-intersection power graph $NIP(G)$ of a finite group G is a graph whose vertex set is G and edge set consists of set of all two element subsets $\{u, v\}$ of G such that $\langle u \rangle \cap \langle v \rangle = \{e\}$.*

The non-intersection power graphs of $\mathbb{Z}_6$ and S_3 are given in Figure 1 and Figure 2 respectively.

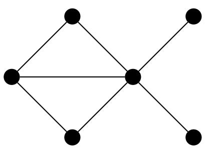

Figure 1 :$NIP(\mathbb{Z}_6)$

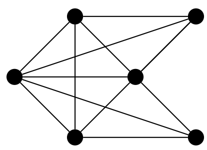

Figure 2 :$NIP(S_3)$

Remark 2.1. *Since every non-identity elements are adjacent to the identity element e, $NIP(G)$ is always connected.*

The very next question is on the completeness. The following theorem gives a characterisation to the completeness of $NIP(G)$.

Theorem 2.1. *For a finite group G, $NIP(G)$ is complete if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.*

Proof. If $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, for every $u \in G$, $\langle u \rangle = \{e, u\}$, then $u \sim v$ for all $v \in G \setminus \{u\}$. Hence, $NIP(G)$ is complete.

Conversely, suppose $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, then there exists an element in G such that $o(u) > 2$ and $u \nsim v$ for all $v \in \langle u \rangle \setminus \{e\}$. □

The following theorem gives a characterization for $NIP(G)$ to be a star graph.

Theorem 2.2. *Let G be a finite group. Then $NIP(G)$ is a star graph if and only if G is either a generalized quaternion 2-group or G is a cyclic p-group.*

Proof. First suppose that $NIP(G)$ is a star graph. If G is not a p-group, then there exists two distinct primes p and q such that p and q divides $|G|$. Then, there are elements $u, v \in G$ such that $o(u) = p$ and $o(v) = q$, and hence $\langle u\rangle \cap \langle v\rangle = \{e\}$. Hence, $u \sim v$ and u, v, e is a cycle in $NIP(G)$, which is a contradiction. Therefore, G is a p-group. Now we show that the subgroup of order p in G is unique . If possible, suppose G_1 and G_2 are subgroups of G such that $G_1 \neq G_2$ and $\mid G_1 \mid=\mid G_2 \mid= p$. Then, $u_1 \sim u_2$ if $u_1 \in G_1$ and $u_2 \in G_2$, and u_1, u_2, e is a cycle in $NIP(G)$, which is not possible. Hence, the subgroup of order p in G is unique , and hence G is either cyclic or generalized quaternion group.

For the converse, suppose G is either a cyclic p-group or a generalized quaternion group. Then G has a unique subgroup of order p [$p = 2$ in case of genaralized quaternion group], say H. If $u, v \in G$, $u \neq v \neq e$, then $o(u) = p^{k_1}$, and $o(v) = p^{k_2}$ for some $k_1, k_2 \in \mathbf{N}$, and $H \subseteq \langle u\rangle$ and $H \subseteq \langle v\rangle$, that implies $\langle u\rangle \cap \langle v\rangle \neq \{e\}$ and hence u and v are not adjacent in $NIP(G)$. □

A vertex in a graph is said to be a dominating vertex if it is adjacent to all of other vertices in the graph.

Proposition 2.1. *A vertex $u \in NIP(G)$ is a dominating vertex if either $u = e$ or u is a maximal involution.*

Proof. If $u \in G$, then u is not adjacent to any vertex in $\langle u\rangle$. Therefore, if $o(u) \geq 3$, then u is not adjacent to at least one other element in G, which implies u is not a dominating vertex. Therefore, if u is a dominating vertex, then $o(u) \leq 2$.

Clearly, e is a dominating vertex. Now suppose that u is a maximal involution in G, that is $\langle u\rangle = \{e, u\}$ is a maximal cyclic subgroup of G. Hence, for every element $v \in G, v \neq u$, $\langle u\rangle \cap \langle v\rangle = \{e\}$, which implies u is adjacent to every other vertex in G. □

A nontrivial connected graph is Eulerian if and only if every vertex has even degree. We use this result to prove the following theorem. Let $\pi(G)$ denote the set of all prime divisors of $|G|$ and ϕ denote the Euler totient function.

Theorem 2.3. *The non-intersection power graph $NIP(G)$ is not Eulerian for any G.*

Proof. To be Eulerian, $|G|$ should be odd since e is adjacent to every other vertices. Let $|G|$ be odd and $u \in G$ such that $u \neq e$, then $o(u) \geq 3$. Let u is adjacent to v for some $v \in G$, then u is adjacent to every generator of $\langle v\rangle$. Since $o(v) \geq 3$, $\phi(o(v))$ is even, also u is adjacent to e, which implies $deg(u)$ is odd. □

Let $\mathcal{M}$ be the set of all minimal subgroups of G. Note that the order of a minimal subgroup of a finite group is prime.

Theorem 2.4. *For a finite group G, the clique number, $\omega(NIP(G)) = |\mathcal{M}| + 1$.*

Proof. Let $\mathcal{M} = \{M_1, M_2, ..., M_k\}$. Take $u_i \in M_i$ such that $\langle u_i\rangle = M_i$. Then $\langle u_i\rangle \cap \langle u_j\rangle = \{e\}, \forall i, j = 1, 2, ..., k$ and $i \neq j$, since if $u \in \langle u_i\rangle \cap \langle u_j\rangle$ such that $u \neq e$, then $\langle u\rangle \subseteq \langle u_i\rangle \cap \langle u_j\rangle$ and therefore $\langle u\rangle = \langle u_i\rangle = \langle u_j\rangle$ which is not possible. Hence $\{u_1, u_2, ..., u_k, e\}$ induces a complete graph. Therefore $\omega(NIP(G)) \geq k + 1$.

Now, if $k + 2$ vertices say $v_1, v_2, ..., v_{k+1}, e$ induce a complete graph in $NIP(G)$, then $\langle v_i\rangle \cap \langle v_j\rangle = \{e\}, \forall i, j = 1, 2, ..., k + 1$, which implies there are $k + 1$ distinct cyclic subgroups in G and hence there are more than k minimal subgroups in G, which is not possible. Therefore, $\omega(NIP(G)) \leq k + 1$. □

Next theorem classifies all groups whose non-intersection power graph is planar.

Theorem 2.5. *Let G be a finite group such that $|G| > 6$, then $NIP(G)$ is planar if and only if G is a cyclic group of order p^n or $2p^n$, where p is a prime and n is a positive integer or G is a generalized quaternion 2-group.*

Proof. First we prove that if $NIP(G)$ is planar then $|\pi(G)| \leq 2$. Let $|\pi(G)| \geq 3$, and let $p, q, r \in \pi(G)$ such that $p < q < r$. Then clearly $\phi(r) > 3$ and $\phi(q) \geq 2$. Take $u_1, u_2, u_3, v_1, v_2 \in G$ such that $o(u_i) = r$, $i = 1, 2, 3$ and $o(v_1) = o(v_2) = q$, then the subgraph induced by the vertex set $\{u_1, u_2, u_3, v_1, v_2, e\}$ contains a $K_{3,3}$. Hence, $|\pi(G)| \leq 2$ and if $|\pi(G)| = 2$ and $\pi(G) = \{p, q\}$ such that $p < q$ then $p = 2$.

Now suppose $|G| = 2^m p^n$, $n \neq 0$ and p is an odd prime. If $m > 1$, then there are 2^m elements in G of orders 2^i, $1 \leq i \leq m$. Take $u_1, u_2, u_3, v_1, v_2 \in G$ such that $o(u_j) = 2^i$ and $o(v_j) = p$, then subgraph induced by the vertex set $\{u_1, u_2, u_3, v_1, v_2, e\}$ contains $K_{3,3}$. Therefore $m \leq 1$.

Consider the case $m = 1$, i.e., $|G| = 2p^n$. If G is not cyclic, then there are at least 4 minimal subgroups in G, otherwise the number of elements will not meet the cardinality of the group. Then by Theorem 2.4, $NIP(G)$ contains a K_5.

Now consider the case $|G| = p^n$, where p is any prime. If the number of subgroups in G of order p is not unique, then there are at least 2 distinct maximal cyclic subgroups, say A and B. If $|A| > 3$, then three elements in A, two elements in B together with identity forms a $K_{3,3}$ in $NIP(G)$. If $|A| \leq 3$ for every cyclic subgroup A of G, then there are at least 6 distinct cyclic subgroups, hence by Theorem 2.4, $NIP(G)$ contains a K_5. Hence, the possibilities left out for $NIP(G)$ to be planar are cyclic groups of orders p^n, $2p^n$ or generalized quaternion 2-group.

For the converse, since the non-intersection power graphs of cyclic p-groups and generalized quaternion 2-groups are star graphs, they are planar. If G is a cyclic group of order $2p^n$, every non-identity elements are adjacent to e, and elements of orders p^i are adjacent to the unique element of order 2, which is also planar. Hence, the statement of the theorem follows.

□

Theorem 2.6. *For a finite group G such that $|G| = n$, the following statements hold*

(1) $\alpha(NIP(G)) = 1$ if and only if G is an elementary abelian group of even order.

(2) $\alpha(NIP(G)) = n - 1$ if and only if G is a cyclic group of order 1 or q^n for some prime q and $n \in \mathbb{N}$ or G is a generalized quaternion 2-group.

Proof. The proof follows from Theorems 2.1 and 2.2. □

An EPPO group is a group in which every element has a prime order.

Theorem 2.7. *Let G be a finite group such that $|G| = n$, then the following statements hold.*

(1) $\alpha(NIP(G)) = 2$ if and only if G is an EPPO group with $|G| = 2^k 3^l$ where l and k are non-negative integers and $l \neq 0$.

(2) $\alpha(NIP(G)) = n - 2$ if and only if G is a cyclic group such that $|G| = 2q^k$ for some odd prime q and positive integer k.

Proof. (1) If some prime $p > 3$ devides $|G|$, then $\alpha \geq \phi(p) > 2$, since the set of all non-identity elements in a cyclic subgroup of prime order is an independent set. Suppose $|G| = 2^k$ for some positive integer k. If $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$, then $NIP(G)$ is complete and $\alpha = 1$. If G has a cyclic subgroup say G_c of order 4, then the non-identity elements in G_c are independent. Hence, $|G| \neq 2^k$.

Now suppose $|G| = 3^k$ for some positive integer k. Let $u \in G$ such that $o(u) = 9$, then the set $\{v : v \in \langle u \rangle\}$ is an independent set with cardinality 8. Therefore, every element in G has order 3.

The next possibility is $|G| = 2^k 3^l$, for some positive integers k, l. If G contains a cyclic subgroup of order greater than or equal to 4, then $\alpha > 2$, which is not possible. Hence an element in G has order either 2 or 3.

Conversely suppose G is an EPPO-group with $|G| = 2^k 3^l$, $l \neq 0$, and k and l are non-negative integers. If u is an element of order 2, then u is adjacent to every other vertex. If v is an element of order 3, then v is adjacent to every other vertex except v^{-1}. Hence, $\{v, v^{-1}\}$ is an independent set of maximum size in $NIP(G)$. Hence the result.

(2) Let I be a maximum independent vertex set in $NIP(G)$ and $|I| = n - 2$. Since $e \notin I$, every non-identity element in G except one belongs to I.

If the orders of two elements are relatively prime, then both elements cannot belong to I together. Hence, $|\pi(G)| \leq 2$. Let q_1 and q_2 be two primes which divide $|G|$. Let Λ_{q_1} be the set of all elements in G whose orders are powers of q_1, and Λ_{q_2} be the set of all elements in G whose orders are powers of q_2. If $u \in \Lambda_{q_1}$ such that $u \in I$, then no element in Λ_{q_2} can belongs to I. Therefore, $|\Lambda_{q_2}| \leq 1$. Also if G_{c_1} and G_{c_2} are two cyclic subgroups of G such that $G_{c_1} \cap G_{c_2} = \{e\}$ and if $u \in G_{c_1}$ such that $u \neq e$ and $u \in I$, then no element in G_{c_2} can belongs to I. Hence $|G_{c_2}| \leq 2$. Thus, the only possibility is G_{c_1} is a cyclic subgroup of order p^k for some odd prime p and positive integer k, G_{c_2} is the cyclic group of order 2 and G is a cyclic group of order $2p^k$.

□

In the next theorem we characterize all finite groups whose non-intersection power graphs and co-prime graphs are same.

Theorem 2.8. *Let G be a finite group and $Co(G)$ be the co-prime graph of G, then $Co(G) = NIP(G)$ if and only if there exists a unique subgroup of order p for every prime $p \mid |G|$.*

Proof. Clearly $Co(G) \subseteq NIP(G)$ for any finite group G, since for two vertices $u, v \in G$, $gcd(o(u), o(v)) = 1$ implies $\langle u \rangle \cap \langle v \rangle = \{e\}$.

First suppose that for some prime p there exist two distinct subgroups say P_1 and P_2 in G of order p each. Then, for the elements $u_1 \in P_1$ and $u_2 \in P_2$, $u_1 \sim u_2$ in $NIP(G)$, but $u_1 \nsim u_2$ in $Co(G)$, which implies $Co(G) \neq NIP(G)$.

For the converse, suppose that G is a finite group such that for every prime divisor p of $|G|$ there exists unique subgroup of order p in G. We prove that if two vertices u and v are adjacent in $NIP(G)$, then they are adjacent in $Co(G)$ also. Assume on the contrary that u and v are not adjacent in $Co(G)$, then $gcd(o(u), o(v)) \neq 1$. Let $p \mid gcd(o(u), o(v))$, then $p \mid o(u)$ and $p \mid o(v)$, which implies $\langle u \rangle$ has a subgroup of order p and $\langle v \rangle$ has a subgroup of order p. Then by the hypothesis G has a unique subgroup of order p, which gives $\langle u \rangle \cap \langle v \rangle \neq \{e\}$, and hence $u \nsim v$ in $NIP(G)$. □

From Theorem 2.8 it is clear that non-intersection power graph and co-prime graphs of every cyclic group are same.

Remark 2.2. *Burnside* [7] *proved that, if G is a p-group and has a unique subgroup of order p, then,*

a) if p is odd, G must be cyclic

b) if $p = 2$, then G is either cyclic or generalized quaternion.

If G is not a p-group with the property that for every prime divisor p of $|G|$ there exists unique subgroup of order p in G, then we can conclude that every Sylow p-subgroup of G is cyclic, if p is odd, and if $p = 2$, Sylow p-subgroup is either cyclic or generalized quaternion group. Burnside [7] proved that, if every Sylow subgroup of a group G are cyclic, then G is solvable, in fact such a group must be meta-cyclic. Suzuki [8] proved that if G is a non-solvable group with the property that every Sylow subgroups of odd order are cyclic, and moreover that the 2-Sylow subgroup is generalized quaternion, then G contains a normal subgroup G_1 such that $[G : G_1] \leq 2$ and $G_1 = G \times H$, where G is a solvable group whose Sylow subgroups are all cyclic, and H is isomorphic with the special linear group $SL(2, p)$.

3. Some topological indices of co-prime graphs of certain groups

Any graph that models a particular molecular structure can be given a topological graph index, also known as molecular descriptor. The physical and chemical qualities of molecules can be further investigated by analysing mathematical values obtained from this index.

When researching the boiling point of paraffin, Harold Wiener [4] developed the first distance-based topological index named 'path number', and later it was renamed as the Wiener index. The Wiener index of a graph G is defined as the sum of the shortest path distance between all pairs of vertices in G, which is denoted by $W(G)$.

ie,

$$W(G) = \sum_{u,v \in V(G)} d(u, v).$$

In 1993, Plavšić et al. [9] and Ivanciuc et al. [10] independently introduced Harary index $H(G)$ of a graph G, and it has been named in honor of Professor Frank Harary. The Harary index is defined as,

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d(u, v)}.$$

Gutman and Trinajstić [5] defined first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$ as follows,

$$M_1(G) = \sum_{u \in V(G)} (deg(u))^2$$

$$M_2(G) = \sum_{uv \in E(G)} deg(u)deg(v).$$

In this section, we find the structure of co-prime graphs of certain finite groups and calculate their Wiener index, Harary index and Zagreb index.

Theorem 3.1. *Let G be a finite group of order qr, where q and r are primes and $q < r$, then,*

$$Co(G) = \begin{cases} K_1 \vee (K_{q-1,r-1} \cup N_{(q-1)(r-1)}) & \text{if } G \text{ is abelian} \\ K_1 \vee K_{q-1,q(r-1)} & \text{if } G \text{ is non-abelian} \end{cases}$$

Proof. Let $\Lambda_q = \{u \in G : o(u) = q\}$, $\Lambda_r = \{u \in G : o(u) = r\}$ and $\Lambda_{qr} = \{u \in G : o(u) = qr\}$.

If $q \nmid (r-1)$, then G is a cyclic group. If $q \mid (r-1)$, then there are only two non-isomorphic groups each of order qr, one of which is abelian (which is again cyclic) and the other is non-abelian.

If G is a cyclic group of order qr, $|\Lambda_q| = q-1$, $|\Lambda_r| = r-1$ and $|\Lambda_{qr}| = (q-1)(r-1)$. Hence the edges set of $Co(G)$, $E(Co(G)) = \{uv : u \in \Lambda_q, v \in \Lambda_r\} \cup \{eu : u \in \Lambda_q \cup \Lambda_r \cup \Lambda_{qr}\}$. Hence the first case follows.

If G is non-abelian, then $G = \mathbb{Z}_r \rtimes \mathbb{Z}_q$ which has a unique Sylow-r subgroup and r number of Sylow-q subgroups. Also, $|\Lambda_q| = r(q-1)$, $|\Lambda_r| = r-1$ and $\Lambda_{qr} = \emptyset$. Hence $E(Co(G)) = \{uv : u \in \Lambda_q, v \in \Lambda_r\} \cup \{eu : u \in \Lambda_q \cup \Lambda_r\}$, and the subgroup induced by the vertex set $\Lambda_q \cup \Lambda_r$ is a complete bipartite graph. Hence the second case also follows. □

We denote N_n as the null graph with n vertices.

Theorem 3.2. *Let G be the cyclic group of order $q^n r^m$, where q and r are prime numbers, $q < r$, and m and n are positive integers. Then,*

$$Co(G) = K_1 \vee (K_{q^n-1, r^m-1} \cup N_{(q^n-1)(r^m-1)})$$

Proof. Let Λ_q be the set of all elements of orders $q, q^2, ...q^n$ and Λ_r be the set of all elements of orders $r, r^2, ...r^m$.

Then,

$$|\Lambda_q| = \phi(q) + \phi(q^2) + ... + \phi(q^n) = q^n - 1,$$

$$|\Lambda_r| = \phi(r) + \phi(r^2) + ... + \phi(r^m) = r^m - 1.$$

Clearly, the vertex set $\Lambda_q \cup \Lambda_r$ induces a complete bipartite subgraph K_{q^n-1, r^m-1}.
Let $\Lambda_0 = \{u \in G : o(u) = q^i r^j, 1 \le i \le n, 1 \le j \le m\}$.

Then,
$$|\Lambda_0| = \sum_{1 \le i \le n, 1 \le j \le m} \phi(q^i r^j)$$

$$= \sum_{i=1}^{m} \phi(qr^j) + \sum_{j=1}^{m} \phi(q^2 r^j) + ... + \sum_{j=1}^{m} \phi(q^n r^j)$$
$$= \big(\phi(q) + \phi(q^2) + ... + \phi(q^n)\big) \sum_{j=1}^{m} \phi(r^j)$$
$$= (q^n - 1)(r^m - 1).$$

Also, elements in Λ_0 are pendant vertices in $Co(G)$. Since e is adjacent to every other vertex, the result follows.

□

Theorem 3.3. *Let G be a finite group of order qr, where q and r are primes and $q < r$, then,*

the Wiener index,

$$W(Co(G)) = \begin{cases} (qr-1)^2 - (r-1)(q-1) & \text{if } G \text{ is abelian} \\ \\ (qr-1)^2 - r(r-1)(q-1) & \text{if } G \text{ is non-abelian} \end{cases}$$

The Harary index,

$$H(Co(G)) = \begin{cases} \frac{1}{2}(q-1)(r-1) + \frac{1}{4}(qr-1)(qr+2) & \textit{if } G \textit{ is abelian} \\ \\ \frac{1}{2}r(r-1)(q-1) + \frac{1}{4}(qr-1)(qr+2) & \textit{if } G \textit{ is non-abelian} \end{cases}$$

Proof. Since the diameter of the co-prime graph is 2, we have to find $d(Co(G), 1)$ and $d(Co(G), 2)$.

Let, $\Omega_0 = \{\{u, v\} : u, v \in V(Co(G)), u \neq v\}$
$\Omega_1 = \{\{u, v\} : u, v \in V(Co(G)), u \neq v, d(u, v) = 1\}$
$\Omega_2 = \{\{u, v\} : u, v \in V(Co(G)), u \neq v, d(u, v) = 2\}$.

Then, $\Omega_0 = \Omega_1 \cup \Omega_2$.

If G is abelian, by Theorem 3.1,

$$|\Omega_0| = \frac{qr(qr-1)}{2}$$

$$|\Omega_1| = d(Co(G), 1) = (q-1)(r-1) + qr - 1$$

$$|\Omega_2| = d(Co(G), 2) = |\Omega_0| - |\Omega_1| = \frac{(qr-1)(qr-2)}{2} - (q-1)(r-1).$$

Hence,

$$\begin{aligned} W(Co(G)) &= (q-1)(r-1) + qr - 1 + 2\left(\frac{(qr-1)(qr-2)}{2} - (q-1)(r-1)\right) \\ &= (qr-1)^2 - (q-1)(r-1) \\ H(Co(G)) &= (q-1)(r-1) + qr - 1 + \frac{1}{2}\left(\frac{(qr-1)(qr-2)}{2} - (q-1)(r-1)\right) \\ &= \frac{1}{2}(q-1)(r-1) + \frac{1}{4}(qr-1)(qr+2). \end{aligned}$$

If G is non-abelian, by Theorem 3.1,

$$|\Omega_1| = r(q-1)(r-1) + qr - 1$$

$$|\Omega_2| = \frac{(qr-1)(qr-2)}{2} - r(q-1)(r-1).$$

Hence by calculation, we get the required result. ☐

Theorem 3.4. *Let G be a finite group of order qr, where q and r are primes and $q < r$, then,*

The 1^{st} Zagreb index,

$$M_1(Co(G)) = \begin{cases} (qr+q+r)(qr-1) - q^2 - r^2 + 2 & \textit{if } G \textit{ is abelian} \\ \\ r^3q(q-1) + r(4q-3)(r-1) & \textit{if } G \textit{ is non-abelian} \end{cases}$$

The 2^{nd} Zagreb index,

$$M_2(Co(G)) = \begin{cases} qr(4qr - 3q - 3r - 1) + 3q + 2r - 1 & \textit{if } G \textit{ is abelian} \\ \\ 3qr^3(q-1) + 3r(r-1) - q^2r^2 + 1 & \textit{if } G \textit{ is non-abelian} \end{cases}$$

Proof. If G is abelian, using first case of Theorem 3.1, we get,

$$deg(u) = \begin{cases} qr-1 & \text{if } u = e \\ r & \text{if } u \in \Lambda_q \\ q & \text{if } u \in \Lambda_r \\ 1 & \text{if } u \in \Lambda_{qr} \end{cases}$$

Therefore,

$$\begin{aligned} M_1(Co(G)) &= (deg(e))^2 + \sum_{u\in\Lambda_q}(deg(u))^2 + \sum_{u\in\Lambda_r}(deg(u))^2 + \sum_{u\in\Lambda_0}(deg(u))^2 \\ &= (qr-1)^2 + (q-1)r^2 + (r-1)q^2 + (q-1)(r-1)1^2 \\ &= (qr+q+r)(qr-1) - q^2 - r^2 + 2. \end{aligned}$$

$$\begin{aligned} M_2(Co(G)) &= \sum_{uv\in E(Co(G))} deg(u)deg(v) \\ &= \sum_{uv\in E, u\in\Lambda_q, v\in\Lambda_r} rq \;+ \sum_{uv\in E, u=e, v\in\Lambda_q}(qr-1)r \;+ \\ &\quad \sum_{uv\in E, u=e, v\in\Lambda_r}(qr-1)q \;+ \sum_{uv\in E, u=e, v\in\Lambda_{qr}}(qr-1) \\ &= (q-1)(r-1)qr + (q-1)(qr-1)r + (r-1)(qr-1)q + (q-1)(r-1)(qr-1) \\ &= qr(4qr-3q-3r-1) + 3q + 2r - 1. \end{aligned}$$

If G is non-abelian, using second case of Theorem 3.1,

$$deg(u) = \begin{cases} qr-1 & \text{if } u = e \\ r & \text{if } u \in \Lambda_q \\ r(q-1)+1 & \text{if } u \in \Lambda_r \end{cases}$$

Therefore,

$$\begin{aligned} M_1(Co(G)) &= (deg(e))^2 + \sum_{u\in\Lambda_q}(deg(u))^2 + \sum_{u\in\Lambda_r}(deg(u))^2 \\ &= (qr-1)^2 + r(q-1)r^2 + (r-1)(r(q-1)+1)^2 \\ &= r^3q(q-1) + r(4q-3)(r-1). \end{aligned}$$

$$\begin{aligned} M_2(Co(G)) &= \sum_{uv\in E, u\in\Lambda_q, v\in\Lambda_r} r(r(q-1)+1) \;+ \sum_{uv\in E, u=e, v\in\Lambda_q}(qr-1)r \;+ \\ &\quad \sum_{uv\in E, u=e, v\in\Lambda_r}(qr-1)(r(q-1)+1) \\ &= (r-1)r(q-1)r(r(q-1)+1) + r(q-1)r(qr-1) + \end{aligned}$$

$$\begin{aligned}&(r-1)(qr-1)(r(q-1)+1)\\&=r^4(q^2-2q+1)+r^3(q^2+q-2)-r^2(q^2+q-3)-2r+1.\end{aligned}$$

□

Theorem 3.5. *Let G be the cyclic group of order $q^n r^m$, where q and r are prime numbers, $q<r$, and m and n are positive integers. Then,*

$$\begin{aligned}W(Co(G))&=(q^nr^m-1)^2-(q^n-1)(r^m-1)\\H(Co(G))&=\frac{1}{4}(q^nr^m-1)(q^nr^m+2)+\frac{1}{2}(q^n-1)(r^m-1)\\M_1(Co(G))&=(q^nr^m+r^m+q^n)(q^nr^m-1)-q^{2n}-r^{2m}+2\\M_2(Co(G))&=q^nr^m(4q^nr^m-3q^n-3r^m-1)+3q^n+2r^m-1.\end{aligned}$$

Proof. Using Theorem 3.2, we have

$$\begin{aligned}d(Co(G),1)&=(q^n-1)(r^m-1)+q^nr^m-1\\d(Co(G),2)&=\frac{1}{2}q^nr^m(q^nr^m-1)-(q^n-1)(r^m-1)-(q^nr^m-1)\\&=\frac{1}{2}(q^nr^m-1)(q^nr^m-2)-(q^n-1)(r^m-1)\\&=\frac{1}{2}(q^nr^m-1)(q^nr^m-2)-(q^n-1)(r^m-1).\end{aligned}$$

Hence,

$$\begin{aligned}W(Co(G))&=(q^n-1)(r^m-1)+q^nr^m-1+\\&\quad 2\left(\frac{1}{2}(q^nr^m-1)(q^nr^m-2)-(q^n-1)(r^m-1)\right)\\&=(q^nr^m-1)^2-(q^n-1)(r^m-1).\\H(Co(G))&=\frac{1}{1}\left((q^n-1)(r^m-1)+q^nr^m-1\right)+\\&\quad\frac{1}{2}\left(\frac{1}{2}(q^nr^m-1)(q^nr^m-2)-(q^n-1)(r^m-1)\right)\\&=\frac{1}{4}(q^nr^m-1)(q^nr^m+2)+\frac{1}{2}(q^n-1)(r^m-1).\end{aligned}$$

Now, $deg(u)=\begin{cases}1 & \text{if } u\in\Lambda_0\\ r^m & \text{if } u\in\Lambda_q\\ q^n & \text{if } u\in\Lambda_r\\ q^nr^m-1 & \text{if } u=e\end{cases}$

Therefore,

$$\begin{aligned}M_1(Co(G))&=\sum_{u\in\Lambda_q}(deg(u))^2+\sum_{u\in\Lambda_r}(deg(u))^2+\sum_{u\in\Lambda_0}(deg(u))^2+(deg(e))^2\\&=(q^n-1)r^{2m}+(r^m-r)q^{2n}+(q^n-1)(r^m-1)+(q^nr^m-1)^2\end{aligned}$$

$$= (q^n r^m + r^m + q^n)(q^n r^m - 1) - q^{2n} - r^{2m} + 2.$$

$$M_2(Co(G)) = \sum_{uv\in E, u\in\Lambda_q, v\in\Lambda_r} r^m q^n + \sum_{uv\in E, u=e, v\in\Lambda_q} (q^n r^m - 1) r^m +$$

$$\sum_{uv\in E, u=e, v\in\Lambda_r} (q^n r^m - 1) q^n + \sum_{uv\in E, u=e, v\in\Lambda_0} (q^n r^m - 1) 1$$

$$= (q^n - 1)(r^m - 1) r^m q^n + (q^n - 1)(q^n r^m - 1) r^m +$$

$$(r^m - 1)(q^n r^m - 1) q^n + (q^n - 1)(r^m - 1)(q^n r^m - 1)$$

$$= q^n r^m (4q^n r^m - 3q^n - 3r^m - 1) + 3q^n + 2r^m - 1.$$

□

4. Conclusions

In this paper, we focused on defining the non-intersection power graphs of finite groups and successfully characterized all groups for which their non-intersection power graphs are complete and star graphs. Furthermore, we established a significant result by proving that the non-intersection power graph and coprime graph coincide for certain groups. This finding highlights a notable connection between these two graph structures in the context of group theory.

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